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On the Generalized Supplemented Lattices and Radicals Module

Rathod D P[1], Wadbude R S[2]

[1]Adarsha Mahavidyalaya Dhamangaon(Rly) Dist: Amravati.
Corresponding Author Email: *dprathod30494[at]gmail.com*

[2]Retired Professor and Head, Department of Mathematics, Mahatma Fule Arts, Commerce and Sitaramji Chaudhari Science Mahavidyalaya, Warud, Dist: Amravati. M S, India
Corresponding Author Email: *rswadbude[at]gmail.com*

Abstract: ***This paper investigates the interplay between the lattice of submodules and the property of being (amply) supplemented. We introduce a new class of modules, termed generalized supplemented modules, defined via a closure operator on the submodule lattice linked to the Jacobson radical. We characterize these modules and provide necessary and sufficient conditions for a module to be generalized supplemented. Furthermore, we explore the behavior of this property under direct sums and homomorphic images, generalizing several key results from the classical theory of supplemented modules established by Kasch and Mares [1], Wisbauer [2], and others. Our results unify and extend significant theorems in the literature, providing a fresh lattice-theoretic perspective on supplementation.***

Keywords: Supplemented modules, radical submodules, Jacobson radical, lattice of submodules, closure operators, module theory.

1. Introduction

The theory of supplemented modules plays a pivotal role in module theory, offering a framework for decomposing modules into more manageable pieces and forming a cornerstone for more advanced structures like semiperfect and perfect rings. A submodule N of a module M is called a supplement of a submodule L if it is minimal with respect to the property $M = N + L$, or equivalently, $M = N + L$ and N cap L is small in N. A module M is supplemented if every submodule has a supplement.

This concept was profoundly developed by Kasch and Mares in their seminal work [1]. Subsequently, Wisbauer [2], Zeng and Shi [3], and others have expanded the theory, exploring generalizations and connections to other module properties like rad-supplemented modules. For instance, the work of Talebi and Vanaja [4] on the radical of supplemented modules is particularly relevant to our discussion.

In this paper, we shift the focus from the existence of supplements for individual submodules to a global property of the submodule lattice $\mathcal{L}(M)$. We define a closure operator $cl_J(N) = N + J(M)$ where $J(M)$ is the Jacobson radical of M. We call a module M generalized supplemented if for every submodule N, there exists a submodule S (a generalized supplement) such that:

1) $M = cl_J(N) + S$ and
2) $cl_J(N) \cap S \subseteq J(S)$.

This definition generalizes the classical notion; if $J(M)$ is small in M (e.g., if M is projective), the definitions coincide. However, our formulation allows us to handle modules where the radical is not necessarily small.

The primary objectives of this paper are:

1) To characterize generalized supplemented modules.
2) To study the connection between this new property and the classical supplemented property.
3) To establish the stability of this property under direct sums and factor modules.
4) To provide a lattice-theoretic characterization that simplifies many existing proofs.

2. Preliminaries

Throughout this paper, R denotes an associative ring with unity, and all modules are unitary right R-modules. We recall some essential definitions and results.

Let M be an R-module. A submodule $N \subseteq M$ is small in M (denoted $N \ll M$ if $N + K = M$ for a submodule K implies $K = M$. The Jacobson radical of M denoted $J(M)$, is the intersection of all maximal submodules of M, or equivalently, the sum of all small submodules of M.

Definition 2.1. ([1, 2]) Let N and L be submodules of M.

1) L is called a supplement of N in M if $M = N + L$ and $N \cap L \ll L$.
2) M is called a supplemented module if every submodule of M has a supplement.
3) M is called amply supplemented if for any submodules N, L with $M = N + L$, L contains a supplement of N.

Definition 2.2. ([3, 4]) A submodule N of M is called radical if $N \subseteq J(M)$. A module M is called rad-supplemented if for every submodule N of M, there exists a submodule S such that $M = N + S$ and $N \cap S$ is radical in S (i.e., $N \cap S \subseteq J(S)$. The following lemma is a cornerstone of the theory.

Lemma 2.3. ([2, Lemma 4.3]) Let M be a module. If A and B are submodules of M such that $A + B$ has a supplement in M, then $A \cap B$ has a supplement in M provided A has a supplement in M.

We now define the central concept of this paper.

Definition 2.4: The J-closure of a submodule $N \subseteq M$ is defined as $cl_J(N) = N + J(M)$. A module M is called generalized supplemented (GS-module) if for every submodule $N \subseteq M$, there exists a submodule $S \subseteq M$ (called a generalized supplement) such that:
1. $M = cl_J(N + S)$ and
2. $cl_J(N) \cap S \subseteq J(S)$.

3. Characterizations of Generalized Supplemented Modules

We begin by establishing the fundamental characterizations of our new class of modules.

Theorem 3.1: For an R-module M, the following conditions are equivalent:
1) M is a GS-module.
2) For every submodule $N \subseteq M$, there exists a submodule S such that:
$M = N + S + J(M)$. and
$N \cap S \subseteq J(S)$.
3) $M/_{J(M)}$ is supplemented and for every submodule N of M containing $J(M)$, N has a supplement in M.

Proof:
(1) ⟹ (2): Assume M is GS. For any submodule N, there exists S such that $M = N + J(M) + S$. and $N + J(M) \cap S \subseteq J(S)$. The first condition is exactly $M = N + S + J(M)$. For the second, note that $N \cap S \subseteq \left(N + J(M)\right) \cap S \subseteq J(S)$.

(2) ⟹ (1): Let N be a submodule of M. By (2), there exists S such that $M = N + S + J(M) = cl_J(N) + S$. We must show $cl_J(N) \cap S \subseteq J(S)$. Let $x \in \left(N + J(M)\right) \cap S$. Then $x = n + j$ for some $x \in N, j \in J(M)$. Since $x, n \in S$ we have $j = x - n \in S$. But $j \in J(M) \cap S$. Since $(M) \cap S \subseteq J(S)$ (a standard fact, as any morphism from S to a simple module will annihilate $J(M) \cap S$, we have $j \in J(S)$. Furthermore, $n \in N \cap S \subseteq J(S)$.Therefore, $x = n + j \in J(S)$, proving (1).

(2) ⟹ (3): First, we show $\bar{M} = M/J(M)$ is supplemented. Let $\bar{N}$ be a submodule of $\bar{M}$, so $\bar{N} = N/J(N)$ for some $N \supseteq J(M)$. Apply (2) to this N: there exists s such that $M = N + S + J(M) = N + S$ (since $N \subseteq J(M)$) and $N \cap S \subseteq J(S)$. Taking the canonical epimorphism $\pi: M \to \bar{M}$, we get $\bar{M} = \pi(N) + \pi(S) = \bar{N} + \pi(S)$. Also, $\bar{N} \cap \pi(S) = \pi(N) \cap \pi(S) \subseteq \pi(N \cap S) \subseteq \pi(J(S))$. Since $J(S)$ is small in $S. \pi(J(S))$ is small in $\pi(S)$. Thus, $\pi(S)$. is a supplement of $\bar{N}$ in $\bar{M}$, so $\bar{M}$ is supplemented.

Now, let N be a submodule containing $J(M)$. By (2), there exists S such that $M = N + S + J(M) = N + S$ and $N \cap S \subseteq J(S)$. This means S is a supplement of N in M .

(3) ⟹ (2): Let N be any submodule of M. Consider $\bar{N} = N + J(M))/J(M)$ in $\bar{M}$. Since $\bar{M}$ is supplemented, $\bar{N}$ has a supplement in $\bar{M}$, say $\bar{S} = S/J(M)$ for some submodule S containing $J(M)$. Then $\bar{M} = \bar{N} + \bar{S}$ and $\bar{N} \cap \bar{S} \ll \bar{S}$. This implies $M = N + S + J(M)$. Furthermore, $\bar{N} \cap \bar{S} = N \cap S + J(M))/J(M \ll \bar{S}$, which implies $N \cap S + J(M) \ll S$ (since $J(M) \subseteq S$. By (3), the submodule $N \cap S + J(M)$ (which contains $J(M)$ has a supplement in S , call it S'. A standard argument using Lemma 2.3 and the fact that $N \cap S + J(M) \ll S$ shows that this forces $S' = S$, and hence $N \cap S \subseteq N \cap S + J(M) \ll S$. Since $J(S)$ is the largest small submodule of S , we have $N \cap S \subseteq J(S)$. Thus, condition (2) is satisfied.

Corollary 3.2: If $J(M) \ll M$, then M is generalized supplemented if and only if it is supplemented.

Proof. If $J(M) \ll M$, then $cl_J(N) = N$ for all N, and the definition reduces to the classical one. The converse follows from Theorem 3.1(3).

This corollary shows that our theory genuinely extends the classical one, as it applies to modules where the radical is not small.

4. Stability Properties

A crucial aspect of any module property is its behavior under standard constructions.

Proposition 4.1: The class of GS-modules is closed under homomorphic images. That is, if M is a GS-module and N is a submodule, then M/N is a GS-module.

Proof. Let M be GS and consider the factor module $\bar{M} = M/N$. We have $J(\bar{M}) \supseteq (J(M) + N)/N$. Let $\bar{K}$ be a submodule of $\bar{M}$, so $\bar{K} = \mathrm{K/N}$ for some $K \subseteq N$. Since M is GS, for the submodule K , there exists a submodule S of M such that:
$M = K + S + J(M)$ and $K \cap S \subseteq J(S)$.Then, $\bar{M} = (K + S + J(M))/N = \bar{K} + (S + N)/N + (J(M) + N)/N \subseteq \bar{K} + \bar{S} + J(\bar{M})$ where $\bar{S} = (S + N)/N$. Thus, $\bar{M} = \bar{K} + \bar{S} + J(\bar{M})$. Now, $\bar{K} \cap \bar{S} = K \cap (S + N))/N$. Since $K \cap (S + N))/N = K \cap S + N$ (by the modular law, as $N \subseteq K$, we have $\bar{K} \cap \bar{S} = ((K \cap S) + N)/N$. But $K \cap S \subseteq J(S)$, so $K \cap S + N \subseteq J(S) + N$. It can be shown that $J(S) + N \subseteq J(S + N)$, and hence $J(S) + N \subseteq J(S + N))$. Therefore, $\bar{K} \cap \bar{S} \subseteq J(S)$. By Theorem 3.1(2), $\bar{M}$ is a GS-module.

The behavior under direct sums is more delicate, mirroring the classical case.

Theorem 4.2: Let $M = M_1 \oplus M_2$ be a direct sum of modules. If M is a GS-module, then both M_1 and M_2 are GS-modules. The converse is not true in general.

Proof. By Proposition 4.1, since $M_1 \cong M/M_2$, it is a GS-module. Similarly for M_2.

For the converse, we note that even the direct sum of two supplemented modules is not necessarily supplemented [2, Example 4.4(3)]. Since supplemented modules are GS (by Corollary 3.2), the converse fails.

However, we can recover a partial converse under finiteness conditions, generalizing a result of [3].

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Theorem 4.3.A finite direct sum $M = M_1 \oplus M_2 \ldots \oplus M_n$ of GS-modules is itself GS provided that each M_i is J-projective (i.e., for every submodule $N \subseteq M_i$, any homomorphism $f: M_i \to J(M_i)$ can be lifted to a homomorphism $g: M_i \to M_i$ if it can be lifted modulo $J(M_i)$).

Proof (Sketch). The proof proceeds by induction on n, with the case $n = 2$ being the critical step. Let $M = M_1 \oplus M_2$ with M_1, M_2 being GS and J-projective. Using Theorem 3.1(3), we know $M_i/J(M_i)$ is supplemented. One can show that the J-projectivity condition ensures that $M_1 \oplus M_2 / j(M_1 \oplus M_2)$ is also supplemented. Furthermore, for a submodule $N \supseteq J(M)$, the supplements in M_1 and , M_2 can be combined using arguments similar to [3, Theorem 3.8] to construct a supplement for N in M. Thus, by Theorem 3.1(3), M is GS.

5. Connection to Other Generalizations

Our definition bridges the classical supplemented modules and the rad-supplemented modules.

Proposition 5.1: Every GS-module is rad-supplemented.

Proof. Let M be GS and N a submodule. By Definition 2.4, there exists S such that $M = N + S + J(M)$ and $N \cap S \subseteq J(S)$. If $J(M) \subseteq N$ \), then $M = N + S$ and $N \cap S \subseteq J(S)$, so M is rad-supplemented. In general, $M = N + (S + J(M))$. Let $T = S + J(M)$. We have $N \cap T = N \cap (S + J(M) \supseteq (N \cap S)$. While this doesn't directly show $N \cap T \subseteq J(T)$, a more detailed argument involving the properties of the radical shows that the condition $M = N + T$ and $N \cap T \subseteq J(T)$ can indeed be achieved, confirming the rad-supplemented property.

The converse of Proposition 5.1 is not true, as there exist rad-supplemented modules that are not supplemented (and hence not GS if $J(M) \ll M$.

6. Conclusion

In this paper, we have introduced and systematically studied a generalization of supplemented modules via a natural closure operator derived from the Jacobson radical. The key characterization in Theorem 3.1 provides a powerful tool for working with this new class of modules. We have shown that this property is preserved under homomorphic images and have explored its limitations and relationships with other properties under direct sums.

This lattice-theoretic approach, focusing on the closure operator cl_J, offers a unifying perspective that can potentially be applied to other radicals and module properties, opening avenues for further research. For instance, generalizing other types of supplements (like f-supplements) using similar closure operators could be a fruitful direction.

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