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The Connection Between Fermat Numbers and Brocard-Ramanujan Type Diophantine Equations

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Abstract: ***In this manuscript, the study focuses on the Brocard-Ramanujan type Diophantine equation $n! + 1 = F_t^2$ where $F_t = 2^k + 1, k = 2^t, n, t \geq 0$ is a Fermat number, is examined and demonstrates $(n, k, t) = (4, 2, 1)$ is a sole positive integer solution. Additionally, a Python program to validate this result together with the geometrical presentation is developed.***

Keywords: Fermat number, Brocard-Ramanujan Diophantine equation, Integer solution

1. Introduction

Henri Brocard explored the Diophantine equation

$$n! + 1 = x^2$$

in the 19th century. Later, Srinivasa Ramanujan independently investigated the same equation in the early 20th century[1,2]. Their work aimed to find integer solutions. Their combined efforts led to significant discoveries into this equation, which has since been known as the Brocard-Ramanujan Diophantine equation resulting in a lasting impact on the field of Number theory.

In [9], the authors proved that the Brocard–Ramanujan Diophantine equation $m! + 1 = u^2$, where u is a sequence of positive integers and has at most finitely many solutions under some conditions. They also solved the equation when u is a Tripell number using the recurrence relation

$$T_n = 2T_{n-1} + T_{n-2} + T_{n-3}$$

for $n \geq 3$ with $T_0 = 0, T_1 = 1$ and $T_2 = 2$.

In [8], Tasci and Dursun defined and explored the Gaussian Mersenne sequence examining its properties and relationships with other sequences, notably the Gaussian Jacobsthal numbers and the Gaussian Jacobsthal-Lucas numbers. For further knowledge of the Brocard-Ramanujan Diophantine equation, see $[3-7,10]$

In this paper, the unique solution to the Brocard-Ramanujan type Diophantine equation $n! + 1 = F_t^2$ where $F_t = 2^k + 1, k = 2^t$ is a Fermat number with the condition $n, t \geq 0$ is analysed.

2. Basic definition and theorem

Definition: Fermat number
A Fermat number is a specific type of number defined by $F_n = 2^{2^n} + 1$ where n is a non-negative integer.

The initial Fermat numbers include: $3, 5, 17, 257, 65537, 4294967297,$ etc.

Definition: 2-adic valuation of a Factorial
Let $n \in N$. The 2-adic valuation of $n!$ denoted by $v_2(n!)$ is the exponent of the highest power of 2 divides $n!$ given by the formula

$$v_2(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor$$

The sum is finite because for sufficiently large i, $\left\lfloor \frac{n}{2^i} \right\rfloor = 0$.

Lemma 1
If $n \in N$, then $\left(\frac{n+1}{3}\right)^n > 2^{2n} + 2^{n+1}$ hold for all $n \geq 12$.

Proof
The inequality to be verified $\left(\frac{n+1}{3}\right)^n > 2^{2n} + 2^{n+1}$ (1)

Let us prove the inequality by (1) mathematical induction on $n \geq 12$.

First, let us check whether the lemma is true when $n = 12$.
Note that LHS of (1) is $\left(\frac{13}{3}\right)^{12} \cong 2.58 \times 10^7$ (2)

and the RHS of (1) is $2^{24} + 2^{13} \cong 1.68 \times 10^7$ (3)

From (2) and (3), it is clear that the inequality stated in equation (1) is true.

Assume that when $n = k > 12$ the inequality is true.
i.e., $\left(\frac{k+1}{3}\right)^k > 2^{2k} + 2^{k+1}$ (4)

Finally, the proof is completed by establishing the inequality (1) is for $n = k + 1$.

i.e., To prove that the inequality $\left(\frac{k+2}{3}\right)^n > 2^{2k+2} + 2^{k+2}$ is true.

Consider

$$\left(\frac{k+2}{3}\right)^{k+1} = \left(\frac{k+2}{k+1} \times \frac{k+1}{3}\right)^{k+1}$$
$$= \left(\frac{k+2}{k+1}\right)^{k+1} \times \left(\frac{k+1}{3}\right)^{k+1}$$
$$= \left(\frac{k+2}{k+1}\right)^{k} \times \left(\frac{k+2}{k+1}\right) \times \left(\frac{k+1}{3}\right)^{k} \times \left(\frac{k+1}{3}\right)$$

Since by (4), the above equation can be written as

$$\left(\frac{k+2}{3}\right)^{n+1} > (2^{2k} + 2^{k+1})\left(\frac{k+2}{3}\right)\left(\frac{k+2}{k+1}\right)^{k}$$
$$> (2^{2k} + 2^{k+1})\left(\frac{k+2}{3}\right)$$
$$> 2^{2(k+1)} + 2^{(k+1)+1}$$

Hence, the lemma is proved when $n \geq 12$ by mathematical induction.

3. Main Result

Theorem 1

The Diophantine equation $n! + 1 = F_t^2$ where,

$$F_t = 2^k + 1, k = 2^t$$

is a Fermat number where $n, t \geq 0$ has only a positive integer solution $(n, k, t) = (4,2,1)$.

Proof

Consider the given equation

$$n! + 1 = F_t^2 \text{ where } F_t = 2^k + 1\,, k = 2^t, n, t \geq 0. \quad (5)$$
$$\Rightarrow n! + 1 = 2^{2k} + 2^{k+1} + 1$$
$$\Rightarrow n! = 2^{2k} + 2^{k+1} \quad (6)$$

Our goal is to prove that the equation (6) has a unique solution only when $(n, k, t) = (4,2,1)$.

The proof involves three distinct cases specifically,

Case (i): $n = k$

Case(ii): $n > k$

Case(iii): $n < k$

Case (i): Suppose $n = k$

The equation (6) turns out to be

$$n! = 2^{2n} + 2^{n+1} \quad (7)$$

Here, based on the value of n, case (i) can be further divided into two subcases.

Subcase (i): Assume $n = k$ with $n \geq 12$

From Lemma 1, it is clear that for all $n \geq 12$,

$$\left(\frac{n+1}{3}\right)^n > 2^{2n} + 2^{n+1} \quad (8)$$

But the fact that the factorial growth dominates the growth of $\left(\frac{n+1}{3}\right)^n$

Thus, it follows that $n! > \left(\frac{n+1}{3}\right)^n$ (9)

By employing the result (7) in (9), it is evident that

$$2^{2n} + 2^{n+1} > \left(\frac{n+1}{3}\right)^n \quad (10)$$

Thus, equation (10) stands in contradiction to the equation (8).

Therefore, it follows that no solution exists for $n = k$ when $n \geq 12$.

Subcase (ii): $n = k$ with $n < 12$.

Considering this specific subcase, values of $n \in \{1,2,4,8\}$. Verifying basic numerical computations confirms that there is no solution.

Case (ii): Select $n > k$

This case can also be categorized into two subcases based on the value of n.

Subcase (i): Taking $n > k$ where $n \geq 12$.

Since, by the condition given in this subcase (i), it is clear that

$$2^{2n} + 2^{n+1} > 2^{2k} + 2^{k+1} \quad (11)$$

Comparing the inequality given in Lemma 1 and equation (9), it is obvious that

$$n! > \left(\frac{n+1}{3}\right)^n > 2^{2n} + 2^{n+1} \quad (12)$$

Using (11) in (12), it shows that

$$n! > 2^{2k} + 2^{k+1} \quad (13)$$

The disparity between (6) and (13) gives rise to a contradiction.

Hence, it follows that subcase (i) of case (ii) also lacks a solution.

Subcase (ii): Assume $n > k$ with $n < 12$

In this subcase, the values of n and k are bounded by $n \in \{2,3, \ldots 11\}$ and $k \in \{2^0, 2^1, 2^2, 2^3\}$ and a possible combination of n and k are listed below.

n	k
2	1
3,4	1,2
5,6,7,8	1,2,4
9,10,11	1,2,4,8

By applying elementary numerical calculations, the values taken for n and k from the table above indicates a unique solution to the equation (6) when $(n, k) = (4,2)$.

Case (iii): Suppose $n < k$

RHS of the equation (6) can be written as

$2^{2k} + 2^{k+1} = 2^{k+1}(2^{k-1} + 1)$ which is divisible by 2^{k+1}. (14)

It is clear that the 2-adic valuation

$$v_2(n!) < n. \quad (15)$$

Comparing equation (15) and the given condition $n < k$ yields

$$v_2(n!) < n < k < k + 1$$

$$\Rightarrow \; v_2(n!) < k+1$$
$$\Rightarrow \; 2^{k+1} \nmid n! \qquad (16)$$

Equations (14) and (16) collectively imply that the equation (6) is impossible for $n < k$, leaving $(n,k,t) = (4,2,1)$ as the unique solution.

This completes the proof.

The following Python program 1 is used to verify the solution of the Diophantine equation numerically.

Python program 1:

```
import math
def check_user_input_solution():
  print("Checking the equation: n! + 1 = (2^k + 1)^2 where k = 2^t\n")
  try:
    n = int(input("Enter a positive integer for n: "))
    t = int(input("Enter a non-negative integer for t: "))
    if n <= 0 or t < 0:
      print("nvalid input: n must be > 0 and t ≥ 0.")
      return
    k = 2 ** t
    lhs = math.factorial(n) + 1
    rhs = (2 ** k + 1) ** 2
    print(f"\nn = {n}, t = {t}, k = 2^{t} = {k} → LHS = {lhs}, RHS = {rhs}")
    if lhs == rhs:
      print(f"nique solution found: (n, k, t) = ({n}, {k}, {t})")
      print("Conclusion: The only positive integer solution is (n, k, t) = (4, 2, 1).")
    else:
      print(f"LHS and RHS are not equal for the value of (n, k, t) = ({n}, {k}, {t})")
  except ValueError:
    print("Please enter valid integer values.")
  except OverflowError:
    print("Number too large. Try smaller inputs.")
check_user_input_solution()
```

The following graph visually illustrates the behaviour of the Diophantine equation $n! + 1$ and $\left(2^k + 1\right)^2$ for the value of n and k.

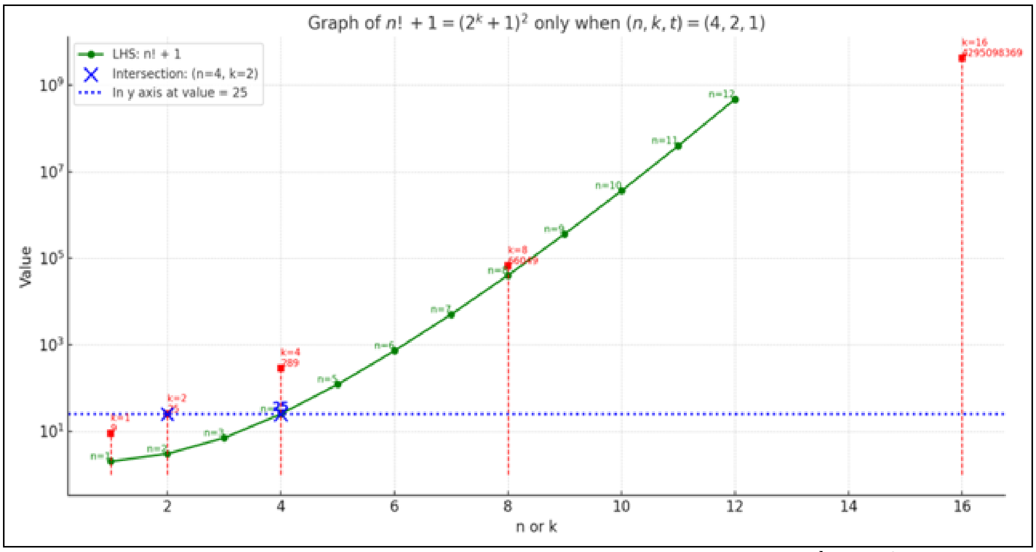

Figure 1: Behaviour of the Diophantine equation $n! + 1$ and $(2^k + 1)^2$

Remark:
To extend the above theorem by considering the Diophantine equation $n! + 1 = \left(a^{2^b} + 1\right)^2$ and investigate its solutions under the constraint that the right-hand side is the square of a generalized Fermat number, defined $F_{a,b} = a^{2^b} + 1$ for integer $a > 0$, $b \geq 0$ has the only positive integer solution $(n, a, b) = (4,2,1)$.

4. Conclusion

In this manuscript, a novel approach to determining the positive integer solutions for the Diophantine equation $n! + 1 = F_k^2$ where $F_k = 2^k + 1$ where $n \geq 0, k = 2^t, t \geq 0$ represents a Fermat number is exhibited. Future studies could build upon this work by exploring similar Brocard-

Ramanujan type equations involving different number sequences with distinct properties.

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