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The Total Circular Number of Graphs

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Abstract: ***Let $G = (V, E)$ be a connected graph with at least two vertices. A set $S \subseteq V(G)$ is called a total circular set of G if S is a circular set of G and the subgraph $G[S]$ induced by S has no isolated vertex. The total circular number $tcr(G)$ of G is the minimum order of its total circular sets and any total circular set of order $tcr(G)$ is called a tcr-set of G. In this article, the total circular number of some standard graphs are determined.***

Keywords: distance, detour number, geodetic number, circular number

AMS Subject Classification: 05C12.

1. Introduction and Preliminaries

A graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1,5]. Two vertices u and v are said to be adjacent in G if $uv \in E(G)$. The neighborhood $N(v)$ of the vertex v in G is the set of vertices adjacent to v. The degree of the vertex v is $deg(v) = |N(v)|$. If $e = \{u, v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then we call v *an end edge*, u a leaf and v a support vertex. We denote by $\Delta(G)$ the maximum degree of a graph G. The subgraph induced by set S of vertices of a graph G is denoted by $G[S]$ with $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : u, v \in S\}$. A vertex v is called an extreme vertex of G if $G[N(v)]$ is complete.

The *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Any $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic of G..A vertex x is an internal vertex of an $u - v$ path P if x is a vertex of P and $x \neq u, v$. The closed interval $I[u, v]$ consists of u, v and all vertices lying on some $u - v$ geodesic of G. For a non-empty set $S \subseteq V(G)$, the set $I[S] = \cup_{u,v \in S} I[u, v]$ is the closure of S. A set $S \subseteq V(G)$ is called a geodetic set if $I[S] = V(G)$. The minimum cardinality of a geodetic set of G is called the geodetic number of G and is denoted by $g(G)$. A geodetic set of minimum cardinality is called g-set of G. For references on geodetic parameters in graphs see [2,3,7].

The detour *distance* $D(u, v)$ between two vertices $u, v \in V(G)$ is the length of a longest path between them. Any $u - v$ path of length $D(u, v)$ is called an $u - v$ detour of G. The closed interval $I_D[u, v]$ consists of u, v and all vertices lying on some $u - v$ detour of G. For a non-empty set $S \subseteq V(G)$, the set $I_D[S] = \cup_{u,v \in S} I_D[u, v]$ is the closure of S. A set $S \subseteq V(G)$ is called a detour set if $I_D[S] = V(G)$. The minimum cardinality of a detour set of G is called the detour number of G and is denoted by $dn(G)$. A detour set of minimum cardinality is called dn-set of G. These concepts were studied in [4,6].

The circular distance between u and v denoted by $D^c(u, v)$ and is defined by

$$D^c(u, v) = \begin{cases} D(u, v) + d(u, v) & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

Where $D(u, v)$ and $d(u, v)$ are detour distance and distance between u and v respectively. The circular diameter D^c is the longest circular distance between 2 vertices on G. Any $u - v$ path of length $D^c(u, v)$ is called an $u - v$ circular of G. The circular diameter D^c is the longest circular distance between 2 vertices on G. For $u, v \in V$, $I_c[u, v]$ represents set of all vertices lying on a $u - v$ circular in G. For $S \subseteq V(G)$, let $I_c[S] = \cup_{u,v \in S} I_c[u, v]$. The minimum cardinality of a circular set of G is called the circular number of G and is denoted by $cr(G)$. Any circular set of cardinality $cr(G)$ is called a cr-set of G. These concepts were studied in [8,9,10,11,12,13,14,15,16].

2. Total Circular Number of Graphs

Definition 2.1. Let $G = (V, E)$ be a connected graph with at least two vertices. A set $S \subseteq V(G)$ is called *a total circular set* of G if S is a *circular* set of G and the subgraph $G[S]$ induced by S has no isolated vertex. The *total circular number* $tcr(G)$ of G is the minimum order of its total *circular* sets and any total *circular* set of order $tcr(G)$ is called a *tcr-set* of G.

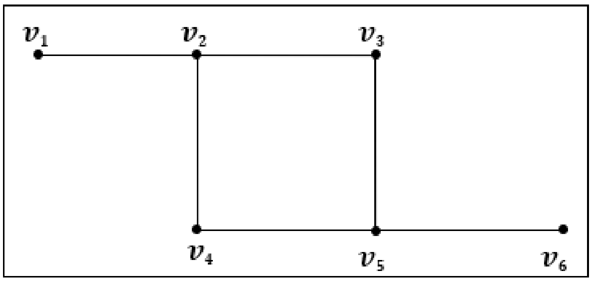

Figure 2.1: A Graph with tcr (G)= 4

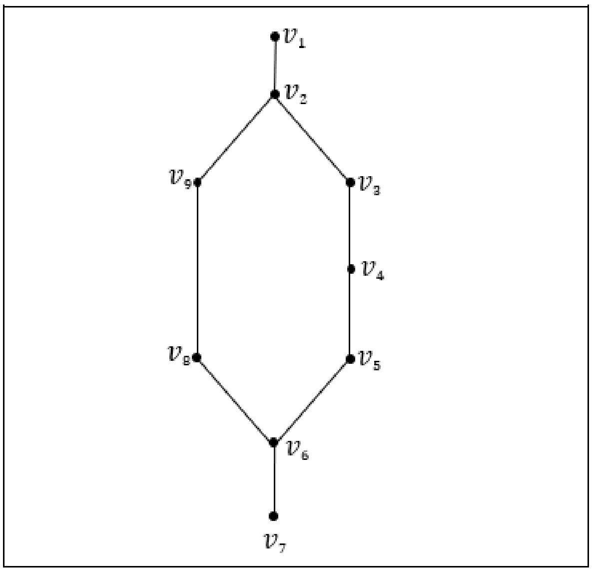

Figure 2.2: A Graph with tcr (G)= 4

Example 2.2 For the graph G given in Figure 2.1, $S_1 = \{v_1, v_6\}$ is the minimum *circular* set of G so that $cr(G) = 2$. Since $G[S_1]$ has isolated vertices,S_1 is not a total *circular* set of G. It is easily verified that no three element subset of $V(G)$ is a total *circular* set of G and so $tcr(G) \geq 4$.However $S_2 = \{v_1, v_2, v_5, v_6\}$ is a minimum total *circular* set of G so that $tcr(G) = 4$.

For the graph G given in Figure 2.2, $S_1 = \{v_1, v_7\}$ is a minimum *circular* sets of G so that cr$(G) = 2$. Also $S_1 = \{v_1, v_2, v_6, v_7\}$ is a minimum total *circular* set of G so that $tcr(G) = 4$.

Theorem 2.3. Every end vertex of G belongs to every total *circular* set of G.

Proof. Since every total *circular* set is a *circular* set of G, the result is obvious.

Theorem 2.4. Let G be a connected graph and uv be an end edge of G.Then $\{u, v\}$ is a subset of every total *circular* set of G.

Proof. Let G be a connected graph. Let uv be an end edge of G and S be a total *circular* set of G. By Theorem 2.3, $v \epsilon S$. Suppose $u \notin S$.Then v is an isolated vertex of $G[S]$, which is a contradiction to S a total *circular* set of G.Therefore $u \in S$. .Hence $\{u, v\}$ is a subset of every total *circular* set of G.

Theorem 2.5. For a graph G of order $n \geq 2$, $2 \leq cr(G) \leq tcr(G) \leq p$.

Proof. A *circular* set needs at least two vertices so that $cr(G) \geq 2$. Since every total *circular* set is also a *circular* set, $cr(G) \leq tcr(G)$. Since the vertex set $V(G)$ is a connected detour set of G, we have $cdn(G) \leq n$ Thus $2 \leq cr(G) \leq tcr(G) \leq n$

Remark 2.6. The bounds in Theorem 2.5 are sharp. For the graph G given in Figure 2.3, $cr(G) = tcr(G) = 2$. Also all the bounds in Theorem 2.5 are strict, For the graph G given in Figure 2.4, $n = 9$, $cr(G) = 3, tcr(G) = 5$ so that $2 < cr(G) < tcr(G) < n$.

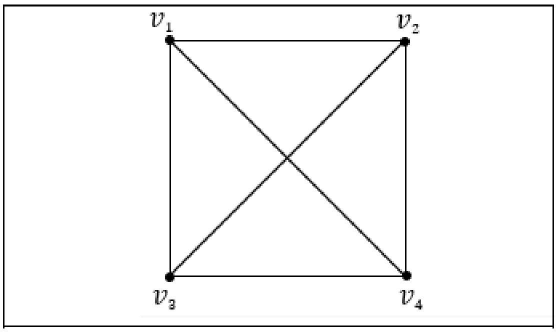

Figure 2.3: A Graph G with tcr (G)= 2

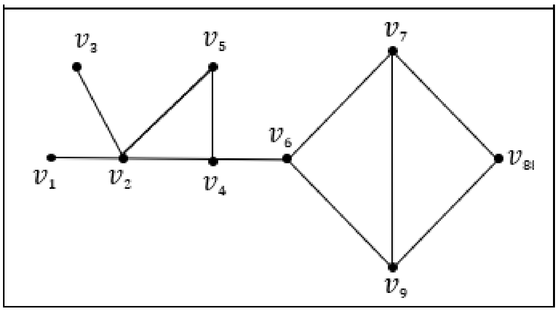

Figure 2.4: A Graph G with tcr (G)= 5

In the following we determine the total *circular* number of some standard graphs.

Theorem 2.7. For the complete graph $G = K_p(n \geq 2)$,$tcr(G) = 2$.

Proof. Let u, v be two vertices of G.Then $S = \{u, v)$ is a total *circular* set of G so that $tcr(G) = 2$.

Theorem 2.8. For the cycle $G = C_p(n \geq 3)$, $tcr(G) = 2$.

Proof. Let u, v be the two adjacent vertices of G. Then $S = \{u, v)$ is a total *circular* set of G so that $tcr(G) = 2$.

Theorem 2.9. For the complete bipartite graph $G = K_{r,s}(r, s \geq 2)$, $tcr(G) = 2$.

Proof. Let $X = \{x_1, x_2, \ldots \ldots . x_r\}$ *and* $Y = \{y_1, y_2, \ldots \ldots . . y_s\}$ be the bipartite sets of G.Then $S = \{x_1, y_1\}$is a total *circular* set of G so that $tcr(G) = 2$.

Theorem 2.10. For the star $G = K_{1,p-1}$, $tcr(G) = n$.

Proof: This follows from Theorems 2.3 and 2.4

Theorem 2.11. For a path $P_n(n \geq 3)$, $tcr(G) = \begin{cases} 3 & if\ n = 3 \\ 4 & if\ n \geq 4 \end{cases}$

Proof. Case (i) Let $n = 3$. Let P_3 be $v_1, v_2, v_{3.}$ Then the result follows from Theorems 2.3 and 2.4.

Case (ii) Let $n \geq 4$, Let P_n be $v_1, v_2, \ldots v_{n-1}, v_{pn}$. Then by Theorems 2.3 and 2.4, $S = \{v_1, v_2, v_{n-1}, v_n\}$ is a subset of

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every total $circular$ set of G and so $tcr(G) \geq 4$ .It is clear that S is a total $circular$ set of G so that $tcr(G) = 4$. Hence the proof.

Theorem 2.12. Let G be a Hamiltonian graph of order $n \geq 3$. Then $tcr(G) = 2$.

Proof. Let C be a Hamiltonian cycle in G. Let $S = \{x, y\}$ be any set of two adjacent vertices in C. Then S is a total $circular$ set of G so that $tcr(G) = 2$.

Remark 2.13. The converse of Theorem 2.12 is not true. For the graph G given in Figure 2.5, $tcr(G) = 2$. However G is not a Hamiltonian graph.

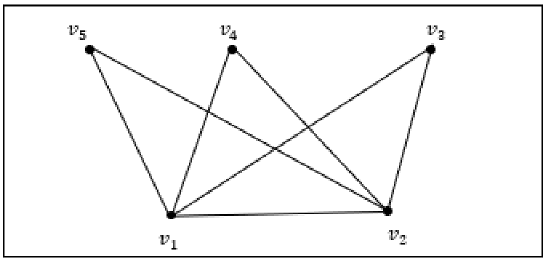

Figure 2.5: A Graph G with tcr (G)= 2

Theorem 2.14. Let G be a connected graph .If every vertex of G lies on $u - v$ $circular$ path P of length diam $_D G$ such that u and v are adjacent, then $tcr(G) = 2$.

Proof. Let P be a $circular$ diametral $u - v$ path of G such that u and v are adjacent. Since every vertex of G lies on the $u - v$ $circular$,$S = \{u, v\}$ is a $circular$ set of G.Since u and v are adjacent $G[S]$ contains no isolated vertices. Hence S is a total $circular$ set of G so that $tcr(G) = 2$.

Remark 2.15. The converse of the Theorem 2.14 is not true. For the graph G given in Figure 2.6, $S = \{v_1, v_2\}$ is a total $circular$ set of G so that $tcr(G) = 2$ .However $v_1 - v_2$ is not a $circular$ diametral path. The $circular$ diametral path is $v_3 - v_4$.

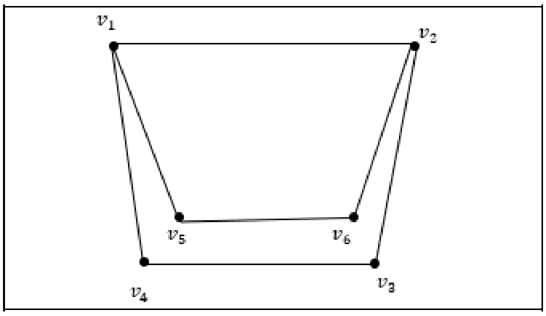

Figure 2.6: A Graph G with tcr (G)= 2

Theorem 2.16. If G is a caterpillar with $deg(v) \geq 3$ for every cut vertex v of G, then $tcr(G) = n$.

Proof. Since $deg(v) \geq 3$ for every cut vertex v of G, there exists a vertex u such that uv is an end edge of G. By Theorems 2.3 and 2.4, $\{u, v\}$ is a subset of every total $circular$ set of G,for every $u, v \in G$. Hence it follows that $tcr(G) = n$.

Remark 2.17. The converse of the Theorem 2.16 is not true. For the graph G given in Figure 2.7, $tcr(G) = 6 = n$. However $deg(v_4) = 2$.

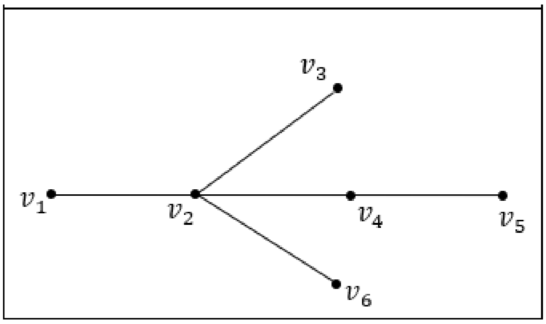

Figure 2.4: A Graph G with tcr (G)= 6

Theorem 2.18. Let G be a connected graph order n without any cut vertex. Let G' be a graph obtained from G by attaching an edge to each vertex of G such that G' has order $2n$. Then $tcr(G') = 2n$.

Proof. Let G be a connected graph without cut vertices. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ .Let G'be a graph obtained from G by adding new vertices $u_i (1 \leq i \leq n)$ and joining $u_i (1 \leq i \leq n)$ with vertex $v_i (1 \leq i \leq n)$.Then $u_i (1 \leq i \leq n)$ is a cut vertex of G' and $v_i (1 \leq i \leq n)$ is an end vertex of G'such that order of $G' is\ 2n$ .By Theorems 2,3 and 2.4, $V(G')$ is the unique total $circular$ set of G so that $tcr(G') = 2n$.

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