Contribution of Laplace Transform & Elzaki Transform in Solving Ordinary Differential Equations with Variable Coefficients

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Abstract: ***There are so many methods for solving ordinary and partial differential equations in literature. As integral transform also plays an important role in solving differential equations with initial conditions. In this paper we present application of Laplace transform & Elzaki transform to solve Ordinary differential equations with variable coefficients and initial condition.***

Keywords: Laplace transform, Elzaki transform, Ordinary differential equation with variable coefficients

1. Introduction

Differential equation is such a field which useful in not only Mathematics but in other Mathematical sciences. There are many traditional methods for solving ordinary and partial differential equations with initial conditions in literature. [6]. Laplace transform is the integral transform which is applicable in solving ordinary and partial differential equations [1]. This transform is also applicable in engineering sciences [1]. There is one more integral transform namely Elzaki transform proposed by Tarig .M. Elzaki to solve differential equations in 2010 [3].Most of the derivations in Mathematical sciences are in the form of ordinary or partial differential equations which we can solve by applying these integral transforms. In this chapter we have applied both transforms Laplace and Elzaki for obtaining solution of ordinary linear differential equations of second order with constant and variable coefficients. Finally we have compared Laplace transform method & Elzaki transform method for solving Ordinary differential equations with variable coefficients.

2. Some definitions and theorems of Laplace transform and Elzaki transform.

Definition1: Laplace transform: The Laplace transform of f (t) is defined by L [f (t)] =F (p) $=\int_0^{\infty} f(t)e^{-pt}dt$

Theorem1:[1]. Let F(t) and its derivatives F'(t),F"t), …………….,$F^{n-1}(t)$ be continuous functions for t≥0 and be of exponential orders as t→ ∞ and $F^n(t)$ is of class A then the Laplace transform of $F^n(t)$ exists when p>α and is given by
$L\{F^n(t)\} = P^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \cdots\cdots\cdots F^{n-1}(0)$

Theorem 2: If $L[f(x)] = F(p)$ then $L[x^n f(x)] = (-1)^n \frac{d^n}{dp^n}[F(p)]$

Definition 2: [3]. Elzaki transform: The Elzaki transform of f (x) denoted by E[f (x)] and is defined by
$E[f(x)] = F(w) = w\int_0^{\infty} f(x)e^{\frac{-x}{w}}dx, x \ge 0,$

Theorem3: [3]. If F(w) is the Elzaki transform of $f(x)$ then

(a) $E\{xf'(x)\} = \left\{w^2 \frac{d}{dw}\left[\frac{F(w)}{w} - w f(0)\right] - w\left[\frac{F(w)}{w} - w f(0)\right]\right\}$

(b) $E\{xf''(x)\} = w^2 \frac{d}{dw}\left[\frac{F(w)}{w^2} - f(0) - w f'(0)\right] - w\left[\frac{F(w)}{w^2} - f(0) - w f'(0)\right]$

(c) $E\{x^2 f'(x)\} = w^4 \frac{d^2}{dw^2}\left\{\frac{F(w)}{w} - w f(0)\right\}$

(d) $E\{x^2 f''(x)\} = w^4 \frac{d^2}{dw^2}\left\{\frac{F(w)}{w^2} - f(0) - w f'(0)\right\}$

Theorem4: [4]. If F(w) is the Elzaki transform of $f(x)$ then

(a) $E\{f'(x)\} = \frac{F(w)}{w} - w f(0)$

(b) $E\{f''(x)\} = \frac{F(w)}{w^2} - f(0) - w f'(0)$

Theorem5: [3]. If F(w) is the Elzaki transform of $f(x)$ then

(a) $E\{xf(x)\} = w^2 \frac{d}{dw}[F(w)] - wF(w)$

(b) $E\{x^2 f(x)\} = w^4 \frac{d^2}{dw^2}[F(w)]$

Fundamental properties of Elzaki Transform are presented [3]. Elzaki transforms can be effectively used in solving ordinary differential equations and control engineering problems [5]. Elzaki transform is closely connected with Laplace transform [4].Elzaki transform is the dual of Laplace transform [4]. Elzaki transforms can be effectively used in solving ordinary differential equations and control engineering problems. Elzaki transform can be effectively applied in solving ordinary differential equations and it is closely connected with Laplace transform [4].

3) Solution of Ordinary differential equation with variable coefficients by applying Laplace transform

Consider second order ordinary differential equation with non-constant variables given by
$y''(t) + mt\, y'(t) - 2m\, y(t) = 1$With $y(0) = y'(0) = 0$ (1)

Applying L.T.to equation (1) we have
$L\{y''(t)\} + m\,L\{ty'(t)\} - 2m\,L\{y(t)\} = L\{1\}$ i.e we have

Applying theorem (2) we obtain

$$p^2F(p) - py(0) - y'(0) - m(-1)^1 \frac{d}{dp}[pF(p) - y(0)] - 2m\,F(p) = \frac{1}{p}$$

Using initial conditions we obtain
$p^2F(p) - m\frac{d}{dp}\big(pF(p)\big) - 2m\,F(p) = \frac{1}{p}$ i.e. we have

$p^2F(p) - mpF'(p) - mF(p) - 2m\,F(p) = \dfrac{1}{p}$

$\{p^2 - 3m\}F(p) - mpF'(p) = \frac{1}{p}$ i.e.

$F'(p) - \frac{\{p^2-3m\}}{mp}F(p) = \frac{-1}{mp^2}$ .This is linear O.D.E.

Integrating factor $=e^{-\int\left(\frac{p^2-3m}{mp}\right)dp} = p^3\ e^{-\frac{p^2}{2m}}$ therefore we have

$F(p)\,e^{-\frac{p^2}{2n}}p^3 = \int \frac{-1}{mp^2}\,e^{-\frac{p^2}{2m}}p^3dp + c$

After simplification we have

$F(p) = \frac{1}{p^3} + \frac{c\,e^{\frac{p^2}{2m}}}{p^3}$

Applying I.L.T. & given initial conditions we obtain
$y(t) = \frac{t^2}{2}$ and this is solution of equation (1).

4) Solution of Ordinary differential equation with variable coefficients by applying Elzaki transforms

Consider second order Ordinary differential equation with non-constant variables given by
$z''(t) + mt\,z'(t) - 2m\,z(t) = 1$ With $z(0) = z'(0) = 0$ (2)

By applying E.T. to equation (2) and using theorem (3) & (4) we have

$E\{z''(t)\} + m\,E\{t\,z'(t)\} - 2m\,E\{z(t)\} = E\{1\}$ i.e. we have
$\frac{F(w)}{w^2} - z(0) - w\,z'(0) \quad +m \quad \left\{w^2\frac{d}{dw}\left[\frac{F(w)}{w} - w\,z(0)\right] - w\left[\frac{F(w)}{w} - w\,z(0)\right]\right\} - 2mF(w) = w^2$

Applying given initial conditions we obtain
$\frac{F(w)}{w^2} + m\left\{w^2\frac{d}{dw}\left[\frac{F(w)}{w}\right] - w\left[\frac{F(w)}{w}\right]\right\} - 2mF(w) = w^2$ i.e.
$\frac{F(w)}{w^2} + mw^2\left[\frac{wF'(w) - F(w)}{w^2}\right] - m\,F(w) - 2mF(w) = w^2$

Simplifying the above equation we obtain

$$F'(w) + \left[\frac{1 - 4mw^2}{mw^3}\right]F(w) = \frac{w}{m}$$

This is the linear differential equation having integrating factor $\frac{1}{w^4}\ e^{\frac{-1}{2mw^2}}$

Solving the above differential equation we obtain
$F(w) = w^4$

Taking I.E.T. we have
$z(t) = \frac{t^2}{2}$. This is a solution of equation (2).

3. Concluding Remarks

We have solved ordinary differential equation with variable coefficients by applying Laplace transform and Elzaki transform. Without finding initial values and arbitrary constants. Hence integral transform method may be better to use in solving such equations as compared to traditional method.

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