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Quantile Based Analysis of Marshall-Olkin Family

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Abstract: ***In this article, we suggest a way to introduce a new parameter to an existing quantile function to produce a new quantile function. We got an identity connecting the two quantile functions which will be useful for studying the reliability properties of the new quantile function. The hazard functions of some distributions and their extended versions have been discussed. Furthermore, we have applied this to Tukey Lambda distribution and an extended version is obtained which is useful for modeling not only symmetric distributions but also skewed distributions. Some of the reliability properties of the extended Tukey Lambda distribution has also been studied. In addition, a simulation study is conducted to estimate the parameters of the extended Tukey lambda distribution. We have also shown that the extended Tukey Lambda distribution can be used to model Covid-19 data.***

Keywords: Marshall Olkin family, Quantile functions, Tukey lambda distribution

1. Introduction

In most of the practical situations, one parameter families of distributions are not sufficient to model the real data. So existing literature is rich in developing new models and methods for adding parameters to the existing models. (Marshall & Olkin, 1997) introduced a general method for obtaining more flexible distributions by adding a new parameter to an existing family of distributions. By using the Marshall and Olkin method for adding a new parameter to an existing distribution, several authors contributed a large number of new distributions to the literature. Some of the recent works using Marshall-Olkin approach are (Mansoor et al., 2018), (Afify et al., 2015), (Ahsan et al., 2021), (Yousof & Afify, 2018), (Afify et al., 2021), (Afify et al., 2020), (Korkmaz et al., 2019), (Mansoor et al., 2019),(Benkhelifa, 2017), (Cakmakyapan et al., 2018),(Castellares & Lemonte, 2016),(Kundu, 2015),(Mathew & Chesneau, 2020),(Saboor & Pogany, 2016),(Güney et al., 2017),(Cui et al., 2020),(Handique & Chakraborty, 2017), (Al-noor & Hadi, 2021), (Chipepa et al., 2020),(Makubate et al., 2021), (Nwezza & Ugwuowo, 2020) etc.

An alternative way of describing a continuous distribution is to use the quantile functions. Let X be a real valued continuous random variable with distribution function $F(x)$ which is continuous from the right. Then, the quantile function $Q(u)$ of X is defined as

$$Q(u) = F^{-1}(u) = \inf(x : F(x) \geq u) \ , \quad 0 \leq u \leq 1 \quad (1)$$

For every $-\infty < x < \infty$ and $0 < u < 1$, we have $F(x) \geq u$ if and only if $Q(u) \leq x$.

A detailed study of quantile functions is available in (*Statistical Modelling with Quantile Functions - Warren Gilchrist - Google Books*, n.d.). Quantile functions possess many advantages which are not shared with distribution functions. For various properties and applications of quantile functions one can refer to (Sankaran et al., 2016), (Nair & Paduthol, 2009), (Kumar Maladan & Sankaran, 2020), (Sankaran et al., 2016), (Nair et al., 2012), (Nair et al., 2013). Even if several authors have discussed methods to obtain new quantile functions, Marhall Olkin approach in the context of quantile function has yet to be discussed. The method suggested by Marshall and Olkin can be used to extend existing quantile functions. In this article we discuss the way to add a tilt parameter to an existing quantile function.

The present study is organized into five sections. After this introduction, Section 2 gives the identity connecting two quantile functions and the corresponding hazard quantile functions. In Section 3, monotonicity of hazard functions of some distributions and their extended versions have been discussed. We have applied this to Tukey lambda distribution and introduced a new distribution which is useful for modeling symmetric as well as skewed distributions. It is discussed in Section 4. In the Section 5 a simulation study is conducted to demonstrate the estimation of parameters of extended Tukey lambda distribution. Finally the new extended Tukey lambda distribution has been used to model Covid 19 data in the last section.

2. Extension of Quantile Function and Hazard Quantile Function

Let X be a continuous non-negative random variable with distribution function $F_X(x)$ and quantile function $Q_X(u)$. (Marshall & Olkin, 1997) devised a new method of introducing a more flexible family of distributions by adding a new parameter to $F(x)$ through defining the survival function

$$\overline{G}(x) = \frac{\theta \overline{F}_X(x)}{1-(1-\theta)\overline{F}_X(x)} \ , \quad \theta > 0 \quad (1)$$

They called θ a tilt parameter and studied the cases when X has exponential and Weibull distributions. Let Y be the random variable with survival function $\overline{G}(x)$. We can rewrite (1) as

$$G_Y(x) = \frac{1-\overline{F}_X(x)}{1-(1-\theta)\overline{F}_X(x)} \quad (2)$$

Setting $F(x) = u$ in (2), we have when F_X is strictly increasing, the quantile function of Y as,

$$G_Y(Q_X(u)) = \frac{u}{1-(1-\theta)(1-u)}$$

$$= \frac{u}{\theta + u(1-\theta)} \quad (3)$$

Or

$$Q_X(u) = Q_Y\left(\frac{u}{\theta + u(1-\theta)}\right) \quad (4)$$

Similarly by putting $G_Y(x) = u$ we get

$$Q_Y(u) = Q_X\left(\frac{u\theta}{1-(1-\theta)u}\right) \quad (5)$$

So we get a new quantile function from the existing quantile function by introducing the tilt parameter θ.

The hazard quantile function is defined as

$$H(u) = h(Q(u)) = \left[(1-u)q(u)\right]^{-1} \quad (6)$$

where $q(u)$ is the quantile density function which is the derivative of the quantile function $Q(u)$. In this definition, $H(u)$ is interpreted as the conditional probability of the failure of a unit in the next small interval of time given the survival of the unit at $100(1-u)\%$ point of the distribution. (Nair & Paduthol, 2009)

Now differentiating (5) with respect to u

$$q_Y(u) = \frac{\theta}{(1-(1-\theta)u)^2} q_X\left(\frac{u\theta}{1-(1-\theta)u}\right)$$

$$(1-u)q_Y(u) = \left(\frac{\theta}{1-(1-\theta)u}\right)\left(1-\frac{u\theta}{1-(1-\theta)u}\right)q_X\left(\frac{u\theta}{1-(1-\theta)u}\right)$$

Therefore

$$H_Y(u) = \left(\frac{1-(1-\theta)u}{\theta}\right)H_X\left(\frac{u\theta}{1-(1-\theta)u}\right) \quad (7)$$

The identity connecting the hazard quantile functions of X and Y in (7) can be used to obtain the reliability properties of Y in terms of X.

3. Monotonicity of hazard quantile functions

Monotonicity of hazard quantile functions can provide the identification of the model in a given data situation. The change in the monotonicity of hazard quantile functions of the extended versions by the introduction of tilt parameter provides better models in several situations. In this section we discuss the monotonicity of some common distributions and their extended versions.

3.1 Exponential and extended exponential

For exponential distribution, $H_X(u) = \lambda$, a constant. But for extended exponential, using (7), $H_Y(u) = \frac{(1-(1-\theta)u)}{\theta}\lambda$.

$$H_Y'(u) = \frac{\lambda}{\theta}(1-\theta) > 0, \quad if\ \theta < 1$$

$$< 0, \quad if\ \theta > 1$$

Hence hazard function is monotonically increasing when $\theta < 1$ and monotonically decreasing if $\theta > 1$.

3.2 Weibull and Extended Weibull

For Weibull distribution, $H_X(u) = \frac{\lambda}{\sigma}(-\log(1-u))^{1-1/\lambda}$

and we get $H_X'(u) > 0 \ if\ \lambda > 1$, $< 0 \ if\ \lambda < 1$ and hence Weibull is IFR if $\lambda > 1$ and DFR if $\lambda < 1$.

$$H_Y(u) = \frac{(1-(1-\theta)u)}{\theta}\frac{\lambda}{\sigma}\left(-\log\left(\frac{1-u}{1-(1-\theta)u}\right)\right)^{1-1/\lambda}.$$

3.3 Half Logistic and extended half logistic

For Half logistic distribution $H_X(u) = \frac{2}{\sigma}(1+u)$ and $H_X'(u) = \frac{2}{\sigma} > 0$. Therefore Half logistic is IFR. But we get extended half logistic as IFR for $\theta > \frac{1}{2}$ and DFR for $\theta < \frac{1}{2}$ For,

$$H_Y(u) = \frac{2(1-u+2\theta u)}{\theta\sigma} \quad \text{and}$$

$$H_Y'(u) = \frac{2}{\theta\sigma}(2\theta - 1) > 0 \ if\ \theta > \frac{1}{2}$$

$$< 0 \ if\ \theta < \frac{1}{2}$$

The change in the monotonic behaviour of hazard quantile functions of the above mentioned distributions and some more distributions are depicted in figure 1.

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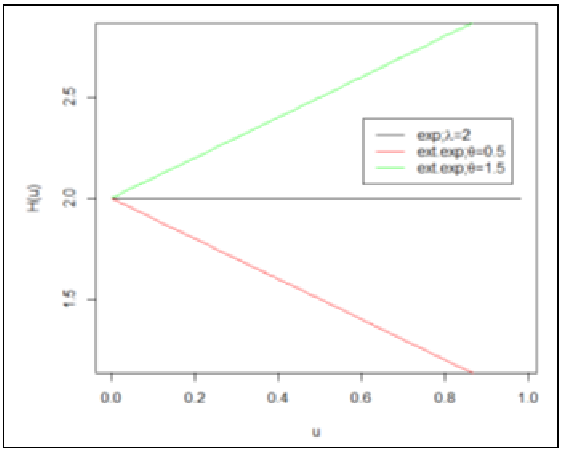
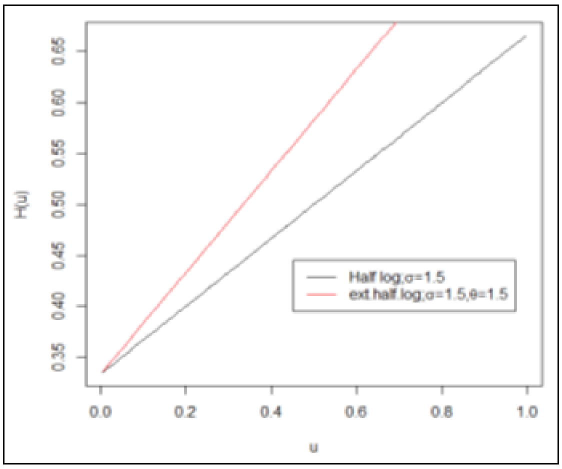
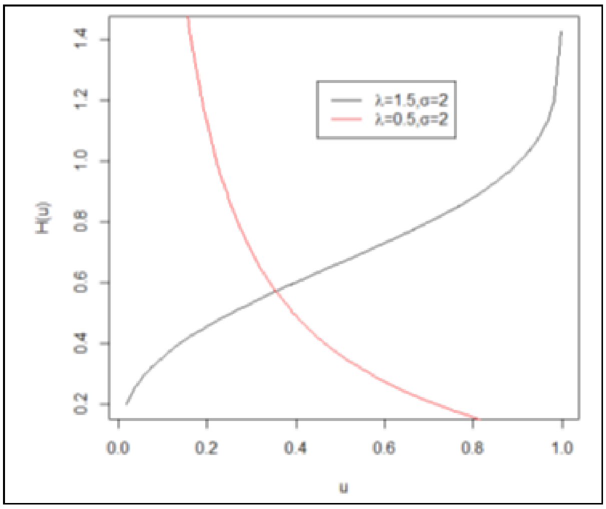


Exponential and Extended Exponential



Half Logistic and Extended Half Logistic



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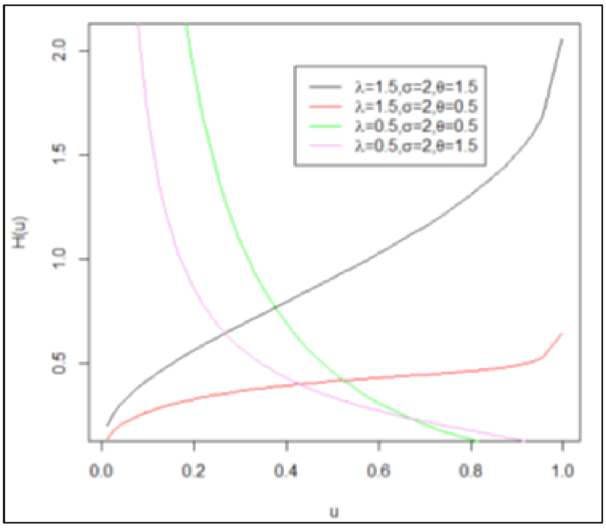

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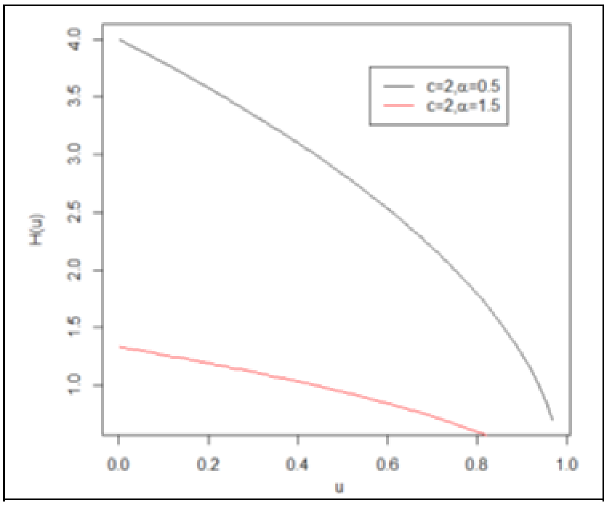
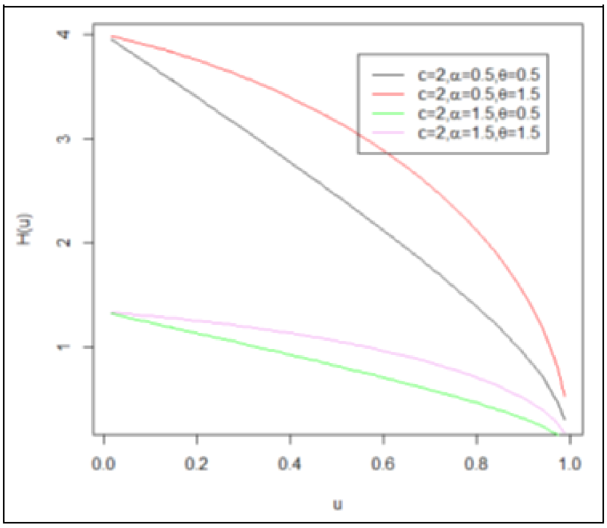


Pareto



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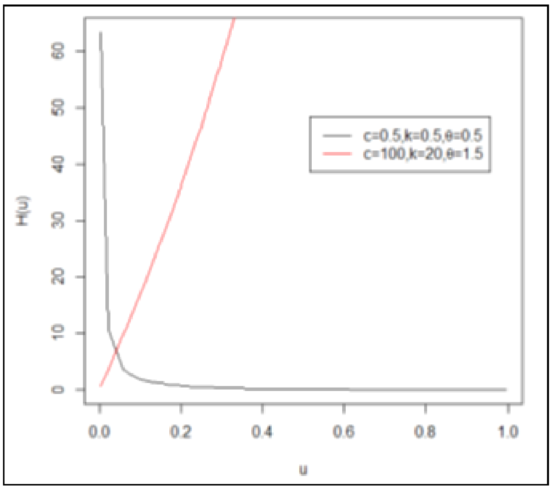
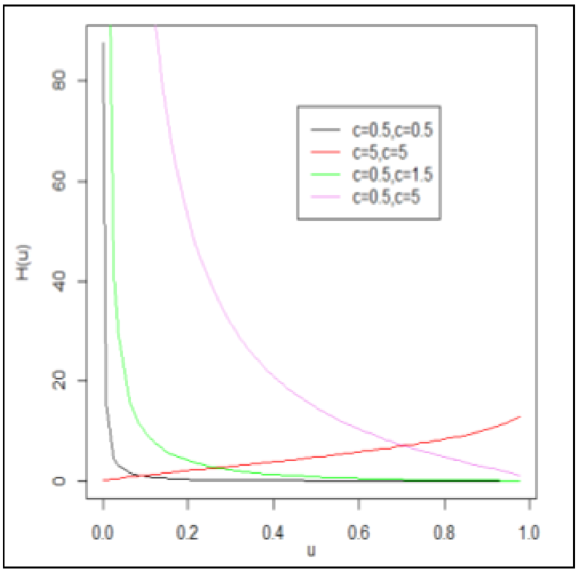


Burr XII



Extended Burr XII

Figure 1: Hazard functions of some distributions and their extended versions

4. Extended Tukey Lambda Distribution

Tukey Lambda distribution was first introduced in the paper (Hastings et al., 1947). Since then, many of the generalisations, properties, estimation procedures and applications of Lambda distribution have been discussed by several authors. Some of the references are (Tarsitano, 2004), (Freimer et al., 1988), (Haritha, 2017) etc.

The quantile function of Tukey Lambda distribution is given by

$$Q_X(u) = \frac{u^{\lambda} - (1-u)^{\lambda}}{\lambda} \tag{8}$$

Using (5) the quantile function of extended Tukey Lambda distribution is obtained as

$$Q_Y(u) = \frac{(u\theta)^{\lambda} - (1-u)^{\lambda}}{\lambda(1-(1-\theta)u)^{\lambda}} \tag{9}$$

When $\theta = 1$ this will reduce to Tukey Lambda distribution. The Tukey Lambda distribution is useful for modeling symmetric distributions only but this extended version also include skewed distributions.

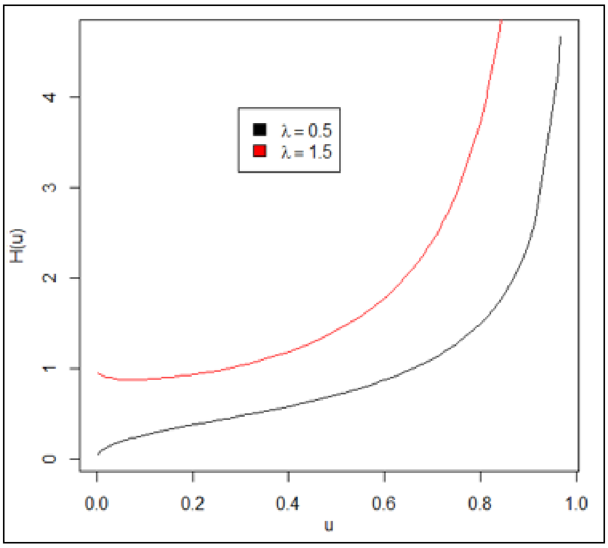
The hazard quantile function of Tukey Lambda distribution is given by

$$H_X(u) = \left\{(1-u)\left(u^{\lambda-1} + (1-u)^{\lambda-1}\right)\right\}^{-1} \tag{10}$$

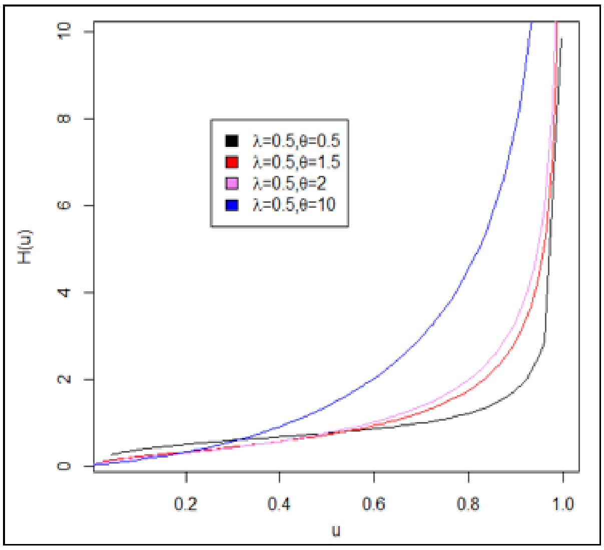
Now using (7) the hazard quantile function of extended Tukey lambda distribution is obtained as

$$H_Y(u) = \frac{(1-(1-\theta)u)^{\lambda+1}}{\theta^{\lambda} u^{\lambda-1}(1-u) + \theta(1-u)^{\lambda}} \tag{11}$$

The monotonicity of hazard functions of Tukey lambda and its extended version is given in figure 2.



Monotonicity of hazard function of Tukey Lambda distribution



Monotonicity of hazard function of Extended Tukey Lambda distribution

Figure 2: Hazard function of Tukey Lambda and its extended version

5. Estimation of parameters of Extended Tukey Lambda Distribution

The methods of estimation like matching 25th and 75th percentiles, matching quantile measures of location and dispersion and the conventional percentile method can be used to estimate parameters of Extended Tukey Lambda distribution. We have compared these three methods using a simulation study.

Thousand samples of sizes $n = 25, 50$ and 100 for two sets of parameter values $\theta = 0.5, \lambda = 1.5$ and $\theta = 0.8, \lambda = 0.05$ are generated using the result that if U has uniform distribution over $(0,1)$ then $Q(u)$ and X have the same distribution. The estimates using matching 25th and 75th percentiles (percentile method 1) are obtained by solving the equations

$$\frac{(0.25\theta)^\lambda - (0.75)^\lambda}{\lambda(1-(1-\theta)0.25)^\lambda} = P_{25} \quad \text{and} \quad \frac{(0.75\theta)^\lambda - 0.25^\lambda}{\lambda(1-(1-\theta)0.75)^\lambda} = P_{75}$$

where P_{25} and P_{75} are the 25th and 75th percentiles of the sample. In matching quantile measures of location and dispersion the estimates are obtained by solving the equations

$$\frac{\theta^\lambda - 1}{\lambda(1+\theta)^\lambda} = Me \quad \text{and}$$

$$\frac{1}{\lambda}\left\{\frac{(0.75\theta)^\lambda - 0.25^\lambda}{(1-(1-\theta)0.75)^\lambda} - \frac{(0.25\theta)^\lambda - 0.75^\lambda}{(1-(1-\theta)0.25)^\lambda}\right\} = IQR$$

where Me and IQR are the sample median and sample interquartile range respectively. In the conventional method of percentiles (percentile method 2) the estimates are obtained by solving the equations $\frac{\theta^\lambda - 1}{\lambda(1+\theta)^\lambda} = Me$ and

$$\frac{1}{\lambda}\left\{\frac{(0.9\theta)^\lambda - 0.1^\lambda}{(1-(1-\theta)0.9)^\lambda} - \frac{(0.1\theta)^\lambda - 0.9^\lambda}{(1-(1-\theta)0.1)^\lambda}\right\} = \hat{Q}(0.9) - \hat{Q}(0.1)$$

where $\hat{Q}(0.9)$ and $\hat{Q}(0.1)$ are 0.9th and 0.1th sample quantiles respectively. The MSEs calculated are summarized in Table 1.

Table 1: MSE of estimates

Method of estimation		Percentile 1	Percentile 2	Quantile measures
MSE of estimates when				
n=25	$\theta = 0.5$	0.0534	0.0517	0.0626
	$\lambda = 1.5$	0.0631	0.0439	0.0852
	$\theta = 0.8$	0.1514	0.1328	0.1522
	$\lambda = 0.05$	0.0874	0.031	0.0853
n=50	$\theta = 0.5$	0.0224	0.0237	0.0213
	$\lambda = 1.5$	0.0292	0.0173	0.0370
	$\theta = 0.8$	0.0544	0.0599	0.063
	$\lambda = 0.05$	0.0441	0.0152	0.0393
n=100	$\theta = 0.5$	0.0116	0.0109	0.0105
	$\lambda = 1.5$	0.012	0.0075	0.0182
	$\theta = 0.8$	0.026	0.0286	0.0289
	$\lambda = 0.05$	0.0188	0.0073	0.0199

We can conclude that all the three methods are suitable for estimating the parameters of extended Tukey Lambda distribution.

6. Application of Extended Tukey Lambda Distribution

To illustrate the application of Extended Tukey Lambda Distribution we have considered a data reported in https://covid19.who.int/. This data represents daily deaths due to COVID 19 in Europe from 1st March to 30th March 2020. 6, 18, 29, 28, 47, 55, 40, 150, 129, 184, 236, 237, 336, 219, 612, 434, 648, 706, 838, 1129, 1421, 118, 116, 1393, 1540, 1941, 2175, 2278, 2824, 2803, 2667.

The sample median and interquartile range of this data is obtained as Median = 336 and IQR= 1290. Equating these values to the median and inter quartile range of Extended Tukey Lambda Distribution we get the following equations.

$$\frac{(0.5\theta)^\lambda - 0.5^\lambda}{\lambda(1-0.5(1-\theta))^\lambda} = 336$$

$$\frac{(0.75\theta)^\lambda - 0.25^\lambda}{\lambda(1-0.75(1-\theta))^\lambda} - \frac{(0.25\theta)^\lambda - 0.75^\lambda}{\lambda\left(1-0.25(1-\theta)\right)^\lambda} = 1290$$

Solving these equations using Newton Raphson method in the R software we have obtained the estimates of the parameters as $\hat{\theta} = 111.027212$ and $\hat{\lambda} = -1.286681$.

To examine the adequacy of the model two goodness of fit techniques such as Kolmogorov Smirnov test and Chi-square test have been applied. The value of KS statistic is 0.16129 and the p value is 0.8235. Instead of conventional chi-square procedure a more easier method suggested by (Voinov, V. et. al., 2013) has been adopted here. The calculated value of chi-square is obtained as 2.854 and $\chi^2_{0.05,30} = 43.77297$. So both the goodness of fit procedures suggest that the Extended Tukey Lambda Distribution is a reasonable model for the given data set.

7. Conclusion

In this paper we have introduced the Marshall Olkin approach to the quantile functions for adding new parameters. We have obtained a new quantile function from an existing quantile function by introducing a tilt parameter θ. The change in the monotonicity of hazard functions of extended versions of several distributions have been discussed. A detailed study of an extended version of Tukey Lambda Distribution has also been carried out in this paper. We have also claimed that this Extended Tukey Lambda Distribution has potential to model COVID 19 data.

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Availability of Data and Materials

To illustrate the application of Extended Tukey Lambda Distribution we have considered a data reported in https://covid19.who.int/. This data represents daily deaths due to COVID 19 in Europe from 1st March to 30th March 2020.

Competing Interest Declaration

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Author Contribution Declaration

Both the authors equally contributed to the manuscript.

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