Tests for Equality of Coefficients of Variation and Inverse Coefficients of Variation in Multivariate Normal Distribution

Aruna Kalkur T.[1], Aruna Rao K.[2]

[1]St. Aloysius (Deemed to be University), Mangaluru, Karnataka, India
Corresponding Author Email: *arunakalkura[at]staloysius.edu.in*

[2]Email: *arunaraomu[at]gmail.com*

Abstract: ***In this paper, we derived the Wald test for testing the equality of the Coefficients of Variation (CV) and Inverse Coefficients of Variation (ICV) of the multivariate normal distribution. The asymptotic null distribution for both proposed test statistics corresponds to the Chi-square distribution. Comprehensive simulation studies demonstrate that the Wald test based on ICV consistently maintains the nominal significance level and offers greater power compared to its CV, particularly as the Euclidean distance between alternative and null parameter points grows. Considering that far-away alternatives are less relevant for applied research, the Wald test for ICV is recommended for practical use.***

Keywords: Coefficient of Variation, Multivariate Normal distribution, Euclidean distance, Wald Test

1. Introduction

Many papers have appeared in the past for testing the equality of the Coefficients of variation of *k* normal distributions. Historically, the first research work on CV dates back to 1932 ([10]). Initially, the researchers were interested in developing an improved Confidence Interval for the CV of the Normal Distribution. Recent references on the subject are available in [19], which also lists earlier works on it.

Although a 100(1-α)% confidence interval and a level α test are interrelated, a formal Likelihood Ratio Test for equality of CVs of independent Normal Distributions was first introduced by Bennett [2] using modified CV. [18] improved these tests using conditional likelihood. Likelihood Ratio (LR) Test for equality of CVs of two independent Normal Distributions was proposed by Miller and Karson (1977). [4] extended the LR test for testing the equality of CVs of more than two independent normal distributions. [15] proposed the Wald test (also see Miller, 1991) for the same hypothesis. Gupta and Ma (1996) proposed a Score test for testing the equality of CVs of two or more Normal Distributions.

Following the generalised variable approach of Tsui and Weerahandi (1989), Jafari and Behboodian (2010) developed generalized test statistic for testing the common CVs of two or more independent normal distributions. The LR, Wald and Score tests and their perturbed versions were not robust against the assumption of normality. This has motivated Cabras et al. (2006) to propose bootstrap tests for the equality of CVs of two distributions. All these tests assume that the samples are independent. But in practice, we often encounter correlated observations. In medical studies, many of the periodic characters are interrelated. For example, in the field of Anatomy, when gender has to be decided using the various measurements of the skull, these measurements are interrelated. In the stock market, the stock prices of various scripts are related, and when testing for equality of volatility for mean return for two or more scripts, the correlation needs to be accounted for. Singh (1993) proposed generalized test for testing equality of CVs of *p*-variate normal distributions. This test is computationally tedious, and it is not appealing to the scientists in the applied disciplines. To overcome this difficulty, Rao and Kalkur (2014) proposed LR, Wald and Score tests for testing equality of CVs and Inverse Coefficients of Variation (ICV) from a Bivariate Normal Distribution. The simulation results indicate that the Wald test based on ICV performs well and has more power compared to the LR and Score tests using CVs and ICVs.

Jafari (2015) proposed a test for equality of CVs in a multivariate normal distribution using the approach of generalized test statistic and *p*-value. Aerts and Haesbroeck (2017) proposed several classical and robust Wald-type tests. The other test for the multivariate normal distribution was due to Kalkur and Rao (2014), where they used a pairwise procedure for testing equality of CV for the multivariate normal distribution. This has motivated us to develop a simple test for testing the equality of CVs and ICVs from a multivariate normal distribution. We use the Wald principle to derive the test. The reason for restricting our attention only to the Wald test is that in a bivariate normal distribution, this test performs very well in terms of maintaining the level of significance and in terms of power of the test compared to the Likelihood Ratio or Score test. The other reason is that it is not clear how we can derive the Likelihood Ratio and Score test for equality of CVs in a multivariate normal distribution. This research work fills a long gap in the area of testing equality of CVs for correlated random variables. Simulation result indicates that, as in the case of bivariate normal distribution Wald test based on ICV maintains the level of significance and is more powerful than the Wald test based on CV.

This paper is divided into 6 sections. The derivations of the tests are presented in Section 1.2. The finite (small) sample performance of the tests is examined using extensive simulation, and the description of the simulation experiment is given in Section 1.3. The finite sample behavior of the test under the null hypothesis is presented in Section 1.4, while the results regarding the power of the test are shown in Section 1.5. The chapter concludes in section 1.6.

2. Tests for Equality of Coefficients of Variation and Inverse Coefficients of Variation of k Variate Normal Distribution

Let $X = [X_1, X_2, \ldots, X_k]$ follow a Multivariate normal distribution with a mean vector $\mu = [\mu_1, \mu_2, \ldots, \mu_k]$ and dispersion matrix Σ. The $(i, j)^{th}$ element of Σ is given by $(\rho_{ij}\sigma_i\sigma_j)$, where σ_i^2 and σ_j^2 denote the variances of X_i and X_j respectively and ρ_{ij} denote the correlation coefficient between X_i and X_j, $i, j = 1, \ldots, k$.

Let $\theta_i = \frac{\sigma_i}{\mu_i}$ and $\theta_{1i} = \frac{\mu_i}{\sigma_i}$ denote the population CV and ICV for the random variable X_i respectively.

Given a random sample of size *n* from the multivariate normal distribution,

Let $\widehat{\theta_i} = \frac{\widehat{\sigma_i}}{\widehat{\mu_i}}$ and $\widehat{\theta_{1i}} = \frac{\widehat{\mu_i}}{\widehat{\sigma_i}}$, where $\hat{\mu}_i = \bar{x}_i = \frac{1}{n}\sum_{i=1}^{n} x_{ij}$ and $\hat{\sigma}_i^2 = s_i^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ij} - \bar{x})^2$ (1)

denotes the sample mean and variance for the i^{th} component of the multivariate normal distribution. Observe that $\widehat{\theta_i}$ and $\widehat{\theta_{1i}}$ are the maximum likelihood estimators (MLEs) of θ_i and θ_{1i} respectively. Let $\hat{\rho}_{ij}$ denote the maximum likelihood estimator of ρ_{ij} and is given by $\hat{\rho}_{ij} = r_{ij}$. Where r_{ij} denote the sample correlation coefficient for the variable X_i and X_j. (The result follows from the invariance property of MLEs (Kale, 1999).

2.1 Tests for Equality of Coefficients of Variation

The null hypothesis is H_0: $\theta_1 = \cdots = \theta_k$

Let $\theta = (\theta_1, \ldots, \theta_k)$

The null hypothesis is equivalent for testing

$h(\theta) = ((\theta_1 - \theta_2)\ (\theta_1 - \theta_3) \ldots (\theta_1 - \theta_k))' = O$ (2)

The Wald test statistic for testing $h(\theta) = \mathbf{O}$ is given by

$W_1 = h(\hat{\theta})'[V(\widehat{h(\hat{\theta})})]^{-1}\, h(\hat{\theta})$, (3)

where $\left[V\left(\widehat{h(\hat{\theta})}\right)\right]$ is the estimated covariance matrix of $h(\hat{\theta})$. For details of the Wald test, see Silvey (1970).

Derivation of $Cov(\hat{\theta}_i, \hat{\theta}_j)$, i, $j=1,\ldots,k$ is given in appendix. $V\left(h(\hat{\theta})\right)$ involves in finding $Cov\ [(\theta_l - \theta_l)\ (\widehat{\theta_l} - \widehat{\theta_m})]$, which can be derived using the expressions given by Kalkur and Rao (2014).

For this purpose, consider,

$Cov\ [(\theta_i - \theta_l), (\widehat{\theta_l} - \widehat{\theta_m})] = V(\widehat{\theta_l}) - Cov\ (\hat{\theta}_l, \hat{\theta}_m) - Cov\ (\hat{\theta}_l, \hat{\theta}_l) + Cov\ (\hat{\theta}_l, \hat{\theta}_m)$ (4)

$V(\widehat{h(\hat{\theta})})$ denotes the estimated variance of $V\left(h(\hat{\theta})\right)$, where the parameters are replaced by their estimators.

2.2 Tests for Equality of Inverse Coefficient of Variation

The null hypothesis is H_0: $\theta_{11} = \ldots, = \theta_{1k}$

Let $\theta_1 = (\theta_{11}, \ldots, \theta_{1k})$

The null hypothesis is equivalent to testing

$h_1(\theta_1) = ((\theta_{11} - \theta_{12})\ (\theta_{11} - \theta_{13}) \ldots (\theta_{11} - \theta_{1k}))' = O$ (5)

The Wald test statistic for testing $h_1(\theta_1) = O$ is given by

$$W_2 = \left(h_1(\widehat{\theta_1})\right)' \left[V\left(\widehat{h_1(\widehat{\theta_1})}\right)\right] (h_1(\widehat{\theta_1})) \quad (6)$$

As in section 1.2.1, the expression for $[V(\widehat{h_1(\widehat{\theta_1})})]$ can easily be derived.

Theorem 1.1: *Under H_0 the asymptotic null distribution of W_1 as well as W_2 follows a central Chi-square distribution with $(k-1)$ degrees of freedom.*

Proof: Follows from standard asymptotics for the maximum likelihood estimators, and the proof is omitted.

A decision can be taken using the upper α^{th} percentile value of the central Chi-square distribution with $(k-1)$ degrees of freedom.

Note: When $k=3$, closed form expression for W_1 and W_2 can be obtained. The expressions are given in Appendix A1.

2.3 A Special Case

Equi correlated observations occur in Split-plot and other hierarchical designs. When the common correlation coefficient is ρ, the above test can be used if we can find MLE of ρ. Since it is difficult, one can use the estimator $\hat{\rho} = \frac{\sum \widehat{\sigma_{ij}}}{\sum \sigma_i \sigma_j}$, where $\widehat{\sigma_{ij}}$ denotes the sample covariance between X_i and X_j and the summation extends over ${}^{K}C_2$ pairs. Since sample covariance is a consistent estimator of population covariance (Kendall and Stuart, 1977), it follows that $\hat{\rho}$ is a consistent estimator of ρ. We have the following theorem.

Theorem 1.2: *The asymptotic null distribution of the test statistic W_1 and W_2 using the estimator $\hat{\rho}$ and other maximum likelihood estimators of other parameters in $[V(\widehat{h(\hat{\theta})})]$ or $[V(\widehat{h_1(\widehat{\theta_1})})]$ is Chi-square with $(k-1)$ degrees of freedom.*

Proof: Since $[V(\widehat{h(\hat{\theta})})]$ or $[V(\widehat{h_1(\widehat{\theta_1})})]$ are consistent estimators of $\left[V\left(h(\hat{\theta})\right)\right]$ or $[V(h_1(\widehat{\theta_1}))]$, standard asymptotic yields the desired result and the proof is omitted.

3. Simulation Experiment

For estimating the Type I error rate, observations are generated from a *k*-variate normal distribution. A common value of CV used in the simulation is 0.1 and 0.3 for each component of the multivariate distribution. In the following, we give the details to the case of common CV=0.1. Under the null hypothesis, the CVs for the components are all set equal to 0.10. The other parameters μ_i and σ_i^2 are adjusted by the relation $\frac{\mu_l}{\sigma_l} = 0.10$; the sample sizes are n=5, 10, 20 and 40.

The value of k is 3. The level of significance α=0.05 and the number of simulations is 10,000. For estimation of the Type I error rate, the upper α^{th} percentile value of the Chi-square distribution with *(k-1)* degrees of freedom is used. The proportion of times the null hypothesis is rejected gives us the estimated type I error rate.

Two scenarios are considered.

Scenario a): In this, we consider the case where the correlation coefficient between the components of the normal distribution is not necessarily equal. The configuration of correlation coefficients $\rho = (\rho_{12,}\ \rho_{13},\ \rho_{23})$ are (0.3, 0.5, 0.7), (-0.5, 0.5, 0.4), (-01, 0.5, -0.3), (-0.1, -0.3, -0.5). The choices of P covers the cases from all positive correlation coefficients to all negative correlation coefficients. The choice of configurations of correlation coefficients is based on the analysis of stock market data reported in Kalkur and Rao (2014).

Scenario b): The case of Equi correlated observations covers situations where Equi-correlated data naturally arise, or for the data sets that are almost Equi correlated. We consider the case of an Equi-correlated normal distribution. The values of correlation coefficients considered are from -0.9 to 0.9 with an increment of 0.2, including an additional value of 0.0.

For the computation of the power of the tests, selected alternative parameter values of CVs are considered. The alternative parameter points for CV (ICV) are chosen to cover the points nearer to the null parameter points (the points which are in the neighbourhood of null parameter points) as well as points which are not in the neighbourhood of the null parameter point. The criterion used for the selection of alternative parameter points is the Euclidean distance from the null parameter points. For the computation of power, the estimated upper α^{th} percentile value of the distribution of the test statistic is used. The proportion of times the null hypothesis is rejected is the estimated power of the test.

4. Behaviour of the Tests Under the Null Hypothesis

We say that a test maintains a level of significance if the estimated Type I error rate falls in the interval $\alpha \pm 0.005$, which is a 10% error. A similar criterion has been used in the past (Guddattu and Rao, 2011).

Scenario (a): Table 1.1 presents the estimated Type I error rate for the Wald test for equality of CV as well as ICV for scenario (a) when k=3. From Table 1.1, it follows that for the case where all the correlation coefficients are positive, even for a small sample size of $n=5$, the Type I error rate are maintained for the Wald test for CV as well as the Wald test for ICV (abbreviated to Wald test for testing equality of CVs and ICVs). The same conclusion is true for the case where two correlation coefficients are positive. When two correlation coefficients are negative, the W_1 test (CV) maintains level significance when the sample sizes are n=20 and 40, while for the W_2 test (ICV), level of significance is maintained for all the sample sizes. When all three correlation coefficients are negative, the level of significance is maintained for the sample sizes *20* and *40* for W_1 test, while for the W_2 test, it is maintained for all sample sizes considered in the simulation. We have carried out the simulation when a common value of CV=0.3. As is to be expected, the results do not change for the two tests across the sample sizes and number of groups. The results are not presented.

Scenario (b): Table 1.2 and 1.3 present the estimated type I error rate for the Equi-correlated case. From Tables 1.2 and 1.3, it follows that the Wald test of ICV maintains a level of significance for the sample sizes n=5, 10, 20 and 40 and for correlation coefficients ranging from -0.7 to 0.7. When the correlation coefficient is ±0.9 the estimated type I error rate is too small. When we have investigated the reason for this, we noticed that for these values of correlation coefficient, the variance matrix becomes semi-definite in several configurations. In such cases, the value of the test statistic is reported as zero or negative, leading to the acceptance of the null hypothesis. Wald test for CV maintains the level of significance for all the sample sizes for the correlation coefficients range from -0.5 to 0.5. These conclusions are true for k=3.

5. Power of the Tests

Scenario (a): Table 1.4 presents the estimated power of W_1 and W_2 tests for various alternative hypotheses when $n=20$ and $k=3$. In the table alternative parameter points shown correspond to the case of the Wald test for CV. The alternative parameter points and Euclidean distance for the W_2 test are suppressed to avoid ambiguity in the cases.

From the table, we notice that for the entire alternative hypothesis and the combinations of correlation coefficients the power of the Wald test for ICV (W_2) is greater than the Wald test for CV (W_1). A peculiar feature that is observed for the W_1 test is that the power of this test increases as the Euclidean distance increases up to a certain point and then starts decreasing. This is true for all the configurations of correlation coefficients. This feature is also observed for W_2 test. The difference is that for the W_2 test, this phenomenon is observed only for one alternative parameter point (Euclidean distance is the largest) among the alternative parameter points, while for W_1 test, this is observed for more than two points.

In order to check whether such a behaviour is related to the number of simulations, the simulation was re-run using 20,000 simulations. But the same phenomenon was observed. To identify the reason for this behavior, we study the estimated variance matrix of $h\ (\hat{\theta})$. We observed that it is negative semi-definite in some cases, leading to the negative value of the test statistic. It may be recalled that $[V\widehat{(h(\hat{\theta}))}]$ depends on the values of CVs as well as the corresponding correlation coefficients. When such cases increase substantially, the power of the test decreases because of the negative value of the test statistic. With reference to the Score test, Sumathi and Rao (2009, also see the papers cited in their paper) referred to it as a finite sample inconsistency of the test.

Scenario (b): In this scenario, in addition to the alternative parameter points considered in scenario (a), additional parameter points are also considered. The alternative parameter point is (0.1 (0.1) *C*, (0.1) C^2), *C*>0. Table 1.5 and 1.6 present the power of the tests for various values of correlation coefficients when *n=20* and *k=3*. Table 1.4 presents the power of the tests when the alternative hypothesis is (0.1 (0.1) *C*, (0.1) C^2). Since the Euclidean distance is only a function of *C*, the power curve is plotted corresponding to the values of *C* in the *x*-axis and the power of the tests along the *y*-axis. *C*=1 corresponds to the case of the null hypothesis. While reading the graph, caution is necessary, as the alternative parameter points are not directly displayed in the two-dimensional graph.

We first discuss the result corresponding to various alternative parameter points of scenario (a) and then discuss the case corresponding to (0.1, (0.1) C, 0.1(C^2)) subsequently.

As in scenario (a), the power of the Wald test W_2 (ICV) has higher power compared to the Wald test W_1 (CV). When the correlation coefficients range from -0.5 to 0.7, the power of the Wald test increases up to a certain value of the alternative hypothesis and starts decreasing as the Euclidean distance of the alternative parameter points increases. It clearly indicates that this behavior has nothing to do with the value of the common correlation coefficient and depends only on the alternative parameter points. As in scenario (a), the same behavior of the power of the test is true for the test W_2 (ICV), along with the observations made.

When the alternative parameter point is (0.1 (0.1)*C*, (0.1)C^2), the power curve is computed only for selected values of the common correlation coefficient. It is computationally tedious to estimate the power function for all values of the common correlation coefficient. Table 1.1 corresponds to the case ρ=0.1. The overall conclusion is based on the power curve for other values of ρ not shown here. The finite sample inconsistency is observed for all values of the common correlation coefficient.

6. Discussion and Conclusion

In this paper, we derived a Wald test for testing the equality of coefficients of variation (CV) and inverse coefficients of variation (ICV) of the multivariate normal distribution. The asymptotic null distribution of both the test statistics is Chi-square with $(k-1)$degrees of freedom. Extensive simulation results indicate that the Wald test based on ICV maintains the level of significance and is more powerful compared to the Wald test based on CV. The power of both the tests increases as the Euclidean distance of the alternative parameter point from the null parameter point increases and then starts decreasing. The Wald test for ICV does not suffer from this drawback to the same extent as the Wald test for CV. As far away alternative approaches are not important from the applied research point, we safely recommend the Wald test for ICV. The test proposed by Jafari (2015) is computationally very intensive and is not included in the present comparison; and can be a topic for future research.

Tables

Table 1.1: Estimated type I error rate for scenario (a) for various combinations of $\rho = (\rho_{12}, \rho_{13}, \rho_{23})$ when k=3 for various sample sizes

n	$P=(0.3, 0.5, 0.7)$		$P=(0.5, 0.5, -0.4)$		$P=(0.5, -01, -0.6)$		$P=(-0.3, -0.5, -0.7)$	
	W_1	W_2	W_1	W_2	W_1	W_2	W_1	W_2
5	0.0490	0.0550	0.0554	0.0485	0.0987	0.0479	0.1133	0.0543
10	0.0498	0.0497	0.0458	0.0513	0.0975	0.0485	0.1020	0.0552
20	0.0458	0.0523	0.0449	0.0478	0.0463	0.0540	0.0515	0.0543
40	0.0465	0.0525	0.0460	0.045	0.0452	0.0504	0.0485	0.0510

Table 1.2: Estimated type I error rate for the Equi-correlated case for the two tests when $\rho = -0.5, -0.3, -0.1, 0$ and $k = 3$

n	$\rho=-0.5$		$\rho=-0.3$		$\rho=-0.1$		$\rho=0$		$\rho=0.1$	
	W_1	W_2	W_1	W_2	W_1	W_2	W_1	W_2	W_1	W_2
5	0.0415	0.0500	0.0475	0.0510	0.0400	0.0523	0.0475	0.0465	0.0465	0.0445
10	0.0540	0.0495	0.0505	0.0475	0.0445	0.0545	0.0540	0.0517	0.0500	0.0510
20	0.0513	0.0473	0.0489	0.0500	0.0510	0.0455	0.0487	0.0525	0.0505	0.0420
40	0.0520	0.0520	0.0515	0.0530	0.0485	0.0540	0.0514	0.0480	0.0476	0.0465

Table 1.3: Estimated type I error rate for Equi-correlated case for the two tests when $\rho = 0.3, 0.5, 0.7, 0.9$ and $k = 3$

n	$\rho=0.3$		$\rho=0.5$		$\rho=0.7$		$\rho=0.9$	
	W_1	W_2	W_1	W_2	W_1	W_2	W_1	W_2
5	0.0445	0.0555	0.0446	0.0519	0.0400	0.0335	0.0320	0.0215
10	0.0545	0.0450	0.0520	0.0445	0.0524	0.0445	0.0405	0.0455
20	0.0474	0.0486	0.0485	0.0500	0.0455	0.0475	0.0445	0.0345
40	0.0445	0.0520	0.0520	0.0375	0.0525	0.0510	0.0200	0.0325

Table 1.4: Estimated Power of the tests for various configurations of $\rho = (\rho_{12}, \rho_{13}, \rho_{23})$ when the sample size n=20 and k=3

Alternative Hypothesis	*Euclidian Distance*	$(0.3, 0.5, 0.7)$		$(0.5, 0.5, -0.4)$		$(0.5, -01, -0.6)$		$(-0.3, -0.5, -0.7)$	
		W_1	W_2	W_1	W_2	W_1	W_2	W_1	W_2
0.1, 0.1, 0.5	0.4000	0.3444	0.4854	0.3994	0.0928	0.0479	0.0553	0.0033	0.1543
0.1, 0.3, 0.5	0.4472	0.4234	0.5891	0.5990	0.6286	0.5985	0.6523	0.1020	0.2955
0.1, 0.5, 0.5	0.5657	0.2735	0.9830	0.6785	0.8867	0.1983	0.9523	0.7150	0.5943
0.3, 0.5, 0.7	0.7483	0.2094	0.9410	0.5714	0.9867	0.1720	0.9829	0.4485	0.9138
0.5, 0.7, 0.9	1.0770	0.0970	0.1660	0.1595	0.0147	0.0734	0.1634	0.0910	0.1216

Table 1.5: Estimated Power of the Tests for Equi-correlated case when n=20 and k = 3

Alternative hypothesis	*Euclidian Distance*	$\rho = -0.5$		$\rho = -0.3$		$\rho = -0.1$		$\rho = 0$	
		W_1	W_2	W_1	W_2	W_1	W_2	W_1	W_2
0.1, 0.1, 0.5	0.4000	0.0967	0.1137	0.1204	0.2817	0.1129	0.1395	0.2194	0.2005
0.1, 0.3, 0.5	0.4472	0.5682	0.6166	0.5567	0.5715	0.5880	0.6381	0.5812	0.5794
0.1, 0.5, 0.5	0.5657	0.6052	0.9107	0.6924	0.8935	0.6021	0.8247	0.6932	0.7278
0.3, 0.5, 0.7	0.7483	0.1238	0.9774	0.1154	0.9505	0.1235	0.9029	0.7980	0.8450
0.5, 0.7, 0.9	1.0770	0.0730	0.1506	0.0797	0.1767	0.0987	0.1494	0.9377	0.9400

Table 1.6 Estimated Power of the Tests for the Equi-correlated case for n=20 and k = 3

Alternative hypothesis	*Euclidian Distance*	$\rho = 0.3$		$\rho = 0.5$		$\rho = 0.7$		$\rho = 0.9$	
		W_1	W_2	W_1	W_2	W_1	W_2	W_1	W_2
0.1, 0.1, 0.5	0.4000	0.3389	0.4005	0.1036	0.3975	0.4098	0.4167	0.1902	0.2473
0.1, 0.3, 0.5	0.4472	0.6024	0.6800	0.6801	0.7001	0.6425	0.8379	0.5552	0.9925
0.1, 0.5, 0.5	0.5657	0.2813	0.9200	0.2499	0.9710	0.2855	0.8858	0.3245	0.9964
0.3, 0.5, 0.7	0.7483	0.2711	0.9635	0.2110	0.3705	0.2590	0.9870	0.2905	0.8666
0.5, 0.7, 0.9	1.0770	0.1257	0.1767	0.1506	0.2133	0.1378	0.3204	0.1600	0.6826

Appendix A1

For *k*=3, the estimated variance matrix of $h(\hat{\theta})$ is

$$V\left(h(\hat{\theta})\right)=\begin{bmatrix} V(\hat{\theta}_1-\hat{\theta}_2) & Cov(\hat{\theta}_1-\hat{\theta}_2,\hat{\theta}_1-\hat{\theta}_3) \\ Cov(\hat{\theta}_1-\hat{\theta}_2,\hat{\theta}_1-\hat{\theta}_3) & V(\hat{\theta}_1-\hat{\theta}_3) \end{bmatrix}$$

Where $V(\hat{\theta}_1-\hat{\theta}_2)$ is computed by Delta method as

$$V(\hat{\theta}_1-\hat{\theta}_2)=V\left(\frac{\hat{\sigma}_1}{\hat{\mu}_1}-\frac{\hat{\sigma}_2}{\hat{\mu}_2}\right)=\left(\frac{\delta A}{\delta\mu_1}\right)^2 V(\hat{\mu}_1)+\left(\frac{\delta A}{\delta\mu_2}\right)^2 V(\hat{\mu}_2)+\left(\frac{\delta A}{\delta\sigma_1}\right)^2 V(\hat{\sigma}_1^2)+$$

$$\left(\frac{\delta A}{\delta\sigma_2}\right)^2 V(\hat{\sigma}_2^2)+2cov(\hat{\mu}_1,\hat{\mu}_2)\frac{\delta A}{\delta\mu_1}\frac{\delta A}{\delta\mu_2}+2cov(\hat{\sigma}_1^2,\hat{\sigma}_2^2)\frac{\delta A}{\delta\sigma_1}\frac{\delta A}{\delta\sigma_2}, \quad where\ A=\frac{\hat{\sigma}_1}{\hat{\mu}_1}-\frac{\hat{\sigma}_2}{\hat{\mu}_2}$$

$$=\frac{1}{n}\hat{\theta}_1^4+\frac{1}{n}\hat{\theta}_2^4+\frac{1}{2n}\hat{\theta}_1^2+\frac{1}{2n}\hat{\theta}_2^2-\frac{2}{n}\rho_{12}\hat{\theta}_1^2\hat{\theta}_2^2-\frac{1}{n}\rho_{12}^2\hat{\theta}_1\hat{\theta}_2$$

$$V(\hat{\theta}_1-\hat{\theta}_3)=\frac{1}{n}\hat{\theta}_1^4+\frac{1}{n}\hat{\theta}_3^4+\frac{1}{2n}\hat{\theta}_1^2+\frac{1}{2n}\hat{\theta}_3^2-\frac{2}{n}\rho_{13}\hat{\theta}_1^2\hat{\theta}_3^2-\frac{1}{n}\rho_{13}^2\hat{\theta}_1\hat{\theta}_3$$

$$V(\hat{\theta}_1)=\frac{1}{n}\hat{\theta}_1^4+\frac{1}{2n}\hat{\theta}_1^2,\ V(\hat{\theta}_2)=\frac{1}{n}\hat{\theta}_2^4+\frac{1}{2n}\hat{\theta}_2^2$$

$$Cov(\hat{\theta}_1-\hat{\theta}_2,\hat{\theta}_1-\hat{\theta}_3)=V(\hat{\theta}_1)\text{-}\ Cov(\hat{\theta}_1,\ \hat{\theta}_2)\text{-}\ Cov(\hat{\theta}_1,\ \hat{\theta}_3)+Cov(\hat{\theta}_2,\ \hat{\theta}_3)$$

$$Cov(\hat{\theta}_1,\ \hat{\theta}_2)=\frac{\rho_{12}\hat{\theta}_1^2\hat{\theta}_2^2}{n}+\frac{{\rho_{12}}^2\hat{\theta}_1^2\hat{\theta}_2^2}{2n},\ Cov(\hat{\theta}_1,\ \hat{\theta}_3)=\frac{\rho_{13}\hat{\theta}_1^2\hat{\theta}_3^2}{n}+\frac{{\rho_{13}}^2\hat{\theta}_1^2\hat{\theta}_3^2}{2n},\ Cov(\hat{\theta}_2,\ \hat{\theta}_3)=\frac{\rho_{23}\hat{\theta}_2^2\hat{\theta}_3^2}{n}+\frac{{\rho_{23}}^2\hat{\theta}_2^2\hat{\theta}_3^2}{2n}$$

For *k*=3, the estimated covariance matrix of $h_1(\widehat{\theta 1})$ is

$$V(h_1(\widehat{\theta 1})=\begin{bmatrix} V(\hat{\theta}_{11}-\hat{\theta}_{12}) & Cov(\hat{\theta}_{11}-\hat{\theta}_{12},\hat{\theta}_{11}-\hat{\theta}_{13}) \\ Cov(\hat{\theta}_{11}-\hat{\theta}_{12},\hat{\theta}_{11}-\hat{\theta}_{13}) & V(\hat{\theta}_{11}-\hat{\theta}_{13}) \end{bmatrix}$$

Where $V(\hat{\theta}_{11}-\hat{\theta}_{12})$ is computed by Delta method as

$$V(\hat{\theta}_1-\hat{\theta}_2)=V\left(\frac{\hat{\mu}_1}{\hat{\sigma}_1}-\frac{\hat{\mu}_2}{\hat{\sigma}_2}\right)=\left(\frac{\delta A1}{\delta\mu_1}\right)^2 V(\hat{\mu}_1)+\left(\frac{\delta A1}{\delta\mu_2}\right)^2 V(\hat{\mu}_2)+\left(\frac{\delta A1}{\delta\sigma_1}\right)^2 V(\hat{\sigma}_1^2)+$$

$$\left(\frac{\delta A1}{\delta\sigma_2}\right)^2 V(\hat{\sigma}_2^2)+2cov(\hat{\mu}_1,\hat{\mu}_2)\frac{\delta A1}{\delta\mu_1}\frac{\delta A1}{\delta\mu_2}+2cov(\hat{\sigma}_1^2,\hat{\sigma}_2^2)\frac{\delta A1}{\delta\sigma_1}\frac{\delta A1}{\delta\sigma_2}, \text{where } A1=\frac{\hat{\mu}_1}{\hat{\sigma}_1}-\frac{\hat{\mu}_2}{\hat{\sigma}_2}$$

$$=\frac{2}{n}+\frac{1}{2n}\hat{\theta}_{11}^2+\frac{1}{2n}\hat{\theta}_{12}^2-\frac{2}{n}\rho_{12}-\frac{1}{n}{\rho_{12}}^2\,\hat{\theta}_{11}^2\hat{\theta}_{12}^2$$

$$V(\hat{\theta}_1-\hat{\theta}_3)=\frac{2}{n}+\frac{1}{2n}\hat{\theta}_{11}^2+\frac{1}{2n}\hat{\theta}_{13}^2-\frac{2}{n}\rho_{12}-\frac{1}{n}{\rho_{13}}^2\,\hat{\theta}_{11}^2\hat{\theta}_{13}^2$$

$$V(\hat{\theta}_1)=\frac{1}{n}+\frac{1}{2n}\hat{\theta}_{11}^2,\ V(\hat{\theta}_2)=\frac{1}{n}+\frac{1}{2n}\hat{\theta}_{12}^2,\ V(\hat{\theta}_3)=\frac{1}{n}+\frac{1}{2n}\hat{\theta}_{13}^2$$

$$Cov(\hat{\theta}_1-\hat{\theta}_2,\hat{\theta}_1-\hat{\theta}_3)=V(\hat{\theta}_1)\text{-}\ Cov(\hat{\theta}_1,\ \hat{\theta}_2)\text{-}\ Cov(\hat{\theta}_1,\ \hat{\theta}_3)+Cov(\hat{\theta}_2,\ \hat{\theta}_3)$$

$$Cov(\hat{\theta}_1,\ \hat{\theta}_2)=\frac{\rho_{12}\hat{\theta}_1^2\hat{\theta}_2^2}{n}+\frac{{\rho_{12}}^2\hat{\theta}_1^2\hat{\theta}_2^2}{2n},\ Cov(\hat{\theta}_1,\ \hat{\theta}_3)=\frac{\rho_{13}\hat{\theta}_1^2\hat{\theta}_3^2}{n}+\frac{{\rho_{13}}^2\hat{\theta}_1^2\hat{\theta}_3^2}{2n},\text{ and } Cov(\hat{\theta}_2,\ \hat{\theta}_3)=\frac{\rho_{23}\hat{\theta}_2^2\hat{\theta}_3^2}{n}+\frac{{\rho_{23}}^2\hat{\theta}_2^2\hat{\theta}_3^2}{2n}$$

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