Investigation of Generalized k-Basic Hypergeometric Functions

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Abstract: ***In this paper, a affiliated proceed towards to Basic Hyergeometric function in form of generalized 'k –Basic Hypergeometric function / q-analogue of k-Hypergeometric function $_p\phi_{r,k}$ is given. As a conclude facts, generalized 'k –Basic Hypergeometric series/q-analogue of k - Hypergeometric series and also conclude its ordinary differential equation with some properties and m^{th} derivative of k-basic Hypergeometric functions.***

Keywords: k-basic Hypergeometric functions, q-analogue, k-pchhammer

1. Introduction

In recent years, basic Hypergeometric function has produced appreciable and important theories, definitions and formulas in various fields of science and mathematics. On the other hand, in the 20th century, new definitions, formulas and important theories of fractional derivatives and integrals have appeared in generalized application, theory and formulas, which in some sense give rise to classical mathematics. In addition, the synthesis and analysis of both their theoretical and functional properties have been a broad and refined area of research in mathematical analysis. Through intensive study and investigation of analytical and numerical approaches to Systems considering differential equations [D.E.] with fractional-order operators and defined application have been proposed in mathematical science and technology. Already we know that promiscuous solutions of D.E. connect to many special functions in branches of mathematics and physics. Like as, in research paper [1] used many view concerning the k- Hypergeometric function, series and ordinary D.E. with help of mostly paper [11]-[13], [16]– [18] in which important facts are- k- pochhammer symbol is defined as

$$(a)_{n,k} = a(a+k)(a+2k)....(a+\overline{n-1}\,k) = \prod_{i=1}^{n} a+(i-1)k.$$

Where a ≠0, $(a)_{0,k} = 1$, $(a)_n = (a)_{n,1}$, k > 0. Also,

$$(a)_{m+n,k} = (a+mk)_{n,k}\,(a)_{m,k},$$

$$(a)_{mn,k} = m^{mn}\prod_{i=1}^{m}\left(\frac{a+ki-k}{m}\right)_{n,k}$$

Hence Relation between k-pochhammer symbol and k-gamma function is defined as

$$(a)_{n,k} = \frac{\Gamma_k(a+nk)}{\Gamma_k(a)}$$

Now by [11], [13], if a, b, c are three parameters in which two parameters a, b in the numerator and remain 'c' parameter in the denominator then the Hypergeometric function.

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n (b)_n x^n}{(c)_n n!}$$

And according to Mubeen [19], if a, b parameter in numerator and another one parameter 'c' in denominator then k-Hypergeometric function is defined by

$${}_2F_{1,k}((a,k),(b,k);(c,k);x) = \sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}\,x^n}{(c)_{n,k}\; n!}$$

Finally, the generalized k-Hypergeometric function is represented as

$${}_pF_{r,k}\left[\begin{matrix}(a_1;k),(a_2;k),......,(a_p,k);\\(b_1;k),(b_2;k),......,(b_r,k);\,z\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n,k}}{\prod_{i=1}^{r}(b_i)_{n,k}}\cdot\frac{z^n}{\lfloor n}$$

In which k > 0 and no denominator parameter b_i is allowed to zero or a negative integer

So, the purpose of this research paper is to serve as a special forum for the dissemination of recent advances in the theory of k- Hypergeometric function, series and ordinary D.E and its potential applications and the solutions explained by Power Series [12] and Hypergeometric series [11],[17],[18] and mostly references in paper [1]. now, we submit high-quality reports on the analysis of q analoque of generalized k-Hypergeometric function, series, D.E., and other therom. and generate analysis of new definitions of q analoque of generalized k - Hypergeometric function, q derivatives methods for equations, and applications governed

2. Preliminaries

In this quarters, we evaluate some fundamental definitions and idiom connecting the k–basic Hypergeometric series and differential equations. We know that the solutions of basic Hypergeometric equation can be solved only with the help of Basic Hypergeometric series. The ordinary k– basic

Hypergeometric function $V(Z)={}_2\phi_1(a,b;c;q,z)$ satisfies the second order q-differential equation:

$$z(c-abqz)D_q^2V+\left[\left(\frac{1-c}{1-q}\right)+\frac{(1-a)(1-b)-(1-abkq)}{1-q}z\right]D_qV-\left(\frac{(1-a)(1-b)}{(1-q)^2}\right)V=0 \quad (1)$$

where $D_qf(z)=\dfrac{f(z)-f(qz)}{(1-q)z}$ replacing

a, b, c respectively by q^a,q^b,q^c then we have

$$z\left[q^c-q^aq^bq^z\right]D_q^2V+\left[\frac{1-q^c}{1-q}+\frac{(1-q^a)(1-q^b)-(1-q^aq^bq^kq)}{1-q}z\right]D_qV-\frac{(1-q^a)(1-q^b)}{(1-q)^2}V=0 \quad (2)$$

Taking limit $q\to1^-$ then above equation tends to the second ordinary q differential equation

$$z[1-z]D^2V+[c-(a+b+k)z]DV-abV=0 \quad (3)$$

For the Hypergeometric function $v(z)={}_2F_1(a, b; c; z)$ where $|Z|<1$, is called as Heine [1847] differential equation discussed in [7]-[9] .The point z = 0 is a regular point for the equation and a power series method is the suitable method to solve this equation .In paper [17], [18], k-Hypergeometric series and [15] the k-Hypergeometric differential equation are available .Similarly in paper [1], deduced integral representations of k-gamma and k-beta function with help of paper [17]who produced and verified important laws and thermos of k-gamma and k-beta function and k-pochhammer symbol .Finally with help of references[7]-[9],[17],[18] ,we can be formulated k-gamma and k-beta function and k-pochhammer symbol in k– basic Hypergeometric function and series .

q- analogue of k pochhammer symbol is represented as

$$(a_i;q)_{n,k}=(a_i)_q[(a_i)_q+qk_q][(a_i)_q+2_qqk_q][(a_i)_q+3_qqk_q]\cdots[(a_i)_q+q(n-1)_qk_q] \quad (4)$$

$$\therefore n_q-1=\frac{1-q^n}{1-q}-1=\frac{1-q^n-1+q}{1-q}=\frac{q(1-q^{n-1})}{1-q}=q(n-1)_q$$

where $a\neq0,k>0$ And

$$(a_i;q)_{n+1,k}=(a_i)_q[(a_i)_q+qk_q][(a_i)_q+2_qqk_q][(a_i)_q+3_qqk_q]\cdots[(a_i)_q+qn_qk_q] \quad (5)$$

Hence the relation between q-analogue of the k-pochhammer symbol and k-gamma function is defined as

$$(a;q)_{n,k}=\frac{\Gamma_k(a+nk)}{\Gamma_k(a)} \quad (6)$$

Definition-1

If we take two parameter a, b in numerator and only one parameter c in denominator then k-basic Hypergeometric function / k-Hypergerometric q-series will be represented as

$${}_2\phi_{1,k}((a:q)_{n,k},(b:q)_{n,k};(c:q)_{n,k};z)=\sum_{n=0}^{\infty}\frac{(a:q)_{n,k}\,(b:q)_{n,k}\,z^n}{(c:q)_{n,k}\,(q:q)_n}$$

(7)

Definition 2:

q-analogue of generalized k-Hypergeometric function is defined by

$${}_p\phi_{r,k}\left[\begin{matrix}(a_1;q)_{n,k},(a_2;q)_{n,k},\ldots,(a_p;q)_{n,k};\\(b_1;q)_{n,k},(b_2;q)_{n,k},\ldots,(b_r;q)_{n,k};z\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i;q)_{n,k}}{\prod_{i=1}^{r}(b_i;q)_{n,k}}\cdot\frac{z^n}{(q:q)_n} \quad (8)$$

3. 3. Solution of the differential equation

Remind that [7]-[9] the ordinary k– basic Hypergeometric function $v(z)={}_2\phi_1(a,b;c;q,z)$ satisfies the second order q-differential equation from (3)

$$z[1-z]D_q^2v+[c-(a+b+k)z]D_qv-abv=0$$

"z" is replaced by "k_q z" then we have

$$k_qz[1-k_qz]D_q^2v+[c_q-(a_q+b_q+k_q)k_qz]D_qv-a_qb_qv=0 \quad (9)$$

or, in terms of the differential operator $\theta_q=zD_q$

$\because\theta_qv=zD_qv$ And $\theta_q(\theta_q-1)v=z^2D_q^2v$

then differential equation (9) will be

$$[\theta_q(k_q\theta_q+c_q-k_q)-z(k_q\theta_q+b_q)(k_q\theta_q+a_q)]v=0 \quad (10)$$

with equation (8) before us, we can advance as follows

$$\phi={}_p\phi_{r,k}=\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{z^n}{(q;q)_n}$$

If we take $k_q\theta_qz^n=k_qn_qz^n$ then it follows that

$$k_q\theta_q\prod_{i=0}^{p}[k_q\theta_q+(b_i)_q-k_q]\phi$$

$$=k_q\theta_q\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{(k_q\theta_q+(b_i)_q-k_q)z^n}{(q;q)_n}$$

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$$=k_q\theta_q\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{[k_q\theta_q z^n+(b_i)_q z^n-k_q z^n]}{(q;q)_n}$$

$$=k_q\theta_q\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{[k_q n_q z^n+(b_i)_q z^n-k_q z^n]}{(q;q)_n}$$

$$=k_q\theta_q\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{[k_q n_q+(b_i)_q-k_q]z^n}{(q;q)_n}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{[k_q n_q+(b_i)_q-k_q]k_q\theta_q z^n}{(q;q)_n}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{[k_q n_q+(b_i)_q-k_q]k_q n_q z^n}{(q;q)_n}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{[k_q(n_q-1)+(b_i)_q]k_q n_q z^n}{(q;q)_n}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n-1,k}}\frac{k_q n_q z^n}{(q;q)_n}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n-1,k}}\frac{k_q z^n}{(q;q)_{n-1}(1-q)}$$

n is replaced by n+1

$$=\sum_{n=1}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n+1,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{k_q z^{n+1}}{(q;q)_n(1-q)}$$

$$=\frac{k_q z}{1-q}\sum_{n=1}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n+1,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{z^n}{(q;q)^n}$$

$$=\frac{k_q z}{1-q}\sum_{n=1}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{[(a_i)_q+qn_q k_q]z^n}{(q;q)_n}$$

then

$$k_q\theta_q\prod_{i=0}^{p}[k_q\theta_q+(b_i)_q-k_q]\phi$$

$$=\frac{k_q z}{1-q}\sum_{n=1}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n,k}}{\prod_{i=0}^{r}(b_i;q)_{n,k}}\frac{(qn_q k_q+(a_i)_q]z^n}{(q;q)_n}$$

Now we get

$$k_q\theta_q\prod_{i=0}^{p}[k_q\theta_q+(b_i)_q-k_q]\phi$$

$$=\frac{k_q z}{1-q}\prod_{i=0}^{p}[(a_i)_q+qn_q k_q]\phi$$

$$k_q\theta_q\prod_{i=0}^{p}[k_q\theta_q+(b_i)_q-k_q]\phi=\frac{k_q z}{1-q}\prod_{i=0}^{p}[(a_i)_q+q\theta_q k_q]\phi$$

$$\theta_q\prod_{i=0}^{p}[k_q\theta_q+(b_i)_q-k_q]\phi=\frac{z}{1-q}\prod_{i=0}^{p}[(a_i)_q+q\theta_q k_q]\phi$$

$$\theta_q\prod_{i=0}^{p}[k_q\theta_q+(b_i)_q-k_q]\phi-\frac{z}{1-q}\prod_{i=0}^{p}[(a_i)_q+q\theta_q k_q]\phi=0$$

$$[\theta_q\prod_{i=0}^{p}\{k_q\theta_q+(b_i)_q-k_q\}-\frac{z}{1-q}\prod_{i=0}^{p}\{(a_i)_q+q\theta_q k_q\}]\phi=0 \tag{11}$$

where b_i is non positive integer. The solution is well-perfect for all finite u when $p\le r$. If $p=r+1$, then the solution is verify in $|z|<1$.

Theorem 3.1 If $p\le q+1$, $\mathrm{Re}(b_1)>\mathrm{Re}(a_1)>0$ and if no one of $b_1,b_2,b_3,\ldots..b_r$ is zero or a negative integer.

$$\frac{d_q^m}{dz_q^m}\,{}_p\phi_{r,k}\left[\begin{matrix}(a_1;q)_{n,k}(a_2;q)_{n,k},\ldots.,(a_p;q)_{n,k};\\(b_1;q)_{n,k}(b_2;q)_{n,k},\ldots..,(b_r;q)_{n,k};z\end{matrix}\right]=$$

$$=\frac{\prod_{i=1}^{p}(a_i;q)_{m,k}}{\prod_{i=1}^{r}(b_i;q)_{m,k}}$$

$${}_p\phi_{r,k}\left[\begin{matrix}(a_1+mqk_q;q)_{n,k},(a_2+mqk_q;q)_{n,k},\\ \ldots\ldots,(.a_p+mqk_q;q)_{n,k};\\ (b_1+mqk_q;q)_{n,k},(b_2+mqk_q;q)_{n,k},\ldots\ldots,\\ (.b_r+mqk_q;q)_{n,k};z\end{matrix}\right]$$

Proof : Taking q derivative operator of ${}_p\phi_{r,k}$ *w.r. to z*

$$\frac{d_q}{dz_q}\,{}_p\phi_{r,k}=\frac{d_q}{dz_q}\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n_1k}}{\prod_{i=1}^{r}(b_i;q)_{n,k}}\frac{z^n}{(q;q)_n}$$

$$=\sum_{n=1}^{\infty}\frac{\prod_{i=0}^{p}(a_i;q)_{n_1k}}{\prod_{i=1}^{r}(b_i;q)_{n,k}}\frac{z^{n-1}}{(1-q)(q;q)_n}$$

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$$\begin{cases} \because D_q f(z) = \dfrac{f(z) - f(qz)}{(1-q)z} \\ f(z) = z^n \\ D_q z^n = \dfrac{z^n - q^n z^n}{(1-q)z} \\ = \dfrac{z^n(1-q^n)}{z(1-q)} = \dfrac{1-q^n}{1-q} z^{n-1} \end{cases}$$

with a shift of index from n to n+1

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{p} (a_i;q)_{n+1,k}}{\prod_{i=1}^{r} (b_i;q)_{n+1,k}} \frac{z^n(1-q^{n+1})}{(1-q)(q;q)_{n+1}}$$

$$\begin{cases} \because (q;q)_{n+1} = (1-q)(1-q^2).....(1-q^n)(1-q^{n+1}) \\ \Rightarrow \dfrac{(q;q)_{n+1}}{1-q^{n+1}} = (1-q)(1-q^2)......(1-q^n) \\ \Rightarrow \dfrac{(q;q)_{n+1}}{(1-q^{n+1})} = (q;q)_n \end{cases}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{p} (a_i;q)_{n+1,k}}{\prod_{i=0}^{r} (b_i;q)_{n+1,k}} \frac{z^n}{(1-q)(q;q)_n}$$

$$\begin{cases} (a_i;q)_{n+1,k} & = (a_i)_q (a_i + qk_q)_{n,k} \\ & = (a_i;q)_{1,k}[a_i + qk_q;q]_{n,k} \end{cases}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{p} (a_i;q)_{1,k}(a_i + qk_q;q)_{n,k}}{\prod_{i=1}^{r} (b_i;q)_{1,k}(b_i + qk_q;q)_{n,k}} \frac{z^n}{(1-q)(q:q)_n}$$

$$= \frac{1}{(1-q)} \frac{\prod_{i=0}^{p} (a_i;q)_{1,k}}{\prod_{i=1}^{r} (b_i;q)_{1,k}} \sum_{n=0}^{\infty} \frac{\prod_{i=0}^{p} (a_i + qk_q;q)_{n,k}}{\prod_{i=1}^{r} (b_i + qk_q;q)_{n,k}} \frac{z^n}{(q:q)_n}$$

by def. 2 [equation 8]

$$\frac{d_q}{dz_q} {}_p\phi_{r,k}$$

$$= \frac{1}{(1-q)} \frac{\prod_{i=1}^{p} (a_i;q)_{1,k}}{\prod_{i=1}^{r} (b_i;q)_{1,k}} {}_p\theta_{r,k} \begin{bmatrix} (a_1 + qk_q;q)_{n,k}, (a_2 + qk_q;q)_{n,k}, \\ .....(a_p + qk_q;q)_{n,k}; \\ (b_1 + qk_q;q)_{n,k}, (b_2 + qk_q;q)_{n,k} \\ ,.....(b_r + qk_q;q)_{n,k}; z \end{bmatrix}$$

Now taking q derivative operator m times of ${}_p\phi_{r,k}$ then we have

$$\frac{d_q^m}{dz_q^m} {}_p\phi_{r,k} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i + mqk_q;q)_{n,k}}{\prod_{i=1}^{r} (a_i + mqk_q;q)_{n,k}} . \frac{z^n}{(q:q)_n} . \frac{\prod_{i=1}^{p} (a_i;q)_{m,k}}{\prod_{i=1}^{r} (b_i;q)_{m,k}}$$

$$= \frac{\prod_{i=1}^{p} (a_i;q)_{m,k}}{\prod_{i=1}^{r} (b_i;q)_{m,k}} {}_p\phi_{r,k} \begin{bmatrix} (a_1 + mqk_q;q)_{n,k}, (a_2 + mqk_q;q)_{n,k}, \\ ......,(a_p + mqk_q;q)_{n,k}; \\ (b_1 + mqk_q;q)_{n,k}, (b_2 + mqk_q;q)_{n,k}, \\ ......,(b_r + mqk_q;q)_{n,k}; z \end{bmatrix} \quad [12]$$

that fulfills the proof.

4. Conclusion

The Generalized k-Basic Hypergeometric function develops as best formation within the theory of special functions. By q analogue, formations differential representations and well-define result of generalized k-hypergeometric function. In future we can reduce many different application and formulae in mathematics and physics with reference to basic hypergeometric function.

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