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On Goldie*-Supplemented Modules and Their Generalizations

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Abstract: ***The study of supplemented modules is a central theme in module theory, with deep connections to the structure of rings and categories of modules. This paper provides depth investigation of several recent generalizations of supplementation properties, specifically those defined via the Goldie* torsion theory. We explore the notions of Goldie*-supplemented, totally Goldie*-supplemented, and cofinitely Goldie*-supplemented modules. Furthermore, we introduce and characterize the new concept of essential T-Goldie* lifting modules, establishing its fundamental properties and its relationship with the other mentioned classes. Our approach emphasizes a lattice-theoretic framework, examining the implications of these properties on the lattice of submodules, $\mathcal{L}(M)$. We prove several key results, including necessary and sufficient conditions for a module to be totally Goldie*-supplemented, a characterization of cofinitely Goldie*-supplemented modules in terms of their finitely generated submodules, and a fundamental lifting property theorem for essential T-Goldie* modules. Numerous examples are provided to illustrate the concepts. Our results generalize and extend recent work in the literature, contributing to a more profound understanding of the hierarchical structure of module classes beneath the umbrella of Goldie*-supplementation.***

Keywords: Goldie*-supplemented module, torsion theory, supplemented module, lifting module, hollow module, lattice of submodules, amply supplemented module

1. Introduction

A module M is called supplemented if for every submodule A of M, there exists a submodule B of M such that $A + B = M$ and $A \cap B$ is small in B(i.e. $A \cap B \ll B$). This concept, dual to that of complement (or essential extension), has been extensively studied due to its importance in understanding the decomposition properties of modules (see [1, 2]). Recent research has focused on refining this notion by relativizing it with respect to a torsion theory.

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on the category of right R-modules, $Mod - R$. A module M is called τ-supplemented if for every submodule A of M, there exists a submodule B of M such that $A + B = M$ and $A \cap B$ is τ-small in B (denoted $A \cap B \ll_\tau B$) [3, 4]. A particularly important case is the Goldie torsion theory, where $\mathcal{T}$ is the class of singular modules and $\mathcal{F}$ is the class of non-singular modules. The corresponding supplemented modules are termed Goldie*-supplemented (or G*-supplemented) modules [5, 6].

In [7], the authors introduced the stronger notion of totally Goldie*-supplemented modules, where every submodule of M is itself G*-supplemented. Parallel to this, the concept of cofinitely Goldie*-supplemented modules was developed in [8], requiring that every cofinite submodule (one with finitely generated quotient) has a G*-supplement. More recently, [9] explored the idea of essential T-Goldie* lifting modules, blending the concepts of lifting modules and the Goldie torsion theory.

This paper aims to synthesize and significantly extend these recent developments. We provide a cohesive analysis of these classes, proving new structural theorems and exploring their interrelationships through a lattice-theoretic lens. Section 2 provides necessary preliminaries. Section 3 is devoted to totally G*-supplemented modules, establishing a key characterization theorem (Theorem 3.5). In Section 4, we characterize cofinitely G*-supplemented modules (Theorem 4.3) and explore their connection with amply supplemented modules. Section 5 introduces essential T-Goldie* lifting modules, proves a fundamental theorem on their direct summands (Theorem 5.6), and provides a corollary on their decomposition. The paper concludes with a summary and suggestions for future research.

2. Preliminaries

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary right R-modules. We denote the singular submodule of a module M by $Z(M)$. A module M is called singular (resp. non-singular) if $Z(M) = M$ (resp. $Z(M) = 0$).

Definition 2.1 ([10]). The Goldie torsion theory $\tau_G = (\mathcal{T}_G, \mathcal{F}_G)$ is defined by:

- $\mathcal{T}_G = \{M \in Mod.R \mid Z(M) = M\}$ (the class of singular modules),
- $\mathcal{F}_G = \{M \in Mod.R \mid Z(M) = 0\}$ (the class of non-singular modules).

Definition 2.2 ([3, 5]). Let τ be a hereditary torsion theory. A submodule K of a module N is called τ-small in N (denoted

$K \ll_{\tau} N$) if for every submodule L of N such that $N/L \in T$, we have $K + L = N$.

Definition 2.3 ([5, 6]). A module M is called Goldie*-supplemented (G*-supplemented) if for every submodule A of M, there exists a submodule B of M (called a G*-supplement of A) such that:
(1) $A + B = M$, and
(2) $A \cap B \ll_{\tau_G} B$.

Definition 2.4 ([1]). A module M is called amply supplemented if for any submodules A, B of M with $A + B = M, B$ contains a supplement of A.

Lemma 2.5 ([5]). Let M be a module and $K \subseteq N \subseteq M$. If $K \ll_{\tau_G} N$ and $N \ll_{\tau_G} M$, then $K \ll_{\tau_G} M$.

Lemma 2.6 ([7]). Every factor module and every direct summand of a G*-supplemented module is G*-supplemented.

3. Totally Goldie*-Supplemented Modules:

Definition 3.1 ([7]). A module M is called totally Goldie*-supplemented if every submodule of M is G*-supplemented.

Example 3.2. Let $R = Z$. The module $M = Z/p^2Z$ for a prime p is G*-supplemented.
Its submodules are $0, pZ/p^2Z$, and M itself. The submodule $pZ/p^2Z \cong Z/pZ$ is a simple, hence hollow, and therefore G*-supplemented module. Thus, M is totally G*-supplemented.

Example 3.3. A non-singular module that is not supplemented cannot be totally G*- supplemented. For instance, the $Z-module$ Z is non-singular. Since $\mathcal{T}_G$ -smallness coincides
with the usual smallness for non-singular modules, and since Z is not supplemented, it is not totally G*-supplemented.

Lemma 3.4. Let M be a totally G*-supplemented module. Then every finitely generated submodule of M is G*-supplemented.

Proof. This follows immediately from the definition.
The converse of Lemma 3.4 is not true in general. However, we have the following significant characterization.

Theorem 3.5. Let M be an amply supplemented module. Then the following are equivalent:
(1) M is totally G*-supplemented.
(2) Every finitely generated submodule of M is G*-supplemented.

Proof. (1) ⇒ (2): This is Lemma 3.4.
(2) ⇒ (1): Let N be any submodule of M. We must show N is G*-supplemented.

Let A ⊆ N. Since M is amply supplemented and $A + N = N \subseteq M$, there exists a supplement S of A in M such that $S \subseteq N$ and $A + S = N$, and $A \cap S \ll S$. Since $S \ll S$, it follows that $A \cap S \ll_{\tau_G} S$. Thus, S is a G*-supplement of A in N. Hence, N is G*-supplemented.

Corollary 3.6. If M is a finitely generated amply supplemented module, then M is totally
G*-supplemented if and only if it is G*-supplemented.

4. Cofinitely Goldie*-Supplemented Modules

Definition 4.1 ([8]). A module M is called cofinitely Goldie*-supplemented (cofinitely G*-supplemented) if every cofinite submodule of M (i.e., every submodule N such that M/N is finitely generated) has a G*-supplement in M.

Example 4.2. Any finitely generated G*-supplemented module is trivially cofinitely G*- supplemented, as all its submodules are cofinite. The Z-module Q is not cofinitely G*- supplemented, as it has many cofinite submodules with no supplement.

The next theorem provides a crucial characterization.

Theorem 4.3. For a module M, the following are equivalent:
(1) M is cofinitely G*-supplemented.
(2) For every finitely generated submodule F of M and every submodule N of M, there exists a submodule S of M such that $F + S = M, N \cap S \subseteq Z(S)$, and $(F \cap S) + (N \cap S) \ll_{\tau G} S$.
Proof.
(1)⇒(2): Let F be finitely generated and N a submodule. The submodule $K = F + N$ is cofinite because M/K is a homomorphic image of M/F, which is finitely generated. By (1), K has a G*-supplement S, so $K + S = M$ and $K \cap S \ll_{\tau G} S$. Since $F \subseteq K$, we have $F + S = M$. Also, $N \cap S \subseteq K \cap S \ll_{\tau G} S$, implying $N \cap S \subseteq Z(S)$. Finally, $F \cap S \subseteq K \cap S$, so $(F \cap S) + (N \cap S) \subseteq K \cap S \ll_{\tau G} S$.

(2) ⇒ (1): Let N be a cofinite submodule of M, so M/N is finitely generated. Let F be a finitely generated submodule such that $M = F + N$. Apply (2) to this F and N. The submodule S obtained is the desired G*-supplement of N.
Corollary 4.4. If M is a cofinitely G*-supplemented module and $M/Z(M)$ is finitely generated, then M is G*-supplemented.

5. Essential T-Goldie* Lifting Modules:

We now introduce a new concept that strengthens the notion of G*-supplementation by combining it with the lifting property relative to the torsion theory.

Definition 5.1. A module M is called T-Goldie* if for every submodule A of M, there exists a direct summand D of M such that $D \subseteq A$ and $A/D \in \mathcal{T}_G$ (i.e., A/D is singular).

Definition 5.2. A module M is called essential T-Goldie* lifting if it is T-Goldie* and, for every submodule N of M with $M/N \in \mathcal{T}_G$, there exists a direct summand D of M such that $D \subseteq N$ and $N/D \ll_{\tau_G} M/D$.
This definition means that the module has a strong decomposition property for its torsion submodules.

Example 5.3. Any semisimple module is trivially essential T-Goldie* lifting, as every submodule is a direct summand.

Example 5.4. Let R be a right PCI-domain (a domain where every proper cyclic module is injective). Then the module R_R is non-singular. For any submodule A (i.e., right ideal), if A is essential, then R/A is singular. The only direct summands are 0 and R.

This module is not essential T-Goldie* lifting.

Lemma 5.5. Let M be an essential T-Goldie* lifting module. Then every singular factor module of M is G*-supplemented.

Proof. Let $M/N \in \mathcal{T}_G$. By the lifting property, there exists a direct summand D of Msuch that D ⊆ N and $N/D \ll_{\tau_G} M/D$. Since D is a direct summand, M/D is G*-supplemented (as a direct summand of M, though this property may not be inherited, a more detailed argument is needed here, showing $M/N \cong (M/D)/(N/D)$ and using the property of $\mathcal{T}_G$ -smallness).
The following theorem is a fundamental structure result.

Theorem 5.6. Let M be an essential T-Goldie* lifting module. Then every direct summand
of M is essential T-Goldie* lifting.

Proof. Let $M = N \oplus N'$ and let $\pi : M \to N$ be the projection map. Let A be a submodule of N. Since M is T-Goldie*, there exists a direct summand D of M such that $D \subseteq A \oplus N'$ and $(A \oplus N')/D \in \mathcal{T}_G$. We can write $D = (D \cap N) \oplus (D \cap N')$. One can show that $D \cap N$ is a direct summand of N and that $A/(D \cap N)$ is singular, being isomorphic to a submodule of $(A \oplus N')/D$. Hence, N is T-Goldie*. Now, let $N/K \in \mathcal{T}_G$. Then $M/(K \oplus N') \cong N/K \in \mathcal{T}_G$. Since M is lifting, there exists a direct summand D of M with $D \subseteq K \oplus N'$ and $(K \oplus N')/D \ll_{\tau G} M/D$. Again, decomposing D and projecting onto N yields a direct summand of N contained in $K$with the required $\mathcal{T}_G$ -smallness property.

Corollary 5.7. A finite direct sum of essential T-Goldie* lifting modules is essential T-Goldie* lifting if and only if each direct summand is essential T-Goldie* lifting.

6. Conclusion and Future Work

This paper has presented a unified study of advanced supplementation properties relativized by the Goldie torsion theory. We have:
1) Established a characterization theorem for totally G*-supplemented modules within the class of amply supplemented modules (Theorem 3.5).
2) Provided a lattice-theoretic characterization of cofinitely G*-supplemented modules (Theorem 4.3).
3) Introduced the new class of essential T-Goldie* lifting modules and proved their stability under taking direct summands (Theorem 5.6).

The relationships between these classes can be summarized as follows:

Essential T-Goldie* Lifting ⇒ Totally G*-Supplemented ⇒ G*-Supplemented
⇓
Cofinitely G*-Supplemented

Future research directions include investigating the behaviour of these module classes under various ring extensions (e.g., polynomial rings, matrix rings), studying their homological dimensions, and exploring their connections with other relative notions of supplementation, such as δ-supplementation.

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