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Fractional Delay Differential Equation with General Initial Conditions and Difference Order Two

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Abstract: ***In this paper we have solved fractional Delay differential equation whose differential order is α with general initial conditions. By using Laplace decomposition method. The Fractional derivative is the Caputo fractional Derivative. We solve both linear and non-linear Delay fractional differential difference equation. we extend the result given by differential difference equation for fractional order differential difference equation where the fractional derivative used is caputo fractional derivative. The obtained result is Compared with differential difference equation by putting $\alpha = 1$ and verify the validity of the result and the result hold good.***

Keywords: Caputo Fractional derivative, Laplace Transform, Laplace Adomian Decomposition, Differential difference equation

1. Introduction

Basic of Fractional calculus and fractional Differential equations is given in [1-4]. Differential difference Equation of order $(\alpha,1)$ *and* $(\alpha,2)$ *is given Ananth in* [5,6]. In [7] Kamble Rajratna M, Kulkarni Pramod Ramakant, have derived “Laplace Transform and Laplace Decomposition Method for of order (α,1) Fractional Differential Difference Equations with Interval Conditions Linear and Nonlinear”. In [8] Kamble Rajratna M., Kulkarni Pramod Ramakant, have proved “Existence and Uniqueness of Continuous Solutions for Conformable Fractional Integro-Differential” Equations in Cone Metric Spaces”. In [9] Kamble, Rajratana, and Pramod Kulkarni. "Numerical Solutions of the SIR Mathematical Model of Computer Viruses Involving Non-linear Fractional Order Differential Equation”. In [10] Rajratana, Maroti Kamble, and P.R. Kulkarni Given new definition of fractional derivative "Extended fractional derivative: Some results involving classical properties and applications”. In [11] Kamble Rajratna M., and Pramod R. Kulkarni have proved "Existence and uniqueness of solutions for exponential fractional differential equations”. In [12] Kamble Rajratna, M., et al. "Gheorghe Săvoiu, Mladen Čudanov and Vesna Tornjanski." have proved “On Some Existence and Uniqueness Results for Nonlinear Fractional Differential Equations with Boundary Conditions”

2. Main Result

Delay fractional differential difference equation given by

$$\frac{d^\alpha \varphi}{dt} = f(t) + P(\varphi(t-w), \varphi^\alpha(t-w)) + Q(\varphi(t-2w), \varphi^\alpha(t-2w)) \quad , \quad t > 2w \quad \dots(1)$$

With Conditions

$$\varphi(t) = p \dots(2)$$

$$\varphi^k(0) = 0, k = 1,2,\dots n-1 \dots(3)$$

Where $n-1 < \alpha < n,\ \ w \in N,\ \ p \in R$

We solve the Above Delay Fractional differential equation with given conditions by Laplace Adomian decomposition Method.

Multiply $e^{-st}, s > 1$ and Integrate equation between 2w to ∞

$$\int_{2w}^{\infty} \varphi^\alpha e^{-st} dt = \int_{2w}^{\infty} f(t) e^{-st} dt + \int_{2w}^{\infty} P(\varphi(t-w), \varphi^\alpha(t-w)) + \int_{2w}^{\infty} Q(\varphi(t-2w), \varphi^\alpha(t-2w))$$

$$L[\varphi^\alpha(t)] = e^{-2w} L[f(t+2w)] + e^{-ws} L[P(\varphi(t), \varphi^\alpha(t))] - \frac{\lambda e^{-ws}}{s}[1 - e^{-ws}] + e^{-2ws} L[Q(\varphi(t), \varphi^\alpha(t))]$$

Where $\lambda = P(p, 0)$

Using Laplace transform of Caputo fractional derivative

$$s^\alpha L[\varphi(t)] - s^{\alpha-1} p = \frac{\lambda e^{-2ws}}{s} - \frac{\lambda e^{-ws}}{s} + e^{-ws} L[f(t+2w)] + e^{-ws} L[P(\varphi(t), \varphi^\alpha(t))] + e^{-2ws} L[Q(\varphi(t), \varphi^\alpha(t))]$$

$$= \frac{p}{s} + \frac{\lambda e^{-2ws}}{s^{\alpha+1}} - \frac{\lambda e^{-ws}}{s^{\alpha+1}} + \frac{e^{-ws} L[f(t+2w)]}{s^\alpha} + \frac{e^{-ws} L[P(\varphi(t), \varphi^\alpha(t))]}{s^\alpha} + \frac{e^{-2ws} L[Q(\varphi(t), \varphi^\alpha(t))]}{s^\alpha}$$

Now consider following type of decomposition

$$L[\varphi(t)] = \sum_{n=0}^{\infty} e^{-nws} L[\varphi_n(t)]$$

Laplace decomposition for

$$P(\varphi(t), \varphi^\alpha(t)), Q(\varphi(t), \varphi^\alpha(t))$$

$$L[P(\varphi(t), \varphi^\alpha(t))] = \sum_{n=0}^{\infty} e^{-nws} L[A_n(t)]$$

$$L[Q(\varphi(t), \varphi^\alpha(t))] = \sum_{n=0}^{\infty} e^{-nws} L[B_n(t)]$$

Where $A_n's, B_n's$ are the n^{th} adomian polynomial for $P(\varphi(t), \varphi^\alpha(t)), Q(\varphi(t), \varphi^\alpha(t))$

Some Adomian Polynomial of the non linear term $P(\varphi(t), \varphi^\alpha(t)), Q(\varphi(t), \varphi^\alpha(t))$ are given below

$$A_0(t) = [P(x,y)]_{\varphi(0),\varphi^\alpha(0)}$$

$$A_1(t) = \frac{\partial P}{\partial x}_{\varphi(0),\varphi^\alpha(0)} \varphi_1(t) + \frac{\partial P}{\partial y}_{\varphi(0),\varphi^\alpha(0)} {\varphi_1}^\alpha(t)$$

$$A_2(t) = A_1(t) = \frac{\partial P}{\partial x}_{\varphi(0),\varphi^\alpha(0)} \varphi_2(t) + \frac{\partial P}{\partial y}_{\varphi(0),\varphi^\alpha(0)} {\varphi_2}^\alpha(t) + \frac{1}{2!}[\frac{\partial^2 P}{\partial x^2}_{\varphi(0),\varphi^\alpha(0)} {\varphi_1}^2(t) + 2\frac{\partial^2 P}{\partial x \partial y}_{\varphi(0),\varphi^\alpha(0)} \varphi_1(t){\varphi_1}^\alpha(t) + \frac{\partial P}{\partial y}_{\varphi(0),\varphi^\alpha(0)} ({\varphi_2}^\alpha(t))^2]$$

And so, on. Similar Laplace Decomposition for $Q(\varphi(t), \varphi^\alpha(t))$

Now the Laplace decomposition main theme is to set iteration as follows

$$\sum_{n=0}^{\infty} e^{-nws} L[\varphi_n(t)] = \frac{p}{s} + \frac{\lambda e^{-2ws}}{s^{\alpha+1}} - \frac{\lambda e^{-ws}}{s^{\alpha+1}} + \frac{e^{-2ws} L[f(t+2w)]}{s^\alpha} + \frac{e^{-ws}}{s^\alpha} \sum_{n=0}^{\infty} e^{-nws} L[A_n(t)] + \frac{e^{-2ws}}{s^\alpha} \sum_{n=0}^{\infty} e^{-nws} L[B_n(t)]$$

$$= \frac{p}{s} + \left[\frac{-\lambda}{s^{\alpha+1}} + \frac{1}{s^\alpha} L[A_0(t)]\right] e^{-ws} + \left[\frac{\lambda}{s^{\alpha+1}} + \frac{1}{s^\alpha} L[f(t+2w)] + \frac{1}{s^\alpha} L[A_1(t)] + \frac{1}{s^\alpha} L[B_0(t)]\right] e^{-2ws} + \sum_{n=3}^{\infty} [\frac{L[A_{n-1}]}{s^\alpha} + \frac{L[B_{n-2}]}{s^\alpha}]$$

Where $A_n's, B_n's$ are the n^{th} adomian polynomial for $P(\varphi(t), \varphi^\alpha(t)), Q(\varphi(t), \varphi^\alpha(t))$

We Calculate $L[\varphi_n(t)]$ iteratively given by

$$L[\varphi_0(t)] = \frac{p}{s}$$

$$L[\varphi_1(t)] = \frac{-\lambda}{s^{\alpha+1}} + \frac{1}{s^\alpha} L[A_0(t)]$$

$$L[\varphi_2(t)] = \frac{\lambda}{s^{\alpha+1}} + \frac{1}{s^\alpha} L[f(t+2w)] + \frac{1}{s^\alpha} L[A_1(t)] + \frac{1}{s^\alpha} L[B_0(t)]$$

$$L[\varphi_n(t)] = \frac{L[A_{n-1}]}{s^\alpha} + \frac{L[B_{n-2}]}{s^\alpha}, \quad n = 3,4,5 \ldots$$

Therefore,

$$\varphi(t) = \sum_{n=0}^{\infty} \varphi_n(t - nw) e(t - nw)$$

Where $e(t - nw) = \begin{cases} 1, & t > nw \\ 0, & otherwise \end{cases}$

3. Applications of the Result

Based on the above method we solve linear and non linear Fractional Delay differential equation with general initial conditions.

1) Consider the equation

$$2\varphi^\alpha(t) - \varphi(t-w) = \varphi(t-2w), t > 2w \ldots 4)$$

$$\varphi(t) = 1, 0 \le t \le 2w \ \ldots 5)$$

$$\varphi^{(k)}(0) = 0, k = 1,2,3, \ldots 6)$$

Where $n - 1 < \alpha \le n, w \in N$

We note that

$$\int_w^{2w} \frac{1e^{-st}dt}{2} = \frac{1}{2s}[-e^{-2ws} + e^{-ws}] \ldots 7)$$

Multiply equation 4 $e^{-st}, s > 1$ and integrate between 2w to ∞

The Laplace transform of Caputo fractional derivative

$$L[\varphi^\alpha(t)] = s^\alpha L[\varphi(t)] - \sum_{k=0}^{\infty} s^{\alpha-k-1} \varphi^k(t) \ldots 8)$$

Where $\varphi^k(t)$ is ordinary derivative at k=1,2,3…,n-1, $n - 1 < \alpha \le n$

Taking the Laplace transform of 4 and using 7 and 8 we get

$$L[\varphi^\alpha(t)] = \frac{1}{s} + \frac{1}{2s^{\alpha+1}}[e^{-2ws} - e^{-ws}] + \frac{e^{-ws}}{2s^\alpha} L[\varphi(t)] + \frac{e^{-2ws}}{2s^\alpha} L[\varphi(t)]$$

Consider the Laplace decomposition for $L[\varphi(t)]$ as

$$L[\varphi(t)] = \sum_{n=0}^{\infty} e^{-nws} L[\varphi_n(t)]$$

$$\sum_{n=0}^{\infty} e^{-nws} L[\varphi_n(t)] = \frac{1}{s} + \frac{1}{2s^{\alpha+1}}[e^{-2ws} - e^{-ws}] + \frac{e^{-ws}}{2s^\alpha} \sum_{n=0}^{\infty} e^{-nws} L[\varphi_n(t)] + \frac{e^{-2ws}}{2s^\alpha} \sum_{n=0}^{\infty} e^{-nws} L[\varphi_n(t)]$$

$$= \frac{1}{s} + \left[\frac{-1}{2s^{\alpha+1}} + \frac{1}{2s^\alpha} L[\varphi_0(t)]\right] e^{-ws} + \left[\frac{1}{2s^{\alpha+1}} + \frac{1}{2s^\alpha} L[\varphi_1(t)]\right] + \sum_{n=3}^{\infty} \frac{L[\varphi_{n-1}(t)]}{2s^\alpha} + \frac{L[\varphi_{n-2}(t)]}{2s^\alpha}$$

Comparing the coefficient of e^{-nws} from both sides

$$L[\varphi_0(t)] = \frac{1}{s} \rightarrow \varphi_0(t) = 1$$

$$L[\varphi_1(t)] = \frac{-1}{2s^{\alpha+1}} + \frac{1}{2s^\alpha} L[\varphi_0(t)] \rightarrow \varphi_1(t) = 0$$

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$$L[\varphi_2(t)] = \frac{1}{2s^{\alpha+1}} + \frac{1}{2s^\alpha}L[\varphi_1(t)] = \frac{1}{2s^{\alpha+1}} + \frac{1}{2s^{\alpha+1}} = \frac{1}{s^{\alpha+1}} \to \varphi_2(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$L[\varphi_3(t)] = \frac{1}{2s^\alpha}L[\varphi_2(t)] + \frac{1}{2s^\alpha}L[\varphi_1(t)] = \frac{1}{s^\alpha s^{\alpha+1}} \to \frac{t^{2\alpha}}{2\Gamma(2\alpha+1)}$$

$$L[\varphi_4(t)] = \frac{1}{2s^\alpha}L[\varphi_3(t)] + \frac{1}{2s^\alpha}L[\varphi_2(t)] = \frac{1}{2s^\alpha}\frac{t^{2\alpha}}{2\Gamma(2\alpha+1)} + \frac{1}{2s^\alpha}\frac{1}{s^{\alpha+1}} \to \varphi_4(t) = \frac{t^{3\alpha}}{4\Gamma(3\alpha+1)} + \frac{t^{2\alpha}}{2\Gamma(2\alpha+1)}$$

For $n \geq 2$

$$\varphi_n(t) = \sum_{r=1}^{\left[\frac{n}{2}\right]} \binom{n-r-1}{r-1} \frac{t^{(n-r)\alpha}}{2^{n-r-1}\Gamma((\mathrm{n}-\mathrm{r})\alpha+1)}$$

Therefore

$$\varphi(t) = \sum_{n=0}^{\infty} \varphi_n\,(t-nw)e(t-nw)$$

Where e(t-nw) is unit step function and $\varphi_1, \varphi_2, \varphi_n, n \geq 2$ calculated as above, We get

$$\varphi(t) = 1 + \sum_{n=2}^{\infty}\sum_{r=1}^{\left[\frac{n}{2}\right]} \binom{n-r-1}{r-1} \frac{t^{(n-r)\alpha}}{2^{n-r-1}\Gamma((\mathrm{n}-\mathrm{r})\alpha+1)}, \quad Nw \leq t \leq (N+1)w$$

Putting $\alpha = 1$ we get

$$\varphi(t) = 1 + \sum_{n=2}^{\infty}\sum_{r=1}^{\left[\frac{n}{2}\right]} \binom{n-r-1}{r-1} \frac{t^{(n-r)\alpha}}{2^{n-r-1}\Gamma((\mathrm{n}-\mathrm{r})+1)}, \quad Nw \leq t \leq (N+1)w$$

So our proved Result is valid for differential difference Eaqution of order $(\alpha, 2)$ given by Ananth in [5].

Letting $w \to 1$, we get

$$\varphi'(t) - \varphi(t) = 0, \quad \varphi(0) = 0$$

Which has solution e^t.

Now we can Solve non linear Fractional Delay differential equation with general initial conditions.

2) Consider the following equation

$$\varphi^\alpha(t) = 2 - \varphi(t-w) + a\varphi^3(t-2w), \ldots 9)$$

$$\varphi(t) = 1\,, 0 \leq t \leq 2w \ldots 10)$$

$$\varphi^{(k)}(0) = 0, \qquad t > 2w, k = 1,2, \ldots (n-1), \ldots 11)$$

Multiply equation 9 by $e^{-st}, s > 1$ and integrate between 2w to ∞

$$L[\varphi^\alpha(t)] = \frac{1}{s} + \frac{e^{-ws}}{s^{\alpha+1}} - \frac{e^{-2ws}}{s^{\alpha+1}} - \frac{e^{-ws}L[\varphi(t)]}{s^\alpha} - a\frac{e^{-2ws}L[\varphi^3(t)]}{s^\alpha}$$

Set Laplace decomposition as

$$L[\varphi(t)] = \sum_{n=0}^{\infty} e^{-nws}\,L[\varphi_n(t)]$$

Set Laplace decomposition as

$$L[\varphi^3(t)] = L[{\varphi_0}^3(t)] + e^{-ws}L[3{\varphi_0}^3(t)\varphi_1(t)] + e^{-2ws}L[3{\varphi_0}^2(t)\varphi_2(t) + 3\varphi_0(t){\varphi_1}^2(t)] + e^{-3ws}L[3{\varphi_0}^2(t)\varphi_3(t) + 3\varphi_0(t)\varphi_1(t)\varphi_2(t) + {\varphi_1}^3(t)] = \sum_{n=0}^{\infty} e^{-nws}A_n(t)$$

Where $A's$ are adomian Polynomial

$$A_0(t) = {\varphi_0}^3(t)$$
$$A_1(t) = 3{\varphi_0}^3(t)\varphi_1(t)$$
$$A_2(t) = 3{\varphi_0}^2(t)\varphi_2(t) + 3\varphi_0(t){\varphi_1}^2(t)$$
$$A_3(t) = 3{\varphi_0}^2(t)\varphi_3(t) + 3\varphi_0(t)\varphi_1(t)\varphi_2(t) + {\varphi_1}^3(t)$$

Now the main decomposition set iteratively as

$$\sum_{n=0}^{\infty} e^{-nws}\,L[\varphi_n(t)] = \frac{1}{s} + \frac{e^{-ws}}{s^{\alpha+1}} - \frac{e^{-2ws}}{s^{\alpha+1}} - \frac{e^{-ws}}{s^\alpha}\sum_{n=0}^{\infty} e^{-nws}\,L[\varphi_n(t)] - a\frac{e^{-2ws}}{s^\alpha}\sum_{n=0}^{\infty} e^{-nws}A_n(t)$$

$$= \frac{1}{s} + \left[\frac{1}{s^{\alpha+1}} - \frac{1L[\varphi_0(t)]}{s^\alpha}\right]e^{-ws} + \left[\frac{-1}{s^{\alpha+1}} - \frac{1L[\varphi_1(t)]}{s^\alpha} - \frac{aA_0(t)}{s^\alpha}\right]e^{-2ws} - \sum_{n=3}^{\infty}\frac{L[\varphi_{n-1}(t)]}{s^\alpha} + \frac{aL[A_{n-2}]}{s^\alpha}$$

Comparing the coefficient of e^{-nws} on both sides

$$n = 0, L[\varphi_0(t)] = \frac{1}{s} \to \varphi_0(t) = 1$$

$$n = 1, L[\varphi_1(t)] = \left[\frac{1}{s^{\alpha+1}} - \frac{1L[\varphi_0(t)]}{s^\alpha}\right] = 0 \to \varphi_1(t) = 0$$

$$n = 2, L[\varphi_2(t)] = \left[\frac{-1}{s^{\alpha+1}} - \frac{1L[\varphi_1(t)]}{s^\alpha} - \frac{aA_0(t)}{s^\alpha}\right] = \frac{-1}{s^{\alpha+1}} - \frac{a}{s^{\alpha+1}} \to \varphi_2(t) = (-1-a)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$n = 3, L[\varphi_3(t)] = \left[-\frac{1L[\varphi_2(t)]}{s^\alpha} - \frac{aA_2(t)}{s^\alpha}\right] = \frac{-1-a}{s^{\alpha+1}}\frac{1}{s^\alpha} \to \varphi_3(t) = (-1-a)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

We obtain approximate solution iteratively ϕ(t), t>0 $3w \leq t \leq 4w$

Therefore

$$\varphi(t) = \sum_{n=0}^{\infty} \varphi_n\,(t-nw)e(t-nw)$$

Where e(t-nw) is unit step function

$$\varphi(t) \approx \sum_{n=0}^{4} \varphi_n\,(t-nw)$$

$$\varphi(t) = 1 += (-1-a)\frac{(t-2w)^{\alpha}}{\Gamma(\alpha+1)} +$$
$$= (-1-a)\frac{(t-3w)^{2\alpha}}{\Gamma(2\alpha+1)}$$

Putting $\alpha = 1$

$$\varphi(t) = 1 += (-1-a)\frac{t-2w}{1!} += (-1-a)\frac{(t-3w)^2}{2!}$$

So, our proved Result is valid for differential difference Equation of order $(\alpha, 2)$ given by Ananth in [5].

Letting $w \to 0$ the Exact Solution

$$\varphi(t) = 1 += (-1-a)\frac{t}{1!} += (-1-a)\frac{t^2}{2!}$$

4. Conclusion

We extend the result given by differential difference equation for fractional order differential difference equation where the fractional derivative used is caputo fractional derivative. The obtained result is Compaired with differential difference equation by putting $\alpha = 1$ and verify the validity of the result and the result hold good.

References

[1] K.S. Miller, An Introduction to Fractional Calculus and Fractional Differential Equations, J. Wiley and Sons, New York, 1993.

[2] K. Oldham, J. Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitrary Order, Academic Press, USA, 1974.

[3] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, in: Math. Studies., North-Holland, New York, 2006.

[4] I. Podlubny, Fractional Differential Equations, Academic Press, USA, 1999.

[5] Ananth Kumar S. R., R. Rangarajan Laplace decomposition method for solving certain differential-difference equations both of order 1 Advanced Studies in Contemporary Mathematics,23 (3) (2013), 501-508

[6] Ananth Kumar S. R, R. Rangarajan. Applicability of Laplace decomposition method for solving certain differential-difference equations of order (1, 2) Bulletin of the International Mathematical Virtual Institute,3 (2013),103-111

[7] Kamble Rajratna M, Kulkarni Pramod Ramakant, Laplace Transform and Laplace Decomposition Method for of order (α,1) Fractional Differential Difference Equations with Interval Conditions Linear and Nonlinear, IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 21, Issue 1, Ser. 1 (Jan.– Feb. 2025), PP 04-29,

[8] Kamble Rajratna M.,Kulkarni Pramod Ramakant, Existence and Uniqueness of Continuous Solutions for Conformable Fractional Integro-Differential Equations in Cone Metric Spaces, Communications on Applied Nonlinear Analysis, ISSN: 1074-133X, Vol 32 No. 9s (2025)

[9] Kamble, Rajratana, and Pramod Kulkarni. "Numerical Solutions of the SIR Mathematical Model of Computer Viruses Involving Non-linear Fractional Order Differential Equation." European Journal of Pure and Applied Mathematics 18.4 (2025): 6693-6693.

[10] RAJRATANA, MAROTI KAMBLE, and PR KULKARNI. "Extended fractional derivative: Some results involving classical properties and applications." THE SCIENTIFIC TEMPER Учредители: The Scientific Temper 15.04 (2024): 3026-3033.

[11] KAMBLE, RAJRATNA M., and PRAMOD R. KULKARNI. "EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR EXPONENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS." THE SCIENTIFIC TEMPER Учредители: The Scientific Temper 15.04 (2024): 3021-3025.

[12] Kamble Rajratna, M., et al. "Gheorghe Săvoiu, Mladen Čudanov and Vesna Tornjanski."