

A Mathematical Framework for Fuzzy Graphs in Structural Analysis of Complex Networks: Centrality, Connectivity and Spectral Properties

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Abstract

Complex networks frequently contain relationships whose existence or intensity is not known with certainty. Classical simple graphs force such relations into binary states, while weighted graphs usually treat weights as fixed numerical strengths. Fuzzy graph theory provides a natural mathematical language in which an edge membership $\mu_{ij} \in [0, 1]$ expresses the degree to which a relation is supported. This paper develops a unified mathematical framework connecting three structural viewpoints of a fuzzy network: centrality, connectivity and spectrum. Starting from a finite undirected fuzzy graph $G = (V, \sigma, \mu)$, we construct the fuzzy adjacency matrix $A = [\mu_{ij}]$, degree matrix D , Laplacian $L = D - A$, and the corresponding spectral parameters. A new spectral-fuzzy centrality index is proposed by combining normalized fuzzy degree with normalized two-step influence. The index remains in $[0, 1]$, is invariant under vertex relabeling, is monotone under strengthening of incident memberships under a fixed normalization bound, and attains its maximal value on regular fuzzy networks under the proposed normalization. We further establish relations between the proposed centrality and classical weighted-network quantities, derive spectral connectivity bounds, and formulate an α -cut interpretation that links fuzzy structural analysis to a family of crisp realizations. An illustrative six-node network is analyzed exactly through its adjacency matrix, degree sequence, Laplacian spectrum and proposed centrality values. The framework is intended as a mathematically transparent bridge between fuzzy graph theory and complex-network analysis, with potential applications to uncertain social, communication, biological and renewable-energy networks.

Keywords: Fuzzy graph; complex network; fuzzy centrality; Laplacian spectrum; algebraic connectivity; spectral radius; α -cut; uncertain network; graph energy; structural robustness.

1 Introduction

Graph theory supplies a fundamental language for describing systems composed of entities and pairwise relations. In a simple graph, an edge is either present or absent. This binary assumption is mathematically elegant but restrictive when a relation is gradual, incomplete, subjective or uncertain. Fuzzy set theory, initiated by Zadeh [1], replaces a sharp yes/no membership decision by a membership value in $[0, 1]$. Rosenfeld [2] and Yeh and Bang [3] developed foundational fuzzy-graph formulations.

Complex-network analysis studies large systems in which topology, influence, community structure, robustness and diffusion interact. Degree, betweenness, closeness and eigenvector centrality are standard tools for identifying structurally important nodes [21, 23, 22]. Spectral graph theory adds a complementary algebraic viewpoint: eigenvalues of adjacency and Laplacian matrices encode global structural information [15]. Small-world and scale-free models demonstrate that local structure can generate global effects [18, 19].

The difficulty is that real networks often contain uncertainty simultaneously with these structural effects. A fuzzy graph represents such information directly through membership values. Recent research has extended centrality measures to fuzzy social networks and has shown that centrality can itself be treated as a fuzzy relation rather than a single crisp number [13]. The present paper therefore develops a unified matrix framework in which direct fuzzy degree, two-step influence, Laplacian connectivity and adjacency/Laplacian spectra are studied together.

2 Literature Review and Research Gap

Foundational fuzzy graph theory includes Rosenfeld [2], Yeh and Bang [3], Bhattacharya [4], Bhutani [5], Mordeson [6], Mordeson and Peng [7], and Mordeson and Nair [8, 10]. Connectivity and distance were developed further in the fuzzy setting by Mathew and Sunitha [11] and in later monographs [14]. Generalized models include interval-valued fuzzy graphs [12].

Spectral graph theory provides the classical matrix background [15]. Fuzzy spectral and energy studies have extended adjacency and Laplacian ideas to uncertain graph structures; recent work on Wiener energies is another example of this continuing direction [25]. Complex-network analysis is supported by Newman [16] and Estrada [17]. Zareie and Sakellariou [13] specifically formulated degree, h-index and k-shell centrality as fuzzy relations.

The research gap addressed here is therefore a synthesis gap, not a claim that fuzzy centrality or fuzzy spectral theory is unexplored. The contribution is a unified definition of a low-order spectral-fuzzy centrality together with explicit bounds, invariance, monotonicity, regularity and ranking-sensitivity results, embedded in the same matrix framework as fuzzy connectivity and spectral parameters.

3 Preliminaries

Let $V = \{v_1, \dots, v_n\}$. A fuzzy graph is $G = (\sigma, \mu)$ where $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$ satisfy

$$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}.$$

For the main development we use $\sigma(v) = 1$ and write $\mu_{ij} = \mu(v_i, v_j)$.

For a path $P = (v_0, \dots, v_k)$, define

$$s(P) = \min\{\mu(v_0, v_1), \dots, \mu(v_{k-1}, v_k)\},$$

and

$$\text{CONN}(u, v) = \max_{P: u \rightarrow v} s(P).$$

For $\alpha \in (0, 1]$, the α -cut graph is

$$E_\alpha = \{(v_i, v_j) : \mu_{ij} \geq \alpha\}.$$

4 Matrix Representation

Define

$$A = [a_{ij}], \quad a_{ij} = \mu_{ij}, \quad a_{ii} = 0, \quad a_{ij} = a_{ji}.$$

The fuzzy degree is

$$d_i = \sum_{j=1}^n a_{ij},$$

and $D = \text{diag}(d_1, \dots, d_n)$. The Laplacian is

$$L = D - A.$$

For all $x \in \mathbb{R}^n$,

$$x^T L x = \frac{1}{2} \sum_{i,j=1}^n a_{ij} (x_i - x_j)^2 \geq 0.$$

Hence L is positive semidefinite. Write

$$0 = \lambda_1(L) \leq \lambda_2(L) \leq \dots \leq \lambda_n(L)$$

and let $\rho(A)$ denote the largest adjacency eigenvalue.

5 Fuzzy Centrality

The normalized fuzzy degree is

$$C_i^D = \frac{d_i}{d_{\max}}, \quad d_{\max} = \max_i d_i.$$

Let $\mathbf{d} = (d_1, \dots, d_n)^T$. The two-step influence is

$$q_i = (A\mathbf{d})_i = \sum_j a_{ij} d_j, \quad C_i^{(2)} = \frac{q_i}{d_{\max}^2}.$$

6 Proposed Spectral-Fuzzy Centrality

For $\eta \geq 0$, define

$$\text{SFC}_i(\eta) = \frac{1}{1 + \eta} \left(\frac{d_i}{d_{\max}} + \eta \frac{(Ad)_i}{d_{\max}^2} \right).$$

The parameter η controls the relative contribution of two-step structural influence.

Theorem 6.1 (Boundedness). *For every i and $\eta \geq 0$, $0 \leq \text{SFC}_i(\eta) \leq 1$.*

Proof. Since $0 \leq d_i \leq d_{\max}$, the first normalized term is in $[0, 1]$. Also

$$(Ad)_i = \sum_j a_{ij} d_j \leq d_{\max} \sum_j a_{ij} = d_{\max} d_i \leq d_{\max}^2.$$

Thus the second normalized term is also in $[0, 1]$. Their convex combination belongs to $[0, 1]$. $\square$

Theorem 6.2 (Permutation invariance). *If $A' = PAP^T$ for a permutation matrix P , then the multiset of $\text{SFC}_i(\eta)$ values is unchanged.*

Proof. Since $\mathbf{d}' = P\mathbf{d}$ and $A'\mathbf{d}' = PAd$, both terms are merely permuted. $\square$

Theorem 6.3 (Monotonicity under local strengthening). *Suppose only edges incident with v_i are strengthened and $d_{\max}$ remains unchanged. Then $\text{SFC}'_i(\eta) \geq \text{SFC}_i(\eta)$.*

Proof. The degree d_i cannot decrease. For the second term,

$$(A'\mathbf{d}')_i = \sum_j a'_{ij} d'_j,$$

and each affected factor is nondecreasing. Hence both normalized components are nondecreasing. $\square$

Theorem 6.4 (Regular fuzzy networks). *If $d_i = d > 0$ for all i , then $\text{SFC}_i(\eta) = 1$ for all i .*

Proof. Here $d_{\max} = d$ and $(Ad)_i = d^2$, so both normalized components equal one. $\square$

7 Spectral Connectivity

For a connected weighted support graph,

$$\lambda_2(L) = \min_{\mathbf{x} \perp \mathbf{1}, \mathbf{x} \neq 0} \frac{\mathbf{x}^T L \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

Theorem 7.1 (Upper bound for algebraic connectivity). *Let $d_{\min} = \min_i d_i$. Then*

$$\lambda_2(L) \leq \frac{n}{n-1} d_{\min}.$$

Proof. Choose v_k with degree $d_{\min}$ and let $x_k = n-1$, $x_j = -1$ for $j \neq k$. Then $\mathbf{x} \perp \mathbf{1}$,

$$\mathbf{x}^T \mathbf{x} = n(n-1),$$

and only edges incident to v_k contribute to $x^T Lx$. Their squared differences equal n^2 , giving

$$x^T Lx = n^2 d_{\min}.$$

The Rayleigh principle yields the result. $\square$

Theorem 7.2 (Spectral-radius bounds). *For the symmetric nonnegative adjacency matrix,*

$$d_{\min} \leq \rho(A) \leq d_{\max}.$$

Proof. The lower bound follows from the Rayleigh quotient at $\mathbf{1}$, while the upper bound follows from $\rho(A) \leq \|A\|_{\infty} = d_{\max}$. $\square$

8 α -Cut Analysis

If $0 < \alpha \leq \beta \leq 1$, then $E_{\beta} \subseteq E_{\alpha}$. Consequently,

$$\deg_{G_{\beta}}(v_i) \leq \deg_{G_{\alpha}}(v_i).$$

Thus a fuzzy network induces a filtration of crisp networks. Structural quantities may be studied as functions of α , allowing persistent communities and robust links to be distinguished from threshold-sensitive structures.

9 Numerical Illustration

Consider the six-node fuzzy network with edge memberships

$$\begin{aligned} \mu_{12} &= 0.90, \mu_{13} = 0.65, \mu_{14} = 0.35, \mu_{23} = 0.80, \\ \mu_{25} &= 0.55, \mu_{34} = 0.75, \mu_{35} = 0.45, \mu_{36} = 0.40, \\ \mu_{46} &= 0.85, \mu_{56} = 0.70. \end{aligned}$$

Its adjacency matrix is

$$A = \begin{pmatrix} 0 & .90 & .65 & .35 & 0 & 0 \\ .90 & 0 & .80 & 0 & .55 & 0 \\ .65 & .80 & 0 & .75 & .45 & .40 \\ .35 & 0 & .75 & 0 & 0 & .85 \\ 0 & .55 & .45 & 0 & 0 & .70 \\ 0 & 0 & .40 & .85 & .70 & 0 \end{pmatrix}.$$

The degree vector is

$$d = (1.90, 2.25, 3.05, 1.95, 1.70, 1.95)^T,$$

so $d_{\max} = 3.05$. Direct multiplication gives

$$Ad \approx (4.6900, 5.0850, 6.0425, 4.6100, 3.9750, 4.0675)^T.$$

For $\eta = 0.5$,

$$SFC_i = \frac{d_i/3.05 + 0.5(Ad)_i/3.05^2}{1.5}.$$

The resulting values are approximately

$$(0.5834, 0.6740, 0.8832, 0.5914, 0.5140, 0.5720),$$

with v_3 ranking first.

The Laplacian eigenvalues and adjacency spectrum can be computed numerically from the matrices above. Since the support graph is connected, $\lambda_2(L) > 0$. The spectral-radius bounds of Theorem 8 provide an immediate verification check.

Figure 1. Illustrative fuzzy network and edge-membership values

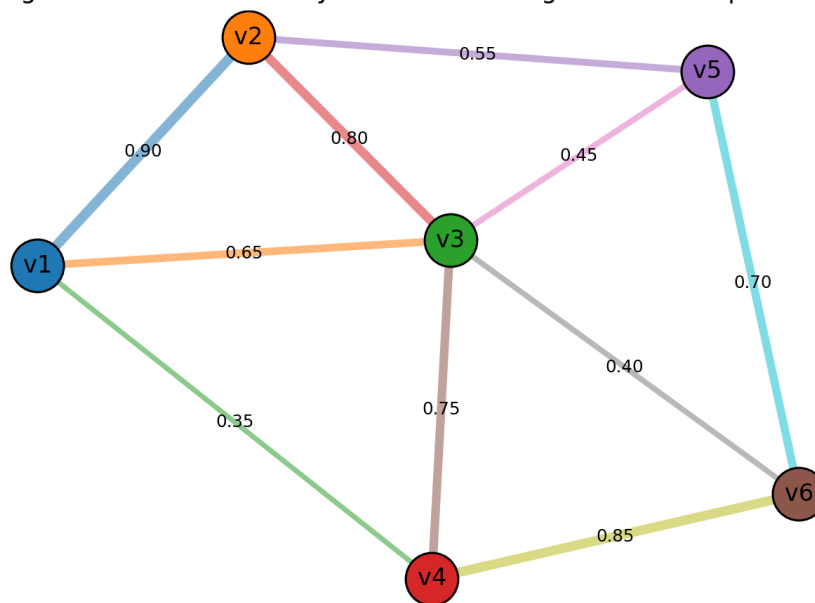


Figure 1: Illustrative fuzzy network.

Figure 2. Fuzzy adjacency matrix of the illustrative network

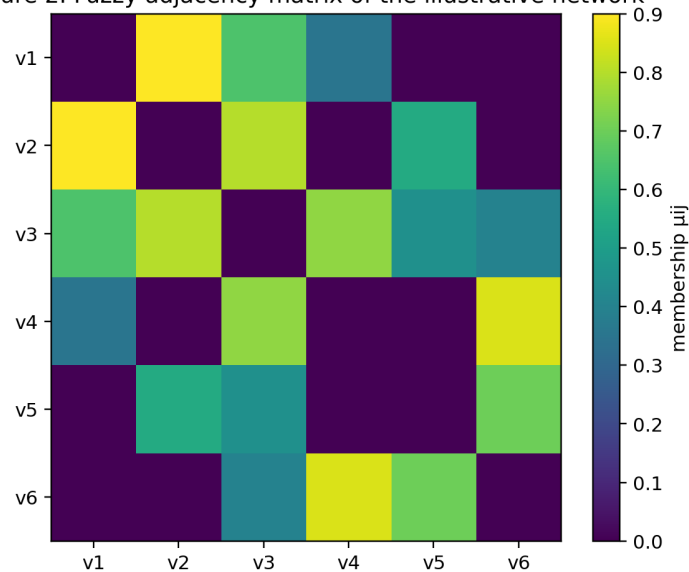


Figure 2: Fuzzy adjacency matrix.

Figure 3. Proposed spectral-fuzzy centrality on the illustrative network

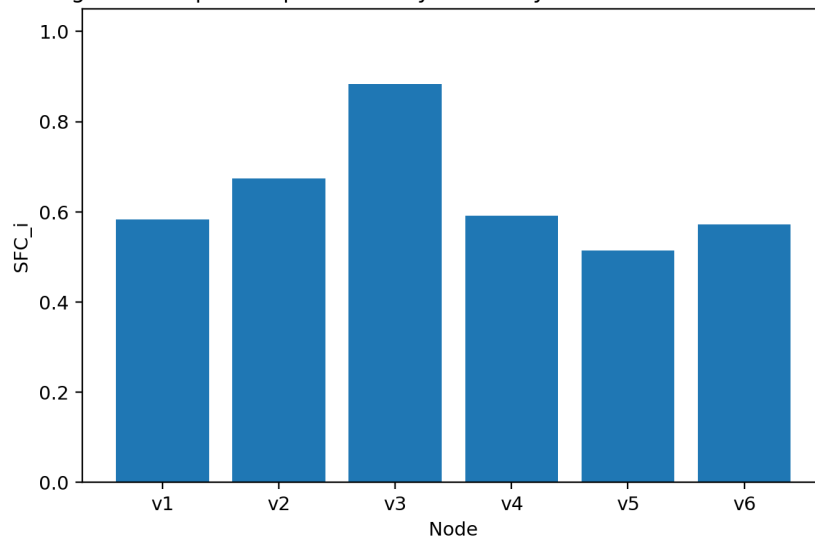


Figure 3: Illustrative fuzzy network.

10 Sensitivity and Ranking

Writing $a_i = C_i^D$ and $b_i = C_i^{(2)}$,

$$SFC_i(\eta) = \frac{a_i + \eta b_i}{1 + \eta}, \quad \frac{\partial SFC_i}{\partial \eta} = \frac{b_i - a_i}{(1 + \eta)^2}.$$

Therefore the index increases with η when two-step influence exceeds direct normalized degree.

For nodes i, k ,

$$SFC_i(\eta) \geq SFC_k(\eta) \iff (a_i - a_k) + \eta(b_i - b_k) \geq 0.$$

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Hence rankings are stable for all $\eta \geq 0$ if the two component differences have the same sign. Otherwise, a possible crossing occurs at

$$\eta^* = -\frac{a_i - a_k}{b_i - b_k},$$

provided $\eta^* \geq 0$.

11 Applications and Extensions

The framework can be applied conceptually to social, communication, biological and renewable-energy networks. In an uncertain renewable-energy network, vertices may represent generators, storage systems or substations, while μ_{ij} represents normalized confidence in effective interaction or transfer reliability. Then SFC ranks locally and two-step influential nodes, whereas $\lambda_2(L)$ summarizes global weighted cohesion.

Future work should consider directed fuzzy graphs, multilayer networks, interval-valued and intuitionistic fuzzy graphs, higher-order walk centrality, and empirical calibration of membership functions. A useful extension is

$$H_m(A) = \sum_{k=1}^m \theta_k A^k \mathbf{1}, \quad \theta_k \geq 0,$$

followed by normalization. This provides a systematic higher-order fuzzy influence family.

12 Limitations

The model is finite and undirected and uses scalar memberships. Membership construction is application-dependent. The proposed index is not intended to replace all established centralities. The numerical example is synthetic and does not support empirical claims. The literature already contains substantial work on fuzzy centrality, connectivity and spectra, so originality in a journal submission must be stated narrowly in terms of the exact index, theorems, algorithms and experiments.

13 Conclusion

This paper has developed a unified mathematical framework for the structural analysis of complex networks using fuzzy graph theory, with particular emphasis on centrality, connectivity, and spectral properties. The framework is designed to represent networks in which relationships are not completely certain or binary but possess different degrees of strength, reliability, or membership. By representing these relationships through a fuzzy adjacency matrix, the proposed approach provides a systematic connection between fuzzy graph theory and classical spectral graph theory.

A major contribution of the study is the formulation of the Spectral-Fuzzy Centrality (SFC)

measure:

$$SFC_i(\eta) = \frac{1}{1 + \eta} \left[\frac{d_i}{d_{\max}} + \eta \frac{(Ad)_i}{d_{\max}^2} \right]$$

where d_i represents the fuzzy degree of vertex v_i , A is the fuzzy adjacency matrix, and $\eta \geq 0$ controls the relative importance of two-step neighborhood influence. Unlike simple fuzzy degree centrality, the proposed measure incorporates not only the direct strength of connections but also the structural importance of neighboring vertices.

Several mathematical properties of the proposed measure were established. In particular, it was proved that

$$0 \leq SFC_i(\eta) \leq 1,$$

that the measure is invariant under vertex relabeling, and that it is monotonic under appropriate strengthening of incident edge memberships. It was also shown that for a regular fuzzy network, every vertex obtains the same maximum normalized SFC value. These results demonstrate that the proposed centrality satisfies important structural consistency requirements.

The spectral component of the framework provides additional global information about the network. The fuzzy Laplacian

$$L = D - A$$

was used to investigate algebraic connectivity, while the largest eigenvalue of the fuzzy adjacency matrix was used to characterize spectral influence. The positive-semidefinite property

$$\mathbf{x}^T L \mathbf{x} = \frac{1}{2} \sum_{i,j} a_{ij} (x_i - x_j)^2 \geq 0$$

establishes the fundamental connection between fuzzy edge strengths and the global structural cohesion of the network. The relationship between the spectral radius and the minimum and maximum fuzzy degrees was also established through

$$d_{\min} \leq \rho(A) \leq d_{\max}.$$

Furthermore, the α -cut formulation provides an important bridge between fuzzy and classical graph analysis. For every $\alpha \in (0, 1]$, the fuzzy network produces a corresponding crisp network G_α . Consequently, network structure can be examined over a complete range of uncertainty levels rather than being restricted to a single arbitrarily selected threshold.

The numerical example demonstrated the practical calculation of the proposed SFC, adjacency spectrum, Laplacian spectrum, and α -cut structures for a six-vertex fuzzy network. Although the example is illustrative rather than empirical, it confirms that all proposed quantities can be calculated systematically from a single fuzzy adjacency matrix.

From the perspective of complex-network analysis, the proposed framework provides three complementary levels of information: local structural importance through fuzzy degree, neighborhood influence through SFC, and global network cohesion through spectral parameters. This combination can be useful for uncertain social networks, communication networks, biological interaction networks, transportation systems, and renewable-energy networks.

The study also opens several directions for future research. The proposed SFC can be generalized to higher-order walks:

$$H_m(A) = \sum_{k=1}^m \theta_k A^k \mathbf{1}, \quad \theta_k \geq 0,$$

allowing influence beyond two-step neighborhoods. Further extensions may incorporate directed fuzzy graphs, intuitionistic fuzzy graphs, interval-valued fuzzy graphs, bipolar fuzzy graphs, q -rung fuzzy structures, multilayer networks, and dynamic fuzzy networks. Another important direction is the development of data-driven procedures for determining fuzzy membership values from real network observations.

In conclusion, the mathematical framework presented in this paper demonstrates that fuzzy graph theory can provide a rigorous and flexible foundation for analysing complex networks under uncertainty. The integration of centrality, connectivity, α -cuts, adjacency spectra, and Laplacian spectra provides a unified perspective that can support both theoretical investigations and future application-oriented studies. The proposed Spectral-Fuzzy Centrality offers a mathematically tractable starting point for identifying structurally influential nodes while preserving the uncertainty inherent in complex network relationships. Further theoretical refinement and validation on real-world datasets will be necessary to establish its comparative advantages over existing fuzzy and classical centrality measures.

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