Hollow R-Annihilator δ-Lifting Modules

Gorle S K[1], Wadbude R S[2]

[1]Hutatma Rashtriya Arts and Science College, Ashti Dist. Wardha.
Corresponding Author Email: *sagargorle99[at]gmail.com*

[2]Ex. Professor, Department of Mathematics, Mahatma fule Arts, Commerce and Sitaramji Choudhari Science Mahavidyalaya, Warud, Dist: Amravati. MS, India
Email: *rswadbude[at]gmail.com*

Abstract: ***In this paper we introduced the new concept hollow R- annihilator-δ-lifting module by the help of R- annihilator-small submodule δ- lifting module and R- annihilator-δ-lifting module. These concepts were introduced by [4], [5], [12], [19]. Let M be an indecomposable module. M is hollow $R-a-\delta$ - lifting module if and only if M is hollow $R-a-\delta$ - lifting module but need not be factor module is $\delta-$lifting module.***

Keywords: δ-small submodule, δ-essential submodule, δ-co-essential submodule, δ-co-closed submodule, R- annihilator co-essential submodule, R- annihilator δ-submodule, δ-hollow module and δ-lifting module

1. Introduction

Through, all rings consider are associative with identity, and modules are unitary right modules. In 1975 P, Fleury introduced the concept of hollow module as follows, A module M is said to be hollow module if $M \neq 0$ and whenever $M = A + B$, where A and B are submodules of M, then either $M = A$ or $M = B$ i.e. it is hollow if $M \neq 0$ and every proper submodule of M is a small submodule of M. A submodule N of an R-module M is called small in M if for every proper submodule L of M, $N + L = M$ (denoted by $N << M$). The concept of R-annihilator small ($R-a-$small) sub module was introduced by [5]. A submodule N of an R-module M is called R- annihilator small if $N+T=M$, T is submodule of M, implies that $ann_R(T)=0$, where $Ann(T)=\{r \in R : r.T=0\}$ and (denoted by $N <<_a M$). A submodule N of an R-module M is said to be essential submodule in M if for any $X \subseteq M$, $X \bigcap N = 0$ implies that $X = 0$ [4].

The singular of an R-module M is the set $Z(M)=\{x \in M : ann(x) \subseteq_e R\}$. If $Z(M)=M$ then M is called singular module, if $Z(M)=0$ then M is called non-singular module, many authors have been intersected in studying different definitions generalization of essential submodules. Let M be an R-module. A submodule N of M is called c-singular (denoted by $N \subseteq_{cs} M$) if $\frac{M}{N}$ is a singular module. [19] A submodule N of an R-module M is called δ-small submodule of an R-module, if $N+K=M$ and $\frac{M}{K}$ is singular then $K=M$. Denoted by $N<<_\delta M$. For $N \subseteq K \subseteq M$, if $N \leq K$, the K is called essential extension in M. the dully concept For $N \subseteq K \subseteq M$, N is said to be a co-essential submodule of K in M, if $\frac{K}{N} << \frac{M}{N}$.(denoted by $N \subseteq_{cs} M$). A submodule N of an R-module M is called co-closed submodule in M if N has no proper submodule of K in M (denoted by $N \subseteq_{cc} K$). A submodule K of M is called δ-co-closed submodule of M if $X \subseteq_{cs} K$ and $X \subseteq_{\delta cs} K$ in M for some $X \subseteq A$, then $X + A$, see [[11]. The concept of R-annihilator-co-essential and R-annihilator-co-closed submodule introduced by [16]. For $N \subseteq K \subseteq M$, N is said to be R-annihilator co-essential submodule of K in M, if $\frac{K}{N} << \frac{M}{N}$.(denoted by $N \subseteq_{ace} M$). A submodule N is said to be R-annihilator co-closed submodule of K in M, if $\frac{K}{N} <<_a \frac{M}{N}$. (denoted by $N \subseteq_{acc} M$) implies that $N = K$. Let N and K submodules of M such that $N \subseteq K \subseteq M$, then N is said to be δ-co-essential submodule of K in M if $\frac{K}{N} <<_\delta \frac{M}{N}$. A submodule N of M is called a fully invariant submodule if $f(N) \subseteq A$, for every $f \in End(M)$, see [8], [12]. An R-module M is called R-annihilator lifting module, if for every submodule K of M there exists a submodule L and N of M such that $M=L \oplus N$ with $L \leq N$ and $K \bigcap N <<_a N$. [9] An R-module M is called δ-lifting module if for every submodule K of M there exists a direct summand N of M such that

Paper ID: SR25904205628 DOI: https://dx.doi.org/10.21275/SR25904205628

$N \subseteq_{\delta ce} K$ in M and $\frac{K}{N} <<_{\delta} \frac{M}{N}$. An R-module M is called hollow-lifting module, if for every submodule K of M with $\frac{M}{K}$ is hollow then there exists a direct summand N of M such that $N \leq_{ce} K$ in M. An R-module M is called hollow R-annihilator lifting module, if for every submodule K of M with $\frac{M}{K}$ is hollow there exists a direct summand N of M such that $N \leq_{ace} K$ in M.

Examples:

The submodule $\{\bar{0}, \bar{3}\}$ of the Z-module Z_6 is a c-singular submodule of Z_6.

The submodule $\{\bar{0}\}$ of the Z_4-module Z_4 is not c-singular submodule of Z_4.

Because $\frac{Z_4}{\{0\}} \cong Z_4$ and Z_4 is not singular, where $ann(\bar{1}) \subseteq_e Z_4$.

Proposition: 1. Let M be an R-module. Let A, B, C are submodules of M with $A \subseteq B \subseteq C$. If $A \subseteq_{ace} C$ then $A \subseteq_{ace} B$.

Proof: Let $A \subseteq X \subseteq M$ with $\frac{M}{A} = \frac{B}{A} + \frac{X}{A}$, thus $M = B + X$. But $B \subset C$, therefore $M = C + X$ and then $\frac{M}{A} = \frac{C}{A} + \frac{X}{A}$. But $A \subseteq_{ace} C$, then $\frac{C}{A} <<_a \frac{M}{A}$ thus $ann(\frac{X}{A}) = 0$ and hence $\frac{B}{A} <<_a \frac{M}{A}$ i.e. $A \subseteq_{ace} B$.

Proposition: 2. Let M be an R-module. Let A, B, N are submodules of M. If $A \subseteq_{\delta ce} B$ and $N << M$, then $A \subseteq_{ace} B + N$ in M.

Proof: Suppose that $A \subseteq X \subseteq M$ with $\frac{M}{A} = \frac{B+N}{A} + \frac{X}{A}$ then $M = (B+N) + X$. But $N << M$, therefore $M = B + X$ and hence $\frac{M}{A} = \frac{B}{A} + \frac{X}{A}$, but $\frac{B}{A} <<_a \frac{M}{A}$. Thus $ann(\frac{X}{A}) = 0$ this means $A \subseteq_{\delta ce} B + N$ in M.

Proposition: 3. Let $A \subseteq X \subseteq B \subseteq M$ and $X \subseteq_{\delta ce} B$ if and only if $\frac{X}{A} \subseteq_{ace} \frac{B}{A}$ in $\frac{M}{A}$.

Proof: Assume that $X \subseteq_{\delta ce} B$ in M. Since by III isomorphism theorem $\frac{B}{X} \cong \frac{B/A}{X/A}$ and $\frac{M}{X} \cong \frac{M/A}{X/A}$. Thus $\frac{B/A}{X/A} <<_a \frac{M/A}{X/A}$ and hence $\frac{X}{A} \subseteq_{ace} \frac{M}{A}$. Since $\frac{B/A}{X/A} \cong \frac{B}{X}$ and $\frac{M/A}{X/A} \cong \frac{M}{X}$. Then $\frac{B}{X} \subseteq_{ace} \frac{M}{X} \Rightarrow X \subseteq_{\delta ce} B$ in M.

Lemma: Let M be an R-module and $A \subseteq B \subseteq C \subseteq M$. If $B \subseteq_{ace} C$ in M, then $A \subseteq_{ace} C$ in M.

Proof: Suppose that $B \subseteq_{ace} C$ in M. Then show that $A \subseteq_{ace} C$ in M. suppose that $\frac{M}{A} = \frac{C}{A} + \frac{T}{A}$, where $A \subseteq T$, thus $M = C + T$ then $\frac{M}{B} = \frac{C}{B} + \frac{T+B}{B}$. But $B \subseteq_{ace} C$ and $\frac{C}{B} \subseteq_a \frac{M}{B}$, therefore $ann(\frac{T+B}{B}) = 0$. To prove that $ann(\frac{T}{B}) = 0$, Let $r \in ann(T)$, thus $r.T \subseteq A$ and $r.T \subseteq B$, since $A \subseteq B$, therefore $r.T + B = B$. Thus $r \in ann(\frac{T+B}{B})$ and this means $ann(\frac{T}{A}) = 0$, therefore $A \subseteq_{ace} C$ in M.

Proposition: 4. If M is an R-module and A, B, C are submodules of an R- module M such that $A + C \subseteq_{ace} B + C$ in M, then $A \subseteq_{ace} B$ in M.

Proof: Let M be an R-module and A, B, C are submodules of an R- module M. Let T be any submodule of an R-module M such that $A \subseteq T$ and $\frac{M}{A} = \frac{B}{A} + \frac{T}{A} \Rightarrow M = B + T$ and $\frac{M}{A+C} = \frac{B+C}{A+C} + \frac{T+C}{A+C}$ where $ann(\frac{T}{A}) = 0$. Let $r \in ann(\frac{T}{A})$, thus $r.T \subset A \subseteq (A+C)$ and hence $r(T + (A+C)) = A + C$. Then $r \in ann(\frac{T+C}{A+C}) = 0$, thus $ann(\frac{T}{A}) = 0$. Therefore $A \subseteq_{ace} B$ in M.

Theorem: 1. Let M be an R-module and A, B, C, X are submodules of an R- module M. Then following statements are equivalent:

a) If $A \subseteq_{ace} A+B$ in M, then $A \bigcap B \subseteq_{ace} A$ in M.

b) If $A \subseteq_{ace} B$ in M and $Y \leq M$, then $A \bigcap Y \subseteq_{ace} B \bigcap Y$ in M.

c) If $A \subseteq_{ace} B$ in M and $X \subseteq_{ace} C$, then $A \bigcap X \subseteq_{ace} B \bigcap C$ in M.

Proof: [16]

Examples and Remarks:

1) Consider Z_8 as a Z-module. It is easy to see that $\{\bar{0}, \bar{4}\} \subseteq_{\delta ce} \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ in Z_8.
2) Consider Z_6 as a Z-module. It is clear that $\{\bar{0}\}$ is not δ-coessential submodule of $\{\bar{0}, \bar{3}\}$ in Z_6. If $\frac{\{\bar{0}, \bar{3}\}}{\{\bar{0}\}} <<_{\delta} \frac{Z_6}{\{\bar{0}\}}$. it is easy to see that $\frac{\{\bar{0}, \bar{3}\}}{\{\bar{0}\}} \cong \{\bar{0}, \bar{3}\}$ and $\frac{Z_6}{\{\bar{0}\}} \cong Z_6$ and hence $\{\bar{0}, \bar{3}\} <<_{\delta} Z_6$, which is a contradiction.
3) Let M be an R-module. Then $A <<_{\delta} M$ if and only if $0 <<_{\delta ce} A$ in M.

Proposition: 5. Let M be an R-module. Let A, B, C are submodules of M with $A \subseteq B \subseteq M$, $A \subseteq_{cs} M$, Then $A \subseteq_{\delta ce} B$ in M if and only if $M = B + X \Rightarrow M = A + X$, for every submodule X of M.

Proof: Suppose that $A \subseteq_{\delta ce} B$ in M and $M = B + X$ where $X \subseteq M$, then $\frac{M}{A} = \frac{B}{A} + \frac{X+A}{A}$. Since $A \subseteq_{cs} M$ and then $X + A \subseteq_{cs} M$, but $\frac{M/A}{X+A/A} \cong \frac{M}{X+A}$ (by III isomorphism theorem) therefore $\frac{X+A}{A} \subseteq_{cs} \frac{M}{A}$. But $\frac{B}{A} \subseteq_{cs} \frac{M}{A}$, therefore $\frac{M}{A} = \frac{X+A}{A}$. Hence $M = X + A$.

Conversely; Let $\frac{M}{A} = \frac{B}{A} + \frac{Y}{A}$ where $A \subseteq Y$ with $\frac{Y}{B} \subseteq_{cs} \frac{M}{B}$. Then $M = B + Y$ and $\frac{Y}{B} \subseteq_{cs} \frac{M}{B}$. Hence (by III isomorphism theorem) $Y \subseteq_{cs} M$. Therefore by our assumption we get $M = A + Y$. But $A \subseteq Y$, therefore $M = Y$. Thus $A \subseteq_{\delta ce} B$ in M.

Proposition: 6. Let M be an R-module. Let A, B, X are submodules of M with $A \subseteq X \subseteq B \subseteq M$. Then $X \subseteq_{\delta ce} B$ in M if and only if $\frac{X}{A} \subseteq_{\delta ce} \frac{B}{A}$ in M.

Proof: Let A, B, X are submodules of M with $A \subseteq X \subseteq B \subseteq M$. Assume that $X \subseteq_{\delta ce} B$ in M. Since $\frac{B}{X} \cong \frac{B/A}{X/A}$ and $\frac{M}{X} \cong \frac{M/A}{X/A}$ (by III isomorphism theorem). Then by $\frac{B/A}{X/A} <<_{\delta} \frac{M/A}{X/A}$. Thus $\frac{X}{A} \subseteq_{\delta ce} \frac{B}{A}$ in M.

Conversely; Suppose that $\frac{X}{A} \subseteq_{\delta ce} \frac{B}{A}$ in M. Since $\frac{B/A}{X/A} \cong \frac{B}{X}$ and $\frac{M/A}{X/A} \cong \frac{M}{X}$ (by III isomorphism theorem). then $\frac{B}{X} <<_{\delta} \frac{M}{X}$. Thus $X \subseteq_{\delta ce} B$ in M.

Proposition: 7. Let M be an R-module. Let A, B, C, are submodules of M with $A \subseteq B \subseteq C \subseteq M$. Then $A \subseteq_{\delta ce} C$ in M if and only if $A \subseteq_{\delta ce} B$ in M and $B \subseteq_{\delta ce} C$ in M.

Proof: Suppose that $A \subseteq_{\delta ce} C$ in M. Since $\frac{B}{A} \subseteq \frac{C}{A} \subseteq \frac{M}{A}$, then $\frac{B}{A} <<_{\delta} \frac{M}{A}$ and hence $A \subseteq_{\delta ce} B$ in M. Now a map $f: \frac{M}{A} \to \frac{M}{B}$ define by $f(m+A) = m + B$ for ever $m \in M$. Since f is epimorphism, therefore and $A \subseteq_{\delta ce} C$ in M. Hence $f(\frac{C}{A}) = \frac{C}{B} <<_{\delta} \frac{M}{B}$. Thus $B \subseteq_{\delta ce} C$ in M.

Conversely; assume that $A \subseteq_{\delta ce} B$ in M and $B \subseteq_{\delta ce} C$ in M. It is clear that $A \subseteq_{\delta ce} C$ in M. Let $\frac{M}{A} = \frac{C}{A} + \frac{X}{A}$ where $\frac{X}{A} \subseteq_{cs} \frac{M}{A}$, then $M = C + X$ and hence $\frac{M}{B} = \frac{C}{B} + \frac{X+B}{B}$. Since $\frac{X}{A} \subseteq_{cs} \frac{M}{A}$, then $X \subseteq_{cs} M$ (by III isomorphism theorem) $\frac{M/B}{X+B/B} \cong \frac{M}{X+B}$ and $X + B \subseteq_{cs} M$, therefore $\frac{X+B}{B} \subseteq_{cs} \frac{M}{B}$, So

Paper ID: SR25904205628 DOI: https://dx.doi.org/10.21275/SR25904205628

$\frac{M}{B}=\frac{X+B}{B}\Rightarrow M=X+B$. Then $\frac{M}{A}=\frac{B}{A}+\frac{X}{A}$. Since $\frac{B}{A}<<_{\delta}\frac{M}{A}$ and $\frac{X}{A}\subseteq_{cs}\frac{M}{A}$, Then $\frac{M}{A}=\frac{X}{A}\Rightarrow M=X$. Hence $A\subseteq_{\delta ce}C$ in M.

Proposition: 8. If M is an R-module. If $A\subseteq_{\delta ce}B$ in M and $X\subseteq_{\delta ce}C$ in M then $A+C\subseteq_{\delta ce}B+C$ in M.

Proof: Suppose that $A\subseteq_{\delta ce}B$ in M and $X\subseteq_{\delta ce}C$ in M, where $X\leq M$. To prove that $A+C\subseteq_{ace}B+C$ in M. Now the map $\phi:\frac{M}{A}\rightarrow\frac{M}{A+X}$ defined by $\phi(m+A)=m+(A+X)$ for each $m\in M$ and $\varphi:\frac{M}{X}\rightarrow\frac{M}{A+X}$ defined by $\phi(m+X)=m+(A+X)$ for each $m\in M$. Since ϕ and φ are epic, therefore $\frac{B}{A}<<_{\delta}\frac{M}{A}$ and $\frac{C}{X}<<_{\delta}\frac{M}{X}$, then $\phi(\frac{B}{A})=\frac{B+X}{A+X}<<_{\delta}\frac{M}{A+X}$ and $\varphi(\frac{C}{X})=\frac{C+X}{A+X}<<_{\delta}\frac{M}{A+X}$. Hence $\frac{B+X}{A+X}+\frac{C+X}{A+X}=\frac{B+C}{A+X}<<_{\delta}\frac{M}{A+X}$, thus $A+X\subseteq_{\delta ce}B+C\Rightarrow A+X\subseteq_{\delta ce}B+C$ in M.

Proposition: 9. If M is an R-module. If $A\subseteq_{\delta ce}B$ in M and $X\subseteq M$ in M then $A+X\subseteq_{\delta ce}B+X$ in M, converse is true for $X<<_{\delta}M$.

Proof: Assume that $A\subseteq_{\delta ce}B$ in M and $X\subseteq M$ in M. Since $X\subseteq_{\delta ce}X$ in M, then $A+X\subseteq_{\delta ce}B+X$ in M. Conversely; Suppose that $A+X\subseteq_{\delta ce}B+X$ in M and $X<<_{\delta}M$. Then show that $A\subseteq_{\delta ce}B$ in M, let $\frac{M}{A}=\frac{B}{A}+\frac{Y}{A}$ where $\frac{Y}{A}\subseteq_{cs}\frac{M}{A}$, then $M=B+Y$. Hence $\frac{M}{A+X}=\frac{B+X}{A+X}+\frac{Y+X}{A+X}$. Since $\frac{Y}{A}\subseteq_{cs}\frac{M}{A}$, Then $Y\subseteq_{cs}M$. But (by III isomorphism theorem) $\frac{M/A+X}{Y+X/A+X}\cong\frac{M}{Y+X}$ and $Y+X\subseteq_{cs}M$, therefore $\frac{Y+X}{A+X}\subseteq_{cs}\frac{M}{A+X}$. Since $\frac{B+X}{A+X}<<_{\delta}\frac{M}{A+X}$, then $\frac{M}{A+X}=\frac{Y+X}{A+X}\Rightarrow M=Y+X$. But $X<<_{\delta}M$ and $Y\subseteq_{cs}M$. Hence $M=X$ and $A\subseteq_{\delta ce}B$ in M.

Proposition: 10. Let M be an R-module and $X<<_{\delta}M$ in M. If $A\subseteq_{\delta ce}B$ in M then $A\subseteq_{\delta ce}B+X$ in M.

Proof: Suppose that $A\subseteq_{\delta ce}B$ in M and $X<<_{\delta}M$. To show that $A\subseteq_{\delta ce}B+X$ in M. Let $\frac{M}{A}=\frac{B+X}{A}+\frac{Y}{A}$ where $\frac{Y}{A}\subseteq_{cs}\frac{M}{A}$. Hence $\frac{M}{A}=\frac{B}{A}+\frac{X+Y}{A}$. Since $\frac{Y}{A}\subseteq_{cs}\frac{M}{A}$, then (by III isomorphism theorem) and $Y\subseteq_{cs}M$. But $\frac{M/A}{Y+X/A}\cong\frac{M}{Y+X}$ (by III isomorphism theorem) and $Y+X\subseteq M$, therefore $\frac{Y+X}{A}\subseteq_{cs}\frac{M}{A}$. Since $\frac{B}{A}<<_{\delta}\frac{M}{A}$, then $\frac{M}{A}=\frac{X+Y}{A}$ and hence $M=X+Y$. But $X<<_{\delta}M$ and $Y\subseteq_{cs}M$, therefore $M=X$. Thus $A\subseteq_{\delta ce}B+X$ in M.

Proposition: 11. Let M and N be an R-modules and let $f:M\rightarrow N$ be an epimorphism. If $A\subseteq_{\delta ce}B$ in M then $f(A)\subseteq_{\delta ce}f(B)$ in N.

Proof: Suppose that $A\subseteq_{\delta ce}B$ in M. To show that $f(A)\subseteq_{\delta ce}f(B)$ in N. Let $\frac{N}{f(A)}=\frac{f(B)}{f(A)}+\frac{X}{f(A)}$ where $\frac{X}{f(A)}\subseteq_{cs}\frac{N}{f(A)}$. Then $N=f(B)+X\Rightarrow f^{-1}(N)=f^{-1}(f(B))+f^{-1}(X)\Rightarrow M=B+f^{-1}(X)$, so $\frac{M}{A}=\frac{B}{A}+\frac{f^{-1}(X)}{A}$. Since $\frac{X}{f(A)}\subseteq_{cs}\frac{M}{f(A)}$, then $X\subseteq_{cs}M$ and hence $f^{-1}(X)\subseteq_{cs}M$ (by III isomorphism theorem) $\frac{M/A}{f^{-1}(X)+A/A}\cong\frac{M}{f^{-1}(X)+A}$ and $f^{-1}(X)+A\subseteq_{cs}M$, therefore $\frac{f^{-1}(X)+A}{A}\subseteq_{cs}\frac{M}{A}$. Since $\frac{B}{A}<<_{\delta}\frac{M}{A}$, then $f^{-1}(X)+A=M\Rightarrow f(f^{-1}(X))+f(A)=f(M)\Rightarrow X+f(A)=N$. But $f(A)\subseteq X$, therefore $N=X$. Thus $f(A)\subseteq_{\delta ce}f(B)$ in N.

Proposition: 12. [12] Let M be an R-module and A, B, C, X are submodules of an R- module M. Then following statements are equivalent:

a) If $A \subseteq_{\delta ce} A+B$ in M, then $A \bigcap B \subseteq_{\delta ce} A$ in M.

b) If $A \subseteq_{\delta ce} B$ in M and $Y \leq M$, then $A \bigcap Y \subseteq_{\delta ce} B \bigcap Y$ in M.

c) If $A \subseteq_{\delta ce} B$ in M and $X \subseteq_{\delta ce} C$, then $A \bigcap X \subseteq_{\delta ce} B \bigcap C$ in M.

Proposition: 13. Let M be an R-module and let $A \subseteq B \subseteq M$. If $B = A+L$ and $L \ll_{\delta} M$, then If $A \subseteq_{\delta ce} B$ in M.

Proof: Assume that $B = A+L$ and $L \ll_{\delta} M$, let $\frac{M}{A} = \frac{B}{A} + \frac{X}{A}$ where $\frac{X}{A} \subseteq_{cs} \frac{M}{A}$, then $M = B+X$ $= A+L+X$. Since $\frac{X}{A} \subseteq_{cs} \frac{M}{A}$, then $X \subseteq_{cs} M$, but $L \ll_{\delta} M$ and $A+X \subseteq_{cs} M$, therefore $M = A+X$ and $A \subseteq X$. Thus $M = X$ and $A \subseteq_{\delta ce} B$ in M.

Examples and Remarks:

1) Consider $Z4$ as a Z-module, clearly $Z4$ is δ-lifting module.
2) Let Q be the set of the rational numbers. It is easy to see that Q as a Z-module is not δ-lifting module.

The dual concept is that R-annihilator δ-lifting module. An R-module M is called R-annihilator δ-lifting module, if for every submodule K of M there exists a direct summand N of M with $\mathrm{N} \leq \mathrm{K}$ and $\frac{\mathrm{K}}{\mathrm{N}} \ll_{a\delta} \frac{\mathrm{M}}{\mathrm{N}}$ with $\mathrm{ann(K)=0}$.

Theorem:2. [12]

Let M be an R-module, then following statements are equivalent:

1) M is δ-lifting.
2) For every submodule A in M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \bigcap M_2 \ll_{\delta} M_2$.
3) Every submodule A in M can be written as $A = B \oplus L$, where B is a direct summand of M and L $L \ll_{\delta} M$.

Proof: (1) ⇒ (2) Suppose that M is δ-lifting and let A is a submodule of M. Then there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $\frac{A}{M_1} \ll_{\delta} \frac{M}{M_1}$. By modular law $A = A \bigcap M$ $= A \bigcap (M_1 \oplus M_2) = M_1 \oplus (A \bigcap M_2)$. Now, let $\varphi : \frac{M}{M_1} \to M_2$ be a map defined by $\varphi(m+M_1) = m_2$ with $m = m_1 \oplus m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$. It is clear that φ is an isomorphism. Since $\frac{A}{M_1} \ll_{\delta} \frac{M}{M_1}$, then $\varphi(\frac{A}{M_1}) \ll_{\delta} M_2$. Since $\varphi(\frac{A}{M_1}) = \{\varphi(a+M_1) : a \in A\}$
$= \{\varphi((x+Y)+M_1 : x \in M_1 \text{ and } y \in (A \bigcap M_2)\}$
$= \{\varphi(x+y)+M_1 : x \in M_1, y \in M_2\}$
$= \{\varphi(x+y)+M_1 : x \in M_1, y \in (A \bigcap M_2)\}$ and
$y \in (A \bigcap M_2) = \{y : y \in (A \bigcap M_2)\} = A \bigcap M_2$, then $A \bigcap M_2 \ll_{\delta} M_2$. Let $L = A \bigcap M_2$, so $A = M_1 \oplus L$ where M_1 is a direct summand of M and $L \ll_{\delta} M$.

(2)⇒(3) Let A be a submodule of M. By our assumption, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \bigcap M_2 \ll_{\delta} M_2$, hence $A \bigcap M_2 \ll_{\delta} M$. By modular law $A = A \bigcap M = A \bigcap (M_1 \oplus M_2) = M_1 \oplus (A \bigcap M_2)$. Let $L = A \bigcap M_2$. Thus $A = M_1 \oplus L$, where M_1 is a direct summand of M and $L \ll_{\delta} M$.

(3)⇒(1) Let A be a submodule of M. By (3) A can be written as $A = B \oplus L$, where B is a direct summand of M and $L \ll_{\delta} M$. To show that $B \subseteq_{\delta ce} A$ in M, let $\frac{M}{B} = \frac{A}{B} + \frac{X}{B}$ where $B \subseteq X$ with $\frac{X}{B} \subseteq_{cs} \frac{M}{B}$, then $M = A+X$ and hence $M = B+L+X = L+X$. Since $\frac{X}{B} \subseteq_{cs} \frac{M}{B}$, then $X \subseteq_{cs} M$, by (the third isomorphism theorem). But $L \ll_{\delta} M$, therefore $M = X$. Thus M is δ-lifting module.

Remark:

Let M be an R-module, then M is δ-lifting if and only if for every submodule A in M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \bigcap M_2 \ll_{\delta} M$.

Proposition: 14. [12] Any direct summand of δ-lifting module is δ-lifting.

Theorem: Let M be an R-module, then the following statements are equivalent:

1) M is δ-lifting module.
2) Every submodule A of M has a δ-supplement B in M such that $A \bigcap B$ is a direct summand in A.

Paper ID: SR25904205628 DOI: https://dx.doi.org/10.21275/SR25904205628

Fully invariant submodule:
Let M be an R-module. If $M = M_1 \oplus M_2$ then $\frac{M}{A} = \frac{A+M_1}{A} \oplus \frac{A+M_2}{A}$, for every fully invariant submodule A of M.

Proposition: 15. Let M be a δ- lifting module. If A is a fully invariant submodule of M, then $\frac{M}{A}$ is a δ-lifting module.

Proof: let $\frac{X}{A}$ be a submodule of $\frac{M}{A}$. Since M is δ-lifting module, then there exists a direct summand B of M such that $B \subseteq_{\delta ce} X$ in M. So $M = B \oplus B_1$, for some $B_1 \subseteq M$. Since A is a fully invariant submodule of M, then $\frac{M}{A} = \frac{B+B_1}{A} = \frac{B+A}{A} \oplus \frac{B_1+A}{A}$. Now, let $f:\frac{M}{B} \to \frac{M}{B+A}$ be a map defined by $f(m+B) = m+(B+A)$ for all $m \in M$. It is clear that f is an epimorphism. Since $\frac{X}{B} <<_\delta \frac{M}{B}$, then $f(\frac{X}{B}) = \frac{X}{B+A} <<_\delta \frac{M}{B+A}$, and hence $B + A \subseteq_{\delta ce} X$ in M. Thus $\frac{B+A}{A} \subseteq_{\delta ce} \frac{X}{A}$ in $\frac{M}{A}$. Thus $\frac{M}{A}$ is δ-lifting module.

Lemma: 1. Let $M = A + B$ be a δ-lifting module. If $A \subseteq_{cs} M$, then there exists a direct summand A of M such that $M = A + X$ and $X \subseteq_{\delta cs} B$ in M.

Proof: Let A and B be submodules of M such that $M = A + B$ and $A \subseteq_{cs} M$ Since M is δ-lifting, then there exists a direct summand X of M such that $X \subseteq_{\delta cs} B$ in M. Then $\frac{M}{X} = \frac{A+X}{X} \oplus \frac{B}{A}$. Since $A \subseteq_{cs} M$, then $A+X \subseteq_{cs} M$, and hence $\frac{A+X}{X} \subseteq_{ce} \frac{M}{X}$ by (the third isomorphism theorem). But $\frac{B}{X} <<_\delta \frac{M}{X}$, therefore $\frac{M}{X} = \frac{A+X}{X}$. Thus $M = A + X$.

Proposition: 16. Let A and B be sub modules of an R-module M such that $A \subseteq B$. If A is a $R-a-$small of B, then A is a $R-a-$small of M.
Proof: Let $M = A + X$, where X is a submodule of M. By modular law $M = A + B = A + (X \cap B)$. Since A is a $R-a-$small of B, then $ann(X \cap B) = 0$. Now $(X \cap B) \subseteq X \Rightarrow ann(X \cap B) \supseteq ann(X)$ for this $r \in ann(X) \Rightarrow r.X = 0$. Since $(X \cap B) \subseteq X$ therefore $r.(X \cap B) \Rightarrow r \in ann(A \cap X)$. So $ann(X) = 0$, thus A is a $R-a-$small of M.

Proposition: 17. Let A and B be sub modules of an R-module M such that $A \subseteq B$. If B is a $R-a-$small of M, then A is a $R-a-$small of M.

Proof: Obvious

Proposition: 18. Let M and N be R- modules and $f: M \to N$ be epimorphism and $B \subseteq N$. If B is a $R-a-$small of N, then $f^{-1}(B)$ is a $R-a-$small of M.

Proof: Let M and N be R- modules and $f: M \to N$ be epimorphism and $B \subseteq N$. Let B is a $R-a-$small of N. To prove that $f^{-1}(B)$ is a $R-a-$small of M. Let $M = f^{-1}(B) + X$, where X is a submodule of M. Then $f(M) = f\{f^{-1}(B) + X\} \Rightarrow f(M) = f(f^{-1}(B) + + f(X)$. Since f is epic. Thereforer $N = B + f(X)$. Since B is a $R-a-$small of N i.e. $annf(X) = 0$. Now claim that $annX \subseteq annf(X)$, fore this let $r \in ann(X) \Rightarrow r.X = 0$. Since $f(X) \subseteq X$, therefore $r.f(X) = f(r.X) = f\{0\} = 0$. thus $r \in f(X)$ means $annX \subseteq annf(X)$. Thus $ann(X) = 0$. hence $f^{-1}(B)$ is a $R-a-$small of M.

Proposition:19. Let K and N be submodules of an R-modules of M with $K \subseteq N$. and $\frac{N}{K}$ is a $R-a-$small of $\frac{M}{K}$. Let $\pi: M \to \frac{M}{K}$ be an natural mapping therefore $\pi^{-1}(\frac{M}{K})$ is a $R-a-$small in M, but $\pi^{-1}(\frac{N}{K}) = N$, then N is a $R-a-$small of M.
Proof: Obvious;

Paper ID: SR25904205628 DOI: https://dx.doi.org/10.21275/SR25904205628

Proposition: 20. Let M be an R-module and $A \subseteq X \subseteq B \subseteq M$ such that $\frac{B}{X}$ is a $R-a-$small of $\frac{M}{X}$, then $\frac{B}{A}$ is a $R-a-$small of $\frac{M}{A}$.

Proof: Consider the map $f : \frac{M}{A} \to \frac{M}{X}$ defined by $f(m+A) = m+X$ for every $m \in M$, So f is epic. Since $\frac{B}{X}$ is a $R-a-$small of $\frac{M}{X}$, therefore $\frac{B}{A} = f^{-1}(\frac{B}{X})$ is a $R-a-$small of $\frac{M}{A}$. Thus $\frac{B}{A}$ is a $R-a-$small of $\frac{M}{A}$.

Proposition: 21.Let $M = M_1 \oplus M_2$ an R- module and let $K_1 \subseteq M_1 \subseteq M$ and $K_2 \subseteq M_2 \subseteq M$. If either K_1 is a $R-a-$small of M_1 or K_2 is a $R-a-$small of M_2, then $K_1 \oplus K_2$ is a $R-a-$small of $M_1 \oplus M_2$.

Proof: Let K be the $R-a-$small of M and the projection map $\rho_1 : M_1 \oplus M_2 \to M_1$. Since K_1 is a $R-a-$small of M_1, then we have $\rho_1^{-1}(K_1) = K_1 \oplus M_1$ is a $R-a-$small of $M_1 \oplus M_2$. Since $K_1 \oplus K_2 \subseteq M_1 \oplus M_2$, therefore $K_1 \oplus K_2$ is a $R-a-$small of $M_1 \oplus M_2$.

Corollary: Let M be an R-module and let $K_1 \subseteq M_1 \subseteq M$ and $K_2 \subseteq M_2 \subseteq M$. If $\frac{N_1+N_2}{K_1+K_2}$ is a $R-a-$small of $\frac{M}{K_1+K_2}$. Then

1) $\frac{N_1+K_2}{K_1}$ is a $R-a-$small of $\frac{M}{K_1}$.
2) $\frac{K_1+N_2}{K_2}$ is a $R-a-$small of $\frac{M}{K_2}$.
3) $\frac{N_1}{K_1} \oplus \frac{N_2}{K_2}$ is a $R-a-$small of $\frac{M}{K_1} \oplus \frac{M}{K_2}$.

Proof: 1. Consider the maps $\pi_1 : \frac{M}{K_1} \to \frac{M}{K_1+K_2}$ defined by $\pi_1(m+K_1) = (m+K_1+K_2)\ \forall m \in M$ and $\pi_2 : \frac{M}{K_2} \to \frac{M}{K_1+K_2}$ defined by $\pi_2(m+K_2) = (m+K_1+K_2),\ \forall m \in M$. So π_1 and π_2 are epimorphisms. Therefore $\frac{N_1+K_2}{K_1+K_2} \subseteq \frac{N_1+N_2}{K_1+K_2}$. Since $\frac{N_1+N_2}{K_1+K_2}$ is a $R-a-$small of $\frac{M}{K_1+K_2}$ therefore $\frac{N_1+K_2}{K_1+K_2}$ is a $R-a-$small of $\frac{M}{K_1+K_2}$. We have by previous prop. $\frac{N_1+K_2}{K_1} = \pi_1^{-1}(\frac{N_1+N_2}{K_1+K_2})$ is a $R-a-$small of $\frac{M}{K_1}$. ..1

2.Since $\frac{K_1+N_2}{K_1+K_2} \subseteq \frac{N_1+N_2}{K_1+K_2}$ and $\frac{N_1+N_2}{K_1+K_2}$ is a $R-a-$small of $\frac{M}{K_1+K_2}$, therefore $\frac{K_1+N_2}{K_1+K_2}$ is a $R-a-$small of $\frac{M}{K_1+K_2}$. We have by previous prop. $\frac{K_1+N_2}{K_2} = \pi_2^{-1}(\frac{K_1+N_2}{K_1+K_2})$ is a $R-a-$small of $\frac{M}{K_2}$. ..2

3.From equation (1) we obtain $\frac{N_1}{K_1} \oplus \frac{N_2}{K_2}$ is a $R-a-$small of $\frac{M}{K_1} \oplus \frac{M}{K_2}$.

Remark: An R- module M is $R-a-$lifting module if and only if for any sub module N of M there exists a submodule L of M such that $M = L \oplus K$ and $L \cap K <<_a M$.
2.Every $R-a-$hollow module is $R-a-$lifting module.

Definition: Hollow R-Annihilator δ-Lifting Modules**:** An R-module M is called hollow R-annihilator δ-lifting module, if for every submodule K of M with $\frac{M}{K}$ is hollow then there exists a direct summand N of M with $N \leq_{ca} K$ and $\frac{K}{N} <<_{a\delta} \frac{M}{N}$ with $ann(K)=0$.

Proposition: 22. Let M be an R-module. Let K_1, K_2 be hollow modules and let $M = K_1 \oplus K_2$ be faithful module, then every hollow $R-a-\delta$- lifting module is -$R-a-\delta$-lifting module.

Proof: Let M be an R-module. Let N be a sub module of M. Consider the two natural projections $\rho_1 : M \to K_1$ and $\rho_2 : M \to K_2$.

Case I; If $\rho_1(N) \neq K_1$ and $\rho_2(N) \neq K_2$. we have $\rho_1(N) <<_{a\delta} K_1$ and $\rho_2(N) <<_{a\delta} K_2$. We obtain $\rho_1(N) \oplus \rho_2(N) <<_{a\delta} K_1 \oplus K_2 = M$ [Waren Anderson]. Now we claim that $N \leq \rho_1(N) \oplus \rho_2(N)$, for this let $t \in N$. Then $t \in M = K_1 \oplus K_2$ and hance $t = (t_1, t_2)$ where $t_1 \in K_1$ and $t_2 \in K_2$. Now $\rho_1(t) = \rho_1(t_1, t_2) = t_1$ and $\rho_2(t) = \rho_2(t_1, t_2) = t_2$. Thus $t = (\rho_1(t), \rho_2(t))$ and we obtain $N \leq \rho_1(N) \oplus \rho_2(N)$, hence $N <<_{a\delta} M$. However, M is faithful thus by $N <<_{a\delta} M$. Hence M is - $R-a-\delta$ - lifting module.

Case II; If $\rho_1(N) = K_1$ and $\rho_1(N) = \rho_1(M)$. This shows $M = N + K_2$ and by second isomorphism theorem $\frac{N+K_2}{N} \cong \frac{K_2}{N \cap K_2}$. Since K_2 is a-δ-hollow module, then $\frac{K_2}{N \cap K_2}$ is *a*-δ-hollow module. (every epimorphic image of a-δ-hollow module is a-δ-hollow). However, M hollow $R-a-$lifting module, therefore there exists an $R-a-\delta$- coessential submodule of N in M, which is direct summand of M. Thus M is $R-a-\delta-$lifting module.

Theorem: 3. Let M be an indecomposable module. The following statements are equivalent.

(a) M is hollow $R-a-\delta$ - lifting module.

(b) M is hollow $R-a-\delta$ - lifting module but need not be factor module is $\delta-$lifting module.

Proof: $\Rightarrow$ Suppose that M has a hollow factor module. Then there exists a proper sub module N of M such that $\frac{M}{N}$ is $\delta-$Hollow. Since M is hollow $R-a-\delta$- lifting module there is a direct summand K of M such that $K <<_{aces} N$ of M. However, M is indecomposable module, therefore $K = 0$, hence $N <<_{a\delta} M$. Thus M is hollow $R-a-\delta$- lifting module.

$\Leftarrow$ obvious.

Proposition: 23. Let M be hollow $R-a-\delta$ - lifting module. Then every $R-a-\delta$ - coclosed submodule K such that $\frac{M}{K}$ is $\delta-$hollow module is a direct summand of M.

Proof: Let M be hollow $R-a-\delta$ - lifting module and let K be $R-a-\delta$ - coclosed submodule in M such that $\frac{M}{K}$ is $\delta-$hollow module. Since M be $R-a-\delta$ - lifting module, then there exists a fully invariant direct summand N of M such that $N <<_{a\delta ac} K$ in M. However, K is $R-a-\delta$ - coclosed submodule in M, so $K = N$, Thus K is fully invariant direct submodule of M.

Proposition: 24. Let M be an *R*-module. If M is $R-a-\delta$ - lifting module. Then for every submodule K of M with $\frac{M}{K}$ is $\delta-$hollow module. There exists a submodule X of K such that $M = X + X_1$, $X_1 \subseteq M$ and $A \cap X_1 <<_{a\delta} X_1$.

Proof: Let M be an *R*-module. Let X be a proper submodule of hollow module M, with $\frac{M}{X}$ is $\delta-$hollow module. Assume that there exists a submodule X of M (where $X \subseteq K$) such that $M = X \oplus X_1$, $X_1 \subseteq_{ace} K$ in M. Now we must show that $A \cap X_1 <<_{a\delta} X_1$. For this suppose that $X_1 = A \cap X_1 + L$, where $L \subseteq X_1$. Again we must prove that $Ann(L) = 0$. By modular law $A = A \cap M = A \cap (X \oplus X_1) = X \oplus (A \cap X_1)$. Since $M = X \oplus X_1$, then $M = X + (A \cap X_1) + L$. Now $\frac{M}{X} = \frac{A}{X} + \frac{L+X}{X}$ and since $X \subseteq_{ace} A$ in M, therefore $Ann(\frac{L+X}{X}) = 0$. To show that $Ann(L) = 0$, let $r \in Ann(L) \Rightarrow r.L = 0$ and hence $r.L + T = T$ i.e. $r \in Ann(\frac{L+X}{X}) = 0$. Hence $A \cap X_1 <<_{a\delta} X_1$.

Corollary: Let M be an *R*-module. If M is hollow $R-a-\delta$- lifting module, then for every submodule K of M, with $\frac{M}{K}$ is $\delta-$hollow module, there exists a submodule X of K such that $M = X + X_1$, $X_1 \subseteq M$ and $A \cap X_1 <<_{a\delta} M$.

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