# Optimality Conditions and Duality Results for Composite Vector Optimization Problem over Cones

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**Abstract**

*This study investigates optimality conditions and duality results for a class of composite vector optimization problems defined over cones. The objective and constraint functions are formulated as compositions of vector functions, and the analysis is conducted within a convex cone framework. Using Fenchel-Lagrange duality and conjugate function techniques, necessary and sufficient conditions are derived for a point to be a weak minimum. A corresponding vector dual problem is formulated, and both weak and strong duality results are established under appropriate convexity and constraint qualification assumptions. The results extend existing theories in composite optimization and provide a unified framework for analyzing such problems using conjugate duality methods.*

**Keywords** Composite vector optimization, Cone optimization, Fenchel-Lagrange duality, Conjugate functions, Weak efficiency, Convex analysis, Multiobjective optimization, Duality theory.

## 1 Introduction

The optimization problems which consist of maximizing or minimizing more than one objective function subject to some constraints where the decision space is ordered by cones are referred to as Vector Optimization Problem over Cones. Such problems arise in engineering, economics, and mathematics. Now, in such a model if we take objective functions and/or constraint functions as composition of functions, then the problem is referred to as Composite Vector Optimization problem over Cones. Many optimization problems arising from various directions can be formulated as optimization of some compositions of functions, such as, minimax problem, penalty method for constrained optimization problem, etc.

In 1991 Jeyakumar [13] initiated the study of composite optimization problems. He took up a nonsmooth optimization problem where the objective function and the constraint functions are compositions of locally Lipschitz and Gâteaux differentiable functions and presented necessary and sufficient optimality conditions. Jeyakumar and Yang [14] considered multi-objective optimization problems where the objective and the constraints are compositions of convex and locally Lipschitz functions and Gâteaux differentiable functions and established optimality and duality results. Jeyakumar and Yang [15] presented second-order optimality conditions for convex composite minimization problems in which the objective function is a composition of a lower semicontinuous convex function and a $C^{1,1}$

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function. Yang and Jeyakumar [16] established first-order optimality conditions for a weakly efficient solution of a convex composite multiobjective optimization problem subject to a closed convex constraint set via scalarization and then extended these conditions to derive second-order optimality conditions. Reddy and Mukherjee [3] proved generalized Karush-Kuhn-Tucker sufficient optimality conditions and duality results for a nonsmooth composite multiobjective programs using $V$-$\rho$-invexity conditions. Boţ et. al. [11] considered a convex composite optimization problem where the objective function is the composition of a cone-increasing convex function and a cone-convex function and the constraint is taken to be a cone-convex function. They established optimality conditions and duality results for this problem using Fenchel-Lagrangian duality results. Then Boţ et. al. [9] studied a convex multiobjective optimization problem where each component of the objective function is the composition of a cone-increasing convex function and a cone-convex function and the constraint is taken to be a cone-convex function. They proved necessary and sufficient optimality conditions for the weakly efficient solutions of this problem and gave duality results for a multiobjective dual. Recently, Tang and Zhao [4] considered a class of composite multiobjective nonsmooth optimization problems with cone constraints and established necessary and sufficient optimality conditions under cone-generalized invexity and generalized null space condition.

In this paper we consider a composite vector optimization problem where both, the objective and the constraint function, are composition of two vector functions. We use Fenchel-Lagrange duality which in turn uses conjugate functions to obtain optimality conditions to our composite vector optimization problem. Fenchel-Lagrange duality is a useful tool in establishing necessary and sufficient optimality condition for optimization problems and deriving duality results without using gradients or subgradients. This concept of Fenchel-Lagrange duality was introduced by Boţ and Wanka [7]. After that Boţ et. al. [6] established strong duality for nearly-convex optimization problems. Then Boţ and Wanka [8] used Fenchel-Lagrange dual problem to present some new Farkas-type results for inequality systems involving a finite as well as an infinite number of convex constraints. In 2006, Boţ et. al. [10] proved that the geometric duality is a special case of Fenchel-Lagrange duality. We have established necessary and sufficient optimality conditions in terms of conjugate functions for a point to be weak minimum of the given problem using cone-increasing and cone-convex functions. We have also formulated a vector dual for our primal problem and proved weak and strong duality results. In the end, we have reduced our composite problem to normal vector optimization problem over cones as special case and hence established optimality conditions and duality results containing conjugate functions.

## 2 Notations and Definitions

Let $K \subseteq \mathbb{R}^m$ and $P \subseteq \mathbb{R}^n$ be convex cones containing origin such that their interiors are nonempty, $h, h_1, h_2, \ldots, h_l : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $F : \mathbb{R}^s \longrightarrow \mathbb{R}^n$, where $f = (f_1, f_2, \ldots, f_m)^T$ and $F = (F_1, F_2, \ldots, F_n)^T$.

The unique smallest affine set containing $S$ (namely, the intersection of all the affine sets containing $S$) is called *affine hull* of $S$, denoted by aff$(S)$, or equivalently we conclude that

$$\operatorname{aff}(S) = \left\{ \sum_{i=1}^{l} t_i x_i : x_i \in S, \text{ for each } i = 1, 2, \ldots, l, \ \sum_{i=1}^{l} t_i = 1 \right\}.$$

The *relative interior* of $S$, denoted by ri$(S)$, is defined as the interior of $S$ with respect to affine hull of $S$. In other words,

$$\operatorname{ri}(S) = \{x \in S : \exists \epsilon > 0, B(x, \epsilon) \cap \operatorname{aff}(S) \subset S\}$$

where $B(x, \epsilon)$ is an open ball centered at $x$ and radius $\epsilon$.

The *indicator function* of a subset $S$ of $\mathbb{R}^n$ denoted by $\delta_S$ is defined as:

$$\delta_S : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}, \quad \delta_S(x) = \begin{cases} 0, & x \in S; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the *effective domain* of the function $h$, is the set

$$\operatorname{dom}(h) = \{x \in \mathbb{R}^n : h(x) < +\infty\}$$

and *effective domain* of the function $f$ is

$$\operatorname{dom}(f) = \bigcap_{i=1}^{m} \operatorname{dom}(f_i).$$

Also, we say that the function $h$ is *proper* if $\operatorname{dom}(h) \neq \phi$.

The *conjugate function of $h$ with respect to the set $S$* is defined as follows,

$$h_S^* : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}, \quad h_S^*(x^*) = \sup_{x \in S}\{{x^*}^T x - h(x)\}.$$

Regarding conjugate functions we have the following inequality known as *Fenchel's Inequality* or *Young-Fenchel Inequality* which holds for every $x \in S$ and $x^* \in \mathbb{R}^n$,

$$h(x) + h_S^*(x^*) - {x^*}^T x \geq 0.$$

Moreover, if $S = \mathbb{R}^n$, then the conjugate function of $h$ with respect to $S$ is called *conjugate function of $h$* and is denoted by $h^*$.

**Definition 2.1.** *The* infimal convolution *of the proper functions $h_1, h_2, \ldots, h_l$ is the function*

$$h_1 \square h_2 \square \ldots \square h_l : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}, \quad (h_1 \square h_2 \square \ldots \square h_l)(x) = \inf\left\{\sum_{i=1}^{l} h_i(x_i) : x = \sum_{i=1}^{l} x_i\right\}$$

We have the following result which includes both conjugate functions and infimal convolution.

**Theorem 2.1.** *[12] Let $h_1, h_2, \ldots, h_l$ be proper convex functions and the set $\bigcap_{i=1}^{l} ri(dom(h_i))$ be nonempty, then*

$$\left(\sum_{i=1}^{l} h_i\right)^*(x) = (h_1^* \square h_2^* \square \ldots \square h_l^*)(x) = \inf\left\{\sum_{i=1}^{l} h_i^*(x_i) : x = \sum_{i=1}^{l} x_i\right\}$$

*and for each $x \in \mathbb{R}^n$ the infimum is attained.*

The positive dual cone $K^+$ of $K$ is defined as follows:

$$K^+ = \{\lambda \in \mathbb{R}^m : \lambda^T x \geq 0, \quad \text{for all } x \in K\}.$$

**Definition 2.2.** *The function $f$ is said to be* $(P, K)$-increasing *(or nondecreasing on $\mathbb{R}^n$ with respect to $(P, K)$ [1, Definition 4.1]), if for all $x, u \in \mathbb{R}^n$,*

$$x \quad u \in P \implies f(x) \quad f(u) \in K.$$

**Definition 2.3.** *The function $f$ is said to be $K$-*convex *if for all $x, u \in \mathbb{R}^n$ and $t \in [0, 1]$,*

$$tf(x) + (1-t)f(u) - f(tx + (1-t)u) \in K.$$

**Lemma 2.1.** *If $f$ is $K$-convex and $(P, K)$-increasing and $F$ is $P$-convex function, then $f \circ F$ is $K$-convex function.*

*Proof.* Let $x, u \in \mathbb{R}^s$ and $t \in [0,1]$ be arbitrary then since $F$ is $P$-convex, therefore

$$tF(x) + (1-t)F(u) - F(tx + (1-t)u) \in P$$

and applying the fact that $f$ is $(P, K)$-increasing, we get

$$f\left(tF(x) + (1-t)F(u)\right) - f\left(F(tx + (1-t)u)\right) \in K.$$

Now using that $f$ is $K$-convex, we have

$$tf(F(x)) + (1-t)f(F(u)) - f(tF(x) + (1-t)F(u)) \in K$$

Adding the above two relations, we get

$$tf(F(x)) + (1-t)f(F(u)) - f\left(F(tx + (1-t)u)\right) \in K$$

that is,

$$t(f \circ F)(x) + (1-t)(f \circ F)(u) - (f \circ F)(tx + (1-t)u) \in K$$

and hence $f \circ F$ is $K$-convex function. □

## 3 Optimality Conditions

Consider the vector optimization problem

**(CVP)** $K$-Minimize $(f \circ F)(x) = ((f_1 \circ F)(x), (f_2 \circ F)(x), \ldots, (f_m \circ F)(x))^T$

subject to

$$-(g \circ G)(x) = -\left((g_1 \circ G)(x), (g_2 \circ G)(x), \ldots, (g_p \circ G)(x)\right)^T \in Q,$$

where $F : \mathbb{R}^s \longrightarrow \mathbb{R}^n$ and $G : \mathbb{R}^s \longrightarrow \mathbb{R}^n$ are vector-valued functions such that $F = (F_1, F_2, \ldots, F_n)^T$ and $G = (G_1, G_2, \ldots, G_n)^T$. For each $i = 1, 2, \ldots, m$, $f_i : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is proper function and hence $f_i \circ F$ is also a proper function. For each $j = 1, 2, \ldots, p$, $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$. $P_1, P_2 \subseteq \mathbb{R}^n$, $Q \subseteq \mathbb{R}^p$ and $K \subseteq \mathbb{R}^m$ are closed convex pointed cones containing origin such that each one has nonempty interior. The feasible set of (CVP) is given by

$$S_0 = \{x \in \mathbb{R}^s : -(g \circ G)(x) \in Q\}.$$

We also assume that

$$S_0 \subseteq Y,$$

where $Y = F^{\ 1}(\mathrm{dom}(f)) = \{x \in \mathbb{R}^s : F(x) \in \mathrm{dom}(f_i), \text{ for every } i = 1, 2, \ldots, m\}$.

**Remark 3.1.**

(i) *Let $m = 1$, $p = m$, $K = \mathbb{R}_+$, $Q = \mathbb{R}^m_+$, $f = f_0$ and $F = F_0$. Also for each $j = 1, 2, \ldots, m$ take $g_j = f_j$ and $G_j = F_j$ then the Problem (CVP) reduces to the problem (P) considered by Jeyakumar [13] where $X = \mathbb{R}^s$.*

(ii) *Taking $m = p$, $p = m$, $K = \mathbb{R}^p_+$, $Q = \mathbb{R}^m_+$ then the Problem (CVP) reduces to the problem (P) considered by Jeyakumar and Yang [14] and problem (CP) considered by Reddy and Mukherjee [3] where $C = X = \mathbb{R}^s$.*

(iii) *If we take $s = n$, $m = k$, $p = m$, $K = \mathbb{R}^k_+$, $Q = K$, $G(x) = x$ and $P_2 = \{0\}$ then the Problem (CVP) reduces to the problem (P) considered by Bot et al. [9] where $X = \mathbb{R}^n$, $n_i = n$, $F_i(x) = F(x)$ and $K_i = P_1$ for $i = 1, 2, \ldots, m$.*

(iv) *If we take $s = n$, $n = k$, $m = 1$, $p = m$, $K = \mathbb{R}_+$, $Q = C$, $G(x) = x$ and $P_2 = \{0\}$ then the Problem (CVP) reduces to the problem $(P_c)$ considered by Bot et al. [11] where $X = \mathbb{R}^n$.*

**Definition 3.1.** *A point $\overline{x} \in S_0$ is called a* weak minimum *of (CVP), if for all $x \in S_0$,*

$$(f \circ F)(\overline{x}) - (f \circ F)(x) \notin intK.$$

For a given $\lambda \in K^+ \setminus \{0\}$, consider the following problem,

$$\textbf{(CVP)}_\lambda \quad \inf \ \lambda^T (f \circ F)(x) = \sum_{i=1}^{m} \lambda_i \left(f_i \circ F\right)(x)$$

subject to

$$-(g \circ G)(x) = -\left((g_1 \circ G)(x), (g_2 \circ G)(x), \ldots, (g_p \circ G)(x)\right)^T \in Q.$$

The feasible set of $(\text{CVP})_\lambda$ is same as that of (CVP), that is, $S_0$.

Below we give a lemma to establish a relation between the optimal point of the problem $(\text{CVP})_\lambda$ and (CVP).

**Lemma 3.1.** *If, for some $\overline{\lambda} \in K^+ \setminus \{0\}$, $\overline{x} \in S_0$ is optimal solution of $(CVP)_{\overline{\lambda}}$, then $\overline{x}$ is a weak minimum of (CVP). Conversely, if $\overline{x}$ is a weak minimum of (CVP) and the following hold*

(i) *$f$ be $K$-convex, $g$ be $Q$-convex, $F$ be $P_1$-convex and $G$ be $P_2$-convex*

(ii) *$f$ be $(P_1, K)$-increasing and $g$ be $(P_2, Q)$-increasing function*

*Then there exists $\overline{\lambda} \in K^+ \setminus \{0\}$ such that $\overline{x} \in S_0$ is optimal solution of $(CVP)_{\overline{\lambda}}$.*

*Proof.* Suppose for some $\overline{\lambda} \in K^+ \setminus \{0\}$, $\overline{x} \in S_0$ is optimal solution of $(\text{CVP})_{\overline{\lambda}}$, then for every $x \in S_0$,

$$\sum_{i=1}^{m} \overline{\lambda}_i \left(f_i \circ F\right)(\overline{x}) \leq \sum_{i=1}^{m} \overline{\lambda}_i \left(f_i \circ F\right)(x). \tag{1}$$

Let, if possible, $\overline{x}$ be not a weak minimum of (CVP). Then there exists $x' \in S_0$ such that

$$(f \circ F)(\overline{x}) - (f \circ F)(x') \in \text{int}K.$$

Since $\overline{\lambda} \in K^+ \setminus \{0\}$, we have

$$\sum_{i=1}^{m} \overline{\lambda}_i \left(f_i \circ F\right)(\overline{x}) > \sum_{i=1}^{m} \overline{\lambda}_i \left(f_i \circ F\right)(x')$$

which is a contradiction to the inequality (1). Hence $\overline{x}$ is a weak minimum of (CVP).

Conversely, suppose $\overline{x}$ is a weak minimum of (CVP), then in view of Definition 3.1 we have

$$\{(f \circ F)(S_0) - (f \circ F)(\overline{x})\} \cap \{-\text{int}K\} = \phi. \tag{2}$$

Now using the conditions (i), (ii) and Lemma 2.1, it follows that $f \circ F$ is $K$-convex and $g \circ G$ is $Q$-convex, hence the feasible set $S_0$ and hence $(f \circ F)(S_0)$ are convex sets.

Therefore from equation (2) we have $(f \circ F)(S_0) - (f \circ F)(\overline{x})$ and $-\text{int}K$ are two disjoint convex sets. Now applying the separation theorem given by Jahn [2, Theorem 3.14], there exists $\overline{\lambda} \in \mathbb{R}^m \setminus \{0\}$ and a real number $t$, such that

$$\overline{\lambda}^T (f \circ F)(x) \quad \overline{\lambda}^T (f \circ F)(\overline{x}) \geq t \quad \text{for every } x \in S_0 \tag{3}$$

and

$$-(\overline{\lambda}^T k) < t \quad \text{for every } k \in \text{int}K. \tag{4}$$


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Substituting $x = \overline{x}$ in (3), we get

$$t \leq 0. \tag{5}$$

Now, for any fix $k' \in \text{int}K$ and $\epsilon > 0$, $\epsilon k' \in \text{int}K$ and substituting this in (4), we get

$$-\epsilon(\overline{\lambda}^T k') < t.$$

Since $\epsilon$ can be arbitrarily small, therefore we have

$$t \geq 0. \tag{6}$$

From (5) and (6), we get

$$t = 0.$$

So, (4) can be written as

$$-(\overline{\lambda}^T k) < 0 \quad \text{for every } k \in \text{int}K.$$

That is,

$$\overline{\lambda}^T k > 0 \quad \text{for every } k \in \text{int}K,$$

which implies that $\overline{\lambda} \in K^+ \setminus \{0\}$.

Also from (3) we have

$$\overline{\lambda}^T (f \circ F)(x) - \overline{\lambda}^T (f \circ F)(\overline{x}) \geq 0 \quad \text{for every } x \in S_0$$

or,

$$\sum_{i=1}^{m} \overline{\lambda}_i \left(f_i \circ F\right)(\overline{x}) \leq \sum_{i=1}^{m} \overline{\lambda}_i \left(f_i \circ F\right)(x) \quad \text{for every } x \in S_0,$$

Hence $\overline{x}$ is an optimal solution of $(\text{CVP})_{\overline{\lambda}}$. □

Next we consider a non-composite problem and establish some results which will be used to prove necessary optimality conditions. The problem is as follows:

$$\begin{aligned} \textbf{(CVP)}^1_\lambda \quad & \inf \ \Phi(x,y,z) \\ & \text{subject to} \\ & \qquad \Psi(x,y,z) \in \overline{Q}, \\ & \qquad (x,y,z) \in S = \mathbb{R}^s \times Y \times \mathbb{R}^n, \end{aligned}$$

where

$$\Phi : S \longrightarrow \overline{\mathbb{R}}, \quad \text{is defined as} \quad \Phi(x,y,z) = (\lambda^T f)(y),$$

$$\Psi : S \longrightarrow \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n, \quad \text{is defined as} \quad \Psi(x,y,z) = \begin{pmatrix} & g(z) \\ F(x) & y \\ G(x) & z \end{pmatrix},$$

$\overline{Q} = Q \times P_1 \times P_2$. Let $S_0^1$ be the feasible set of the problem $(\text{CVP})^1_\lambda$, that is,

$$S_0^1 = \left\{(x,y,z) \in S : -g(z) \in Q, -(F(x) - y) \in P_1, -(G(x) - z) \in P_2\right\}.$$

We assume here that $\text{ri}(S) \neq \phi$.

Note that when $f$ is $K$-convex, $g$ is $Q$-convex, $F$ is $P_1$-convex and $G$ is $P_2$-convex, then for any $\lambda \in K^+ \setminus \{0\}$, $(\lambda^T f) = \Phi$ is a convex function and $\Psi$ is a $\overline{Q}$-convex function.

We use the following constraint qualification for proving the next lemma.

**Definition 3.2.** *The constraint qualification (CQ) is said to hold if there exists* $(x', y', z') \in S$ *such that*

$$-g(z') \in intQ, \quad -(F(x') - y') \in intP_1 \quad and \quad -(G(x') - z') \in intP_2.$$

**Lemma 3.2.** *Let for some* $\overline{\lambda} \in K^+ \setminus \{0\}$, $(\overline{x}, \overline{y}, \overline{z}) \in S_0^1$ *be an optimal solution of* $(CVP)_{\frac{1}{\overline{\lambda}}}$. *Suppose the constraint qualification (CQ) holds,* $f$ *is* $K$*-convex,* $g$ *is* $Q$*-convex,* $F$ *is* $P_1$*-convex and* $G$ *is* $P_2$*-convex, then*

$$\Phi(\overline{x}, \overline{y}, \overline{z}) = \sup_{\gamma \in \overline{Q}^+} \inf_{\substack{x \in \mathbb{R}^s, \\ y \in Y, \\ z \in \mathbb{R}^n}} \left\{ \Phi(x, y, z) + (\gamma^T \Psi)(x, y, z) \right\},$$

*where the supremum is attained.*

*Proof.* Since $f$ is $K$-convex, $g$ is $Q$-convex, $F$ is $P_1$-convex, $G$ is $P_2$-convex and $\overline{\lambda} \in K^+ \setminus \{0\}$, we have $\Phi$ is convex and $\Psi$ is $\overline{Q}$-convex. Also $(\overline{x}, \overline{y}, \overline{z})$ is an optimal solution of $(\text{CVP})_{\frac{1}{\overline{\lambda}}}$. Therefore by Theorem 3 given by Bazaraa [5], there exist $\overline{\tau} \geq 0$, $\overline{\gamma}' \in \overline{Q}^+$ with $(\overline{\tau}, \overline{\gamma}') \neq 0$, such that

$$\begin{aligned} \overline{\tau}\Phi(x, y, z) + (\overline{\gamma}'^T \Psi)(x, y, z) &\geq \overline{\tau}\Phi(\overline{x}, \overline{y}, \overline{z}) + (\overline{\gamma}'^T \Psi)(\overline{x}, \overline{y}, \overline{z}) \\ &\geq \overline{\tau}\Phi(\overline{x}, \overline{y}, \overline{z}) + (\gamma'^T \Psi)(\overline{x}, \overline{y}, \overline{z}) \quad \text{for all } (x, y, z) \in S \ \& \ \gamma' \in \overline{Q}^+. \end{aligned} \tag{7}$$

If we substitute $\gamma' = 0 \in \overline{Q}^+$ in right hand side of the inequality (7), we get $(\overline{\gamma}'^T \Psi)(\overline{x}, \overline{y}, \overline{z}) \geq 0$ and due to the fact that $(\overline{x}, \overline{y}, \overline{z}) \in S_0^1$ and $\overline{\gamma}' \in \overline{Q}^+$, we have $(\overline{\gamma}'^T \Psi)(\overline{x}, \overline{y}, \overline{z}) \leq 0$, therefore combining the two we have

$$(\overline{\gamma}'^T \Psi)(\overline{x}, \overline{y}, \overline{z}) = 0. \tag{8}$$

Substituting this equation (8) in right hand side of inequality (7), we get

$$\overline{\tau}\Phi(x, y, z) + (\overline{\gamma}'^T \Psi)(x, y, z) \geq \overline{\tau}\Phi(\overline{x}, \overline{y}, \overline{z}) \quad \text{for all } (x, y, z) \in S. \tag{9}$$

Using the above constraint qualification (CQ), we prove that $\overline{\tau} \neq 0$. Let, if possible, $\overline{\tau} = 0$, then we have $\overline{\gamma}' \neq 0$ and

$$(\overline{\gamma}'^T \Psi)(x, y, z) \geq 0 \quad \text{for all } x \in \mathbb{R}^s, y \in Y, z \in \mathbb{R}^n. \tag{10}$$

Since $\overline{\gamma}' \in \overline{Q}^+ = Q^+ \times P_1^+ \times P_2^+$, we can write $\overline{\gamma}' = (\overline{\mu}', \overline{\alpha}', \overline{\beta}') \in Q^+ \times P_1^+ \times P_2^+$. Using the constraint qualification (CQ), we have $(x', y', z') \in S$ such that

$$-g(z') \in \text{int}Q, -(F(x') - y') \in \text{int}P_1 \text{ and } -(G(x') - z') \in \text{int}P_2.$$

Using the fact that $\overline{\mu}' \in Q^+, \overline{\alpha}' \in P_1^+, \overline{\beta}' \in P_2^+$ and $(\overline{\mu}', \overline{\alpha}', \overline{\beta}') \neq 0$, we have

$$(\overline{\gamma}'^T \Psi)(x', y', z') = (\overline{\mu}'^T g)(z') + \overline{\alpha}'^T (F(x') - y') + \overline{\beta}'^T (G(x') - z') < 0$$

which is a contradiction to (10). Therefore $\overline{\tau} \neq 0$. Now dividing (9) by $\overline{\tau}$ and taking $\frac{1}{\overline{\tau}}\overline{\gamma}' = \overline{\gamma}$, we get

$$\Phi(x, y, z) + (\overline{\gamma}^T \Psi)(x, y, z) \geq \Phi(\overline{x}, \overline{y}, \overline{z}) \quad \text{for all } (x, y, z) \in S \text{ or } x \in \mathbb{R}^s, y \in Y, z \in \mathbb{R}^n, \tag{11}$$

hence,

$$\Phi(\overline{x}, \overline{y}, \overline{z}) \leq \inf_{\substack{x \in \mathbb{R}^s, \\ y \in Y, \\ z \in \mathbb{R}^n}} \left\{ \Phi(x, y, z) + (\overline{\gamma}^T \Psi)(x, y, z) \right\} \leq \sup_{\gamma \in Q^+} \inf_{\substack{x \in \mathbb{R}^s, \\ y \in Y, \\ z \in \mathbb{R}^n}} \left\{ \Phi(x, y, z) + (\gamma^T \Psi)(x, y, z) \right\},$$

that is,

$$\Phi(\overline{x}, \overline{y}, \overline{z}) \leq \sup_{\gamma \in \overline{Q}^+} \inf_{\substack{x \in \mathbb{R}^s, \\ y \in Y, \\ z \in \mathbb{R}^n}} \left\{ \Phi(x, y, z) + (\gamma^T \Psi)(x, y, z) \right\}. \tag{12}$$


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Also for any $(x,y,z) \in S_0^1$ and any $\gamma \in \overline{Q}^+$, $(\gamma^T\Psi)(x,y,z) \le 0$ and hence

$$\Phi(x,y,z) \ge \Phi(x,y,z) + (\gamma^T\Psi)(x,y,z),$$

thus,

$$\begin{aligned}\Phi(\overline{x},\overline{y},\overline{z}) = \inf_{(x,y,z)\in S_0^1} \Phi(x,y,z) \quad &\ge \quad \inf_{(x,y,z)\in S_0^1} \left\{\Phi(x,y,z) + (\gamma^T\Psi)(x,y,z)\right\} \\ &\ge \quad \inf_{\substack{x\in\mathbb{R}^s,\\ y\in Y,\\ z\in\mathbb{R}^n}} \left\{\Phi(x,y,z) + (\gamma^T\Psi)(x,y,z)\right\},\end{aligned}$$

and consequently we get

$$\Phi(\overline{x},\overline{y},\overline{z}) \ge \sup_{\gamma\in\overline{Q}^+} \inf_{\substack{x\in\mathbb{R}^s,\\ y\in Y,\\ z\in\mathbb{R}^n}} \left\{\Phi(x,y,z) + (\gamma^T\Psi)(x,y,z)\right\}. \tag{13}$$

From (12) and (13), we have

$$\Phi(\overline{x},\overline{y},\overline{z}) = \sup_{\gamma\in\overline{Q}^+} \inf_{\substack{x\in\mathbb{R}^s,\\ y\in Y,\\ z\in\mathbb{R}^n}} \left\{\Phi(x,y,z) + (\gamma^T\Psi)(x,y,z)\right\}. \tag{14}$$

Now putting $\frac{1}{\overline{\tau}}\overline{\gamma}' = \overline{\gamma}$ in (8), we get

$$(\overline{\gamma}^T\Psi)(\overline{x},\overline{y},\overline{z}) = 0. \tag{15}$$

From above two equations we have

$$\Phi(\overline{x},\overline{y},\overline{z}) + (\overline{\gamma}^T\Psi)(\overline{x},\overline{y},\overline{z}) = \sup_{\gamma\in\overline{Q}^+} \inf_{\substack{x\in\mathbb{R}^s,\\ y\in Y,\\ z\in\mathbb{R}^n}} \left\{\Phi(x,y,z) + (\gamma^T\Psi)(x,y,z)\right\}.$$

Hence the supremum is attained in (14). □

**Lemma 3.3.** *Let $f$ is $K$-convex, $g$ is $Q$-convex, $F$ is $P_1$-convex and $G$ is $P_2$-convex, then for any $\lambda \in K^+ \setminus \{0\}$, $\mu \in Q^+$, $\alpha \in P_1^+$ and $\beta \in P_2^+$, we have*

$$\inf_{\substack{x\in\mathbb{R}^s,\\ y\in Y,\\ z\in\mathbb{R}^n}} \Phi(x,y,z) + (\gamma^T\Psi)(x,y,z) = \sup_{v\in\mathbb{R}^s} \left\{-(\lambda^T f)^*(\alpha) - (\mu^T g)^*(\beta) - (\alpha^T F)^*(v) - (\beta^T G)^*(-v)\right\},$$

*where the supremum is attained.*

*Proof.* Under the given conditions, we have $\Phi$ is a proper convex function on $S$ and $\Psi$ being $\overline{Q}$-convex, $(\gamma^T\Psi)$ is a convex function on $S$, for any $\gamma \in \overline{Q}^+$. So, we have to work on their convex extensions to $\mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^n$, say $\widehat{\Phi}$ and $\widehat{\Psi}$, defined as follows

$$\widehat{\Phi}(x,y,z) = \begin{cases} \Phi(x,y,z), & (x,y,z) \in S \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$\widehat{\Psi}(x,y,z) = \begin{cases} \Psi(x,y,z), & (x,y,z) \in S \\ +\infty, & \text{otherwise} \end{cases}$$

Also $\mathrm{dom}(\widehat{\Phi}) = \mathrm{dom}(\gamma^T\widehat{\Psi}) = S$. Then by Theorem 31.1 of [12], we have for any $\gamma \in \overline{Q}^+$,

$$\inf_{\substack{x\in\mathbb{R}^s,\\ y,z\in\mathbb{R}^n}} \left\{\widehat{\Phi}(x,y,z) + (\gamma^T\widehat{\Psi})(x,y,z)\right\} = \sup_{\substack{x^*\in\mathbb{R}^s,\\ y^*,z^*\in\mathbb{R}^n}} \left\{ \quad \widehat{\Phi}^*(x^*,y^*,z^*) \quad (\gamma^T\widehat{\Psi})^*( \quad x^*, \quad y^*, \quad z^*)\right\}$$


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and since $\mathrm{ri}(S) \neq \phi$, so the supremum is attained at some $(x^*, y^*, z^*)$.

So, for any $\gamma \in \overline{Q}^+$ we get

$$\begin{aligned}
\inf_{\substack{x\in\mathbb{R}^s,\\ y\in Y,\\ z\in\mathbb{R}^n}} \{\Phi(x,y,z) + (\gamma^T\Psi)(x,y,z)\} &= \inf_{\substack{x\in\mathbb{R}^s,\\ y,z\in\mathbb{R}^n}} \left\{\widehat{\Phi}(x,y,z) + (\gamma^T\widehat{\Psi})(x,y,z)\right\} \\
&= \sup_{\substack{x^*\in\mathbb{R}^s,\\ y^*,z^*\in\mathbb{R}^n}} \left\{-\widehat{\Phi}^*(x^*,y^*,z^*) - (\gamma^T\widehat{\Psi})^*(-x^*,-y^*,-z^*)\right\} \\
&= \sup_{\substack{x^*\in\mathbb{R}^s,\\ y^*,z^*\in\mathbb{R}^n}} \left\{-\Phi_S^*(x^*,y^*,z^*) - (\gamma^T\Psi)_S^*(-x^*,-y^*,-z^*)\right\}. \qquad (16)
\end{aligned}$$

Now, using the definition of conjugate functions, we have

$$\begin{aligned}
\Phi_S^*(x^*,y^*,z^*) &= \sup_{\substack{x\in\mathbb{R}^s\\ y\in Y,\\ z\in\mathbb{R}^n}} \left\{{x^*}^T x + {y^*}^T y + {z^*}^T z - \Phi(x,y,z)\right\} \\
&= \sup_{x\in\mathbb{R}^n} \left\{{x^*}^T x\right\} + \sup_{y\in Y} \left\{{y^*}^T y - (\lambda^T f)(y)\right\} + \sup_{z\in\mathbb{R}^n} \left\{{z^*}^T z\right\} \\
&= \sup_{x\in\mathbb{R}^n} \left\{{x^*}^T x\right\} + (\lambda^T f)_Y^*(y^*) + \sup_{z\in\mathbb{R}^n} \left\{{z^*}^T z\right\} \\
&= \sup_{x\in\mathbb{R}^n} \left\{{x^*}^T x\right\} + (\lambda^T f)^*(y^*) + \sup_{z\in\mathbb{R}^n} \left\{{z^*}^T z\right\}.
\end{aligned}$$

Now we must have $x^* = 0 = z^*$, otherwise $\sup_{x\in\mathbb{R}^n} \left\{{x^*}^T x\right\} = +\infty$ and $\sup_{z\in\mathbb{R}^n} \left\{{z^*}^T z\right\} = +\infty$.

Therefore,

$$\Phi_S^*(x^*,y^*,z^*) = (\lambda^T f)^*(y^*).$$

Also for any $\gamma = (\mu,\alpha,\beta) \in \overline{Q}^+ = Q^+ \times P_1^+ \times P_2^+$, we have

$$\begin{aligned}
&(\gamma^T\Psi)_S^*(-x^*,-y^*,-z^*) \\
&\quad = \sup_{\substack{x\in\mathbb{R}^s\\ y\in Y\\ z\in\mathbb{R}^n}} \left\{-{x^*}^T x - {y^*}^T y - {z^*}^T z - (\gamma^T\Psi)(x,y,z)\right\} \\
&\quad = \sup_{x\in\mathbb{R}^s} \left\{-{x^*}^T x - (\alpha^T F)(x) - (\beta^T G)(x)\right\} + \sup_{y\in Y} \left\{-{y^*}^T y + \alpha^T y\right\} + \sup_{z\in\mathbb{R}^n} \left\{-{z^*}^T z - (\mu^T g)(z) + \beta^T z\right\}.
\end{aligned}$$

From above we have $x^* = 0 = z^*$, and hence

$$\begin{aligned}
&(\gamma^T\Psi)_S^*(-x^*,-y^*,-z^*) \\
&\qquad = \sup_{x\in\mathbb{R}^s} \{-(\alpha^T F)(x) - (\beta^T G)(x)\} + \sup_{y\in Y} \left\{-{y^*}^T y + \alpha^T y\right\} + \sup_{z\in\mathbb{R}^n} \{\beta^T z - (\mu^T g)(z)\} \\
&\qquad = \left(\alpha^T F + \beta^T G\right)^* (0) + \delta_Y^* (\alpha - y^*) + \left(\mu^T g\right)^* (\beta).
\end{aligned}$$

Due to condition (i) and the fact that $\alpha \in P^+$ and $\beta \in P^+$, $\alpha^T F$ and $\beta^T G$ are convex functions. So by using Theorem 2.1, we get

$$\begin{aligned}
(\gamma^T\Psi)_S^*(-x^*,-y^*,-z^*) &= \inf_{v\in\mathbb{R}^s} \{(\alpha^T F)^*(v) + (\beta^T G)^*(-v)\} + \delta_Y^* (\alpha - y^*) + \left(\mu^T g\right)^* (\beta) \\
&= -\sup_{v\in\mathbb{R}^s} \left\{-(\alpha^T F)^*(v) - (\beta^T G)^*(-v) - \delta_Y^* (\alpha - y^*) - \left(\mu^T g\right)^* (\beta)\right\}
\end{aligned}$$

and the supremum is attained.

Therefore, for any $\gamma = (\mu, \alpha, \beta) \in \overline{Q}^+$

$$\sup_{\substack{x^* \in \mathbb{R}^s \\ y^*, z^* \in \mathbb{R}^n}} \left\{ -\Phi_S^*(x^*, y^*, z^*) - (\gamma^T \Psi)_S^*(-x^*, -y^*, -z^*) \right\}$$

$$= \sup_{\substack{v \in \mathbb{R}^s, \\ y^* \in \mathbb{R}^n}} \left\{ -(\lambda^T f)^*(y^*) - \delta_Y^*(\alpha - y^*) - (\alpha^T F)^*(v) - (\beta^T G)^*(-v) - (\mu^T g)^*(\beta) \right\}. \quad (17)$$

Now, as $f$ is a $K$-convex function, $\lambda \in K^+ \setminus \{0\}$ and for each $i = 1, 2, \ldots, m$, $f_i$ is a proper function, so $\lambda^T f$ is a proper convex function. Therefore by Theorem 2.1, we have

$$\begin{aligned} \sup_{y^* \in \mathbb{R}^n} \left\{ -(\lambda^T f)^*(y^*) - \delta_Y^*(\alpha - y^*) \right\} &= -\inf_{y^* \in \mathbb{R}^n} \left\{ (\lambda^T f)^*(y^*) + \delta_Y^*(\alpha - y^*) \right\} \\ &= -\left( \lambda^T f + \delta_Y \right)^*(\alpha) \\ &= -(\lambda^T f)^*(\alpha). \end{aligned}$$

Therefore, (17) can also be written as follows

$$\sup_{\substack{x^* \in \mathbb{R}^s \\ y^*, z^* \in \mathbb{R}^n}} \left\{ -\Phi_S^*(x^*, y^*, z^*) - (\gamma^T \Psi)_S^*(-x^*, -y^*, -z^*) \right\}$$

$$= \sup_{v \in \mathbb{R}^s} \left\{ -(\lambda^T f)^*(\alpha) - (\mu^T g)^*(\beta) - (\alpha^T F)^*(v) - (\beta^T G)^*(-v) \right\}. \quad (18)$$

Combining (16) and (18), for any $\gamma = (\mu, \alpha, \beta) \in \overline{Q}^+ = Q^+ \times P_1^+ \times P_2^+$ we get

$$\inf_{\substack{x \in \mathbb{R}^s, \\ y \in Y, \\ z \in \mathbb{R}^n}} \Phi(x, y, z) + (\gamma^T \Psi)(x, y, z) = \sup_{v \in \mathbb{R}^s} \left\{ -(\lambda^T f)^*(\alpha) - (\mu^T g)^*(\beta) - (\alpha^T F)^*(v) - (\beta^T G)^*(-v) \right\}$$

and the supremum is attained. $\square$

Now we arrive at the main results of this paper. In the following theorems we will prove the necessary and sufficient optimality conditions for the problem (CVP).

**Theorem 3.1.** *Let $\overline{x} \in S_0$ be a weak minimum of the problem (CVP) and the following conditions hold:*

*(i) $f$ is $K$-convex, $g$ is $Q$-convex, $F$ is $P_1$-convex and $G$ is $P_2$-convex function*

*(ii) $f$ is $(P_1, K)$-increasing and $g$ is $(P_2, Q)$-increasing function*

*(iii) constraint qualification (CQ) is satisfied*

*Then there exists $\overline{\lambda} \in K^+ \setminus \{0\}$, $\overline{\mu} \in Q^+$, $\overline{\alpha} \in P_1^+$, $\overline{\beta} \in P_2^+$ and $\overline{v} \in \mathbb{R}^s$ such that*

$$\overline{\lambda}^T (f \circ F)(\overline{x}) + (\overline{\lambda}^T f)^*(\overline{\alpha}) + (\overline{\alpha}^T F)^*(\overline{v}) + \overline{\mu}^T (g \circ G)(\overline{x}) + (\overline{\mu}^T g)^*(\overline{\beta}) + (\overline{\beta}^T G)^*(-\overline{v}) = 0 \quad (19)$$

*and*

$$(\overline{\mu}^T g \circ G)(\overline{x}) = 0. \quad (20)$$

*Proof.* Since $\overline{x}$ is a weak minimum of the problem (CVP) and conditions (i) and (ii) hold, therefore, by Lemma 3.1, there exists $\overline{\lambda} \in K^+ \setminus \{0\}$ such that $\overline{x}$ is an optimal solution of the problem $(\text{CVP})_{\overline{\lambda}}$.

We claim that $(\overline{x}, \overline{y} = F(\overline{x}), \overline{z} = G(\overline{x}))$ is an optimal solution of $(\text{CVP})_{\overline{\lambda}}^1$.

Since $(\overline{x},\overline{y},\overline{z}) \in S_0^1$. Now suppose, if possible, $(\overline{x},\overline{y},\overline{z})$ be not an optimal solution of $(\mathrm{CVP})^1_{\overline{\lambda}}$, then there exists $(x,y,z) \in S_0^1$ such that

$$\Phi(\overline{x},\overline{y},\overline{z}) > \Phi(x,y,z)$$

that is,

$$(\overline{\lambda}^T f)(F(\overline{x})) = (\overline{\lambda}^T f)(\overline{y}) > (\lambda^T f)(y). \tag{21}$$

Since $(x,y,z) \in S_0^1$, therefore

$$-\Psi(x,y,z) \in \overline{Q},$$

that is,

$$-g(z) \in Q \quad \text{and} \quad z - G(x) \in P_2.$$

Using the fact that $g$ is $(P_2,Q)$-increasing function, we have

$$-g(z) \in Q \quad \text{and} \quad g(z) - g(G(x)) \in Q$$

that is,

$$-(g \circ G)(x) \in Q.$$

Hence $x \in S_0$.

Similarly using the fact that $(x,y,z) \in S_0^1$, we get

$$y - F(x) \in P_1,$$

and using $f$ is $(P_1,K)$-increasing function and $\overline{\lambda} \in K^+ \setminus \{0\}$, we have

$$(\overline{\lambda}^T f)(y) \geq (\overline{\lambda}^T f)(F(x)).$$

Now using the above inequality and (21), we have

$$(\overline{\lambda}^T f)(F(\overline{x})) > (\overline{\lambda}^T f)(F(x))$$

which is a contradiction to the fact that $\overline{x}$ is an optimal solution of $(\mathrm{CVP})_{\overline{\lambda}}$.

Therefore $(\overline{x},\overline{y},\overline{z})$ is an optimal solution of the problem $(\mathrm{CVP})^1_{\overline{\lambda}}$ and hence by Lemma 3.2 we have

$$\Phi(\overline{x},\overline{y},\overline{z}) = \sup_{\gamma \in \overline{Q}^+} \inf_{\substack{x\in\mathbb{R}^s,\\ y\in Y,\\ z\in\mathbb{R}^n}} \left\{\Phi(x,y,z) + (\gamma^T\Psi)(x,y,z)\right\}, \tag{22}$$

where the supremum is attained at $\overline{\gamma} = (\overline{\mu},\overline{\alpha},\overline{\beta}) \in \overline{Q}^+$ and by equation (15) of the same lemma, we have

$$(\overline{\gamma}^T\Psi)(\overline{x},\overline{y},\overline{z}) = 0,$$

which implies that

$$(\overline{\mu}^T g)(\overline{z}) + \overline{\alpha}^T\left(F(\overline{x}) - \overline{y}\right) + \overline{\beta}^T\left(G(\overline{x}) - \overline{z}\right) = 0.$$

But since $F(\overline{x}) = \overline{y}$ and $G(\overline{x}) = \overline{z}$, so we get

$$(\overline{\mu}^T g)(G(\overline{x})) = 0. \tag{23}$$

Also, by Lemma 3.3, we have for any $\gamma = (\mu,\alpha,\beta) \in \overline{Q}^+$,

$$\inf_{\substack{x\in\mathbb{R}^s,\\ y\in Y,\\ z\in\mathbb{R}^n}} \Phi(x,y,z) + (\gamma^T\Psi)(x,y,z) = \sup_{v\in\mathbb{R}^s} \left\{-(\overline{\lambda}^T f)^*(\alpha) - (\mu^T g)^*(\beta) - (\alpha^T F)^*(v) - (\beta^T G)^*(-v)\right\} \tag{24}$$

where the supremum is attained at $\overline{v} \in \mathbb{R}^s$.

From (22) and (24), we get

$$\Phi(\overline{x}, \overline{y}, \overline{z}) = \sup_{\substack{\mu \in Q^+, \alpha \in P_1^+, \\ \beta \in P_2^+, v \in \mathbb{R}^s}} \left\{ -(\overline{\lambda}^T f)^*(\alpha) - (\mu^T g)^*(\beta) - (\alpha^T F)^*(v) - (\beta^T G)^*(-v) \right\},$$

where the supremum is attained at $(\overline{\mu}, \overline{\alpha}, \overline{\beta}, \overline{v})$.

Hence we have $\overline{\mu} \in Q^+$, $\overline{\alpha} \in P_1^+$, $\overline{\beta} \in P_2^+$ and $\overline{v} \in \mathbb{R}^s$ such that

$$\overline{\lambda}^T (f \circ F)(\overline{x}) = -(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(\overline{v}) - (\overline{\beta}^T G)^*(-\overline{v}). \tag{25}$$

From equations (25) and (23), we arrive at

$$\overline{\lambda}^T (f \circ F)(\overline{x}) + (\overline{\lambda}^T f)^*(\overline{\alpha}) + (\overline{\alpha}^T F)^*(\overline{v}) + \overline{\mu}^T (g \circ G)(\overline{x}) + (\overline{\mu}^T g)^*(\overline{\beta}) + (\overline{\beta}^T G)^*(-\overline{v}) = 0$$

and

$$\overline{\mu}^T (g \circ G)(\overline{x}) = 0.$$

□

**Theorem 3.2.** *Let $\overline{x} \in S_0$, $f$ is $K$-convex, $g$ is $Q$-convex, $F$ is $P_1$-convex and $G$ is $P_2$-convex. Also, for some $\overline{\lambda} \in K^+ \setminus \{0\}$, $\overline{\mu} \in Q^+$, $\overline{\alpha} \in P_1^+$, $\overline{\beta} \in P_2^+$ and $\overline{v} \in \mathbb{R}^s$, (19) and (20) are satisfied. Then $\overline{x}$ is a weak minimum of (CVP).*

*Proof.* Since $\overline{x}$ is feasible for (CVP), therefore for $\overline{\lambda} \in K^+ \setminus \{0\}$, $(\overline{x}, \overline{y} = F(\overline{x}), \overline{z} = G(\overline{x}))$ is feasible for $(\text{CVP})^1_{\overline{\lambda}}$.

Now following the lines of proof of above theorem in reverse order, (19) and (20) imply that

$$\overline{\lambda}^T (f \circ F)(\overline{x}) = -(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(\overline{v}) - (\overline{\beta}^T G)^*(-\overline{v}). \tag{26}$$

We will prove that $(\overline{x}, \overline{y}, \overline{z})$ is an optimal solution of $(\text{CVP})^1_{\overline{\lambda}}$. Suppose it is not true, then there exist $(x', y', z') \in S_0^1$ such that

$$\Phi(x', y', z') < \Phi(\overline{x}, \overline{y}, \overline{z}). \tag{27}$$

If we take $(\overline{\mu}, \overline{\alpha}, \overline{\beta}) = \overline{\gamma}$, then by Lemma 3.3, we get

$$\inf_{\substack{x \in \mathbb{R}^s, \\ y \in Y, \\ z \in \mathbb{R}^n}} \left\{ \Phi(x, y, z) + (\overline{\gamma}^T \Psi)(x, y, z) \right\} = \sup_{v \in \mathbb{R}^s} \left\{ -(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(v) - (\overline{\beta}^T G)^*(-v) \right\}.$$

Then for given $\overline{v} \in \mathbb{R}^s$, we have

$$\inf_{\substack{x \in \mathbb{R}^s, \\ y \in Y, \\ z \in \mathbb{R}^n}} \left\{ \Phi(x, y, z) + (\overline{\gamma}^T \Psi)(x, y, z) \right\} \geq -(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(\overline{v}) - (\overline{\beta}^T G)^*(-\overline{v})$$

and for $(x', y', z') \in S_0^1$, we have

$$\Phi(x', y', z') \geq \Phi(x', y', z') + (\overline{\gamma}^T \Psi)(x', y', z') \geq -(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(\overline{v}) - (\overline{\beta}^T G)^*(-\overline{v}). \tag{28}$$

From (27) and (28), we get

$$\Phi(\overline{x}, \overline{y}, \overline{z}) = \overline{\lambda}^T (f \circ F)(\overline{x}) > -(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(\overline{v}) - (\overline{\beta}^T G)^*(-\overline{v})$$

which is a contradiction to (26). Hence $(\overline{x}, \overline{y}, \overline{z})$ is an optimal solution of $(\text{CVP})^1_{\overline{\lambda}}$.

Next we claim that $\overline{x}$ is an optimal solution of $(\text{CVP})_{\overline{\lambda}}$.

Now let, if possible, $\overline{x}$ be not an optimal solution of $(\text{CVP})_{\overline{\lambda}}$, then some $x \in S_0$, we have

$$(\overline{\lambda}^T f)(F(\overline{x})) > (\overline{\lambda}^T f)(F(x)). \tag{29}$$

Take $F(x) = y$ and $G(x) = z$, then $(x, y, z) \in S_0^1$. Also (29) can be written as

$$(\overline{\lambda}^T f)(\overline{y}) > (\overline{\lambda}^T f)(y)$$

which is a contradiction to the fact that $(\overline{x}, \overline{y}, \overline{z})$ is an optimal solution of $(\text{CVP})^1_{\overline{\lambda}}$. Hence $\overline{x}$ is optimal solution of $(\text{CVP})_{\overline{\lambda}}$. Finally using Lemma 3.1, we have $\overline{x}$ is a weak minimum of (CVP). □

In this section we considered a vector optimization problem in which the objective as well as the constraint function are taken to be composition of two vector functions. We studied the minimization with respect to a convex cone and hence proved necessary and sufficient optimality conditions in terms of conjugate functions for a point to be a weak minimum of the given problem using cone-increasing and cone-convex functions. To prove these optimality conditions we have used Fenchel-Lagrangian duality results for scalar problem. Now we will formulate a vector dual for our primal problem and prove weak and strong duality results.

# 4 Vector Dual

We consider the following dual:

**(CVD)** $K$-Maximize $f(F(u))$

subject to

$\lambda^T (f \circ F)(u) + (\lambda^T f)^*(\alpha) + (\alpha^T F)^*(v) + \mu^T (g \circ G)(u) + (\mu^T g)^*(\beta) + (\beta^T G)^*(-v) = 0$

$\mu^T (g \circ G)(u) \geq 0.$

$u, v \in \mathbb{R}^s, \lambda \in K^+ \setminus \{0\}, \mu \in Q^+, \alpha \in P_1^+, \beta \in P_2^+.$

**Theorem 4.1.** *[**Weak Duality**] Let $f$ be $K$-convex, $g$ be $Q$-convex, $F$ be $P_1$-convex and $G$ be $P_2$-convex functions. Also, suppose that $\overline{x}$ is feasible for primal problem (CVP) and $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{\beta})$ is feasible for the dual problem (CVD), then*

$$(f \circ F)(\overline{u}) - (f \circ F)(\overline{x}) \notin intK.$$

*Proof.* Let, if possible,

$$(f \circ F)(\overline{u}) - (f \circ F)(\overline{x}) \in \text{int}K$$

using $\overline{\lambda} \in K^+ \setminus \{0\}$, we have

$$(\overline{\lambda}^T f)(F(\overline{u})) - (\overline{\lambda}^T f)(F(\overline{x})) > 0.$$

Since $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{\beta})$ is feasible for the dual problem (CVD), we get that

$$-(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(\overline{v}) - (\overline{\beta}^T G)^*(-\overline{v}) = (\overline{\lambda}^T f)(F(\overline{u})).$$

From above two inequalities, we can infer that

$$-(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(\overline{v}) - (\overline{\beta}^T G)^*(-\overline{v}) > (\overline{\lambda}^T f)(F(\overline{x})). \tag{30}$$

By Lemma 3.3, for $\overline{\gamma} = (\overline{\mu}, \overline{\alpha}, \overline{\beta}) \in \overline{Q}^+$, we have

$$\sup_{v\in\mathbb{R}^s} \left\{-(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(v) - (\overline{\beta}^T G)^*(-v)\right\} = \inf_{\substack{x\in\mathbb{R}^s,\\ y\in Y,\\ z\in\mathbb{R}^n}} \left\{\Phi(x,y,z) + (\overline{\gamma}^T\Psi)(x,y,z)\right\}.$$

Hence, if we take $F(\overline{x}) = \overline{y}$ and $G(\overline{x}) = \overline{z}$ then

$$-(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(\overline{v}) - (\overline{\beta}^T G)^*(-\overline{v}) \leq \Phi(\overline{x},\overline{y},\overline{z}) + (\overline{\gamma}^T\Psi)(\overline{x},\overline{y},\overline{z}) \leq \Phi(\overline{x},\overline{y},\overline{z}),$$

that is

$$-(\overline{\lambda}^T f)^*(\overline{\alpha}) - (\overline{\mu}^T g)^*(\overline{\beta}) - (\overline{\alpha}^T F)^*(\overline{v}) - (\overline{\beta}^T G)^*(-\overline{v}) \leq (\overline{\lambda}^T f)(F(\overline{x})),$$

which is a contradiction to (30). Hence

$$(f\circ F)(\overline{u}) - (f\circ F)(\overline{x}) \notin \text{int}K.$$

□

**Theorem 4.2.** *[Strong Duality] Let $\overline{x} \in S_0$ be a weak minimum of the problem (CVP) and the following conditions hold:*

*(i) $f$ is $K$-convex, $g$ is $Q$-convex, $F$ is $P_1$-convex and $G$ is $P_2$-convex function*

*(ii) $f$ is $(P_1, K)$-increasing and $g$ is $(P_2, Q)$-increasing function*

*(iii) constraint qualification (CQ) is satisfied*

*Then there exists $\overline{\lambda} \in K^+ \setminus \{0\}, \overline{\mu} \in Q^+, \overline{\alpha} \in P_1^+, \overline{\beta} \in P_2^+, \overline{v} \in \mathbb{R}^s$ such that $(\overline{x}, \overline{v}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{\beta})$ is weak maximum for (CVD).*

*Proof.* Since $\overline{x}$ is a weak minimum of the problem (CVP) and conditions (i), (ii) and (iii) hold, therefore by Theorem 3.1, there exists $\overline{\lambda} \in K^+ \setminus \{0\}$, $\overline{\mu} \in Q^+$, $\overline{\alpha} \in P_1^+$, $\overline{\beta} \in P_2^+$ and $\overline{v} \in \mathbb{R}^s$ such that

$$\overline{\lambda}^T(f\circ F)(\overline{x}) + (\overline{\lambda}^T f)^*(\overline{\alpha}) + (\overline{\alpha}^T F)^*(\overline{v}) + \overline{\mu}^T(g\circ G)(\overline{x}) + (\overline{\mu}^T g)^*(\overline{\beta}) + (\overline{\beta}^T G)^*(-\overline{v}) = 0$$

and

$$\overline{\mu}^T(g\circ G)(\overline{x}) = 0$$

hence $(\overline{x}, \overline{v}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{\beta})$ feasible for (CVD).

Let, if possible, $(\overline{x}, \overline{v}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{\beta})$ is not a weak maximum of (CVD), then there exists $(u, v, \lambda, \mu, \alpha, \beta)$ feasible for (CVD) such that

$$(f\circ F)(u) - (f\circ F)(\overline{x}) \in \text{int}K. \tag{31}$$

Using the fact that $\overline{x}$ is feasible for the problem (CVP) and $(u, v, \lambda, \mu, \alpha, \beta)$ feasible for (CVD), by Weak Duality Theorem 4.1 we have

$$(f\circ F)(u) - (f\circ F)(\overline{x}) \notin \text{int}K,$$

which is a contradiction to relation (31) and hence $(\overline{x}, \overline{v}, \overline{\lambda}, \overline{\mu}, \overline{\alpha}, \overline{\beta})$ is a weak maximum of (CVD). □

# 5 Conclusions

This paper developed necessary and sufficient optimality conditions for composite vector optimization problems over cones using conjugate function techniques. By employing Fenchel-Lagrange duality, a corresponding dual problem was formulated and weak as well as strong duality results were established under convexity and constraint qualification assumptions. The findings generalize existing results in vector optimization and provide a unified analytical framework for composite structures. Future work may explore extensions to nonconvex settings or applications in multi-criteria decision-making and engineering optimization.

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