Computation of Atom Bond Sum ConnectivityUphill and Multiplicative Atom Bond Sum Connectivity Uphill Indices of Graphs

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Abstract: ***In this paper, we introduce the atom bond sum connectivity uphill and the multiplicative atom bond sum connectivityuphill indices of a graph. Furthermore, we compute these newly defined uphill indices for some standard graphs, wheel graphs, helm graphs, tadpole graphs.***

Keywords: atom bond sum connectivity uphill index, multiplicative atom bond sum connectivity uphill index, graph.

1. Introduction

The simple graphs which are finite, undirected, connected graphs without loops and multiple edges are considered. Let *G* be such a graph with vertex set *V*(*G*) and edge set *E*(*G*). The degree $d_G(u)$ of a vertex *u* is the number of vertices adjacent to *u*.

A *u*-*v* path *P* in *G* is a sequence of vertices in *G*, starting with *u* and ending at *v*, such that consecutive vertices in *P* are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \ldots v_{k+1}$ in *G* is a uphill path if for every *i*, $1 \leq i \leq k$, $d_G(v_i) \leq d_G(v_{i+1})$.

A vertex *v* is uphill dominates a vertex *u* if there exists an uphill path originated from *u* to *v*. The uphill neighborhood of a vertex *v* is denoted by $N_{up}(v)$ and defined as: $N_{up}(v) = \{u: v$ uphill dominates $u\}$. The uphill degree $d_{up}(v)$ of a vertex *v* is the number of uphill neighbors of *v*, see [1,2].

In [3], Ali et al. introduced the atom bond sum connectivity index and this index is defined as

$$ABS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u) + d_G(v) - 2}{d_G(u) + d_G(v)}}.$$

Recently, some atom bond connectivity indices were studied in [4-13].

Motivated by the atom bond sum connectivity index, the atom bond sum connectivity uphill index of a graph *G* is defined as

$$ABSU(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_{up}(u) + d_{up}(v) - 2}{d_{up}(u) + d_{up}(v)}}.$$

We also define the multiplicative atom bond sum connectivity uphill index of a graph *G* as

$$ABSUII(G) = \prod_{uv \in E(G)} \sqrt{\frac{d_{up}(u) + d_{up}(v) - 2}{d_{up}(u) + d_{up}(v)}}.$$

Recently, some uphill indices were studied such as the Nirmala uphill index [14], F-uphill index [15], Sombor uphill index [16], inverse sum indeg uphill index [17], geometric-arithmetic uphill index [18].

In this research, the atom bond sum connectivity uphill index and multiplicative atom bond sum connectivity uphill index for some standard graphs, wheel graphs, helm graphsand tadpole graphs are determined.

2. Results for Some Standard Graphs

Proposition 1. Let *G*be r-regular with *n* vertices and $r \geq 2$. Then

$$ABSU(G) = \frac{nr}{2}\sqrt{\frac{(n-2)}{(n-1)}}.$$

Proof: Let *G* be an *r*-regular graph with *n* vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{up}(v) = n - 1$ for every *v* in *G*.

From definition,

$$ABSU(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_{up}(u) + d_{up}(v) - 2}{d_{up}(u) + d_{up}(v)}}$$

$$= \frac{nr}{2}\sqrt{\frac{(n-1)+(n-1)-2}{(n-1)+(n-1)}}$$

$$= \frac{nr}{2}\sqrt{\frac{(n-1)+(n-1)-2}{(n-1)+(n-1)}}$$

$$= \frac{nr}{2}\sqrt{\frac{(n-2)}{(n-1)}}.$$

Corollary 1.1. Let C_n be a cycle with $n \geq 3$vertices. Then

$$ABSU(C_n) = n\sqrt{\frac{(n-2)}{(n-1)}}.$$

Corollary 1.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$ABSU(K_n) = \frac{n}{2}\sqrt{(n-1)(n-2)}.$$

Proposition 2. Let G be r-regular with n vertices and $r\geq 2$. Then

$$ABSUII(G)=\left(\sqrt{\frac{(n-2)}{(n-1)}}\right)^{\frac{nr}{2}}.$$

Proof: Let G be an r-regular graph with n vertices and $r\geq 2$ and $\frac{nr}{2}$ edges. Then $d_{up}(v)=n-1$ for every v in G.

$$ABSUII(G)=\prod_{uv\in E(G)}\sqrt{\frac{d_{up}(u)+d_{up}(v)-2}{d_{up}(u)+d_{up}(v)}}$$

$$=\left(\sqrt{\frac{(n-1)+(n-1)-2}{(n-1)+(n-1)}}\right)^{\frac{nr}{2}}$$

$$=\left(\sqrt{\frac{(n-2)}{(n-1)}}\right)^{\frac{nr}{2}}.$$

Corollary 2.1. Let C_n be a cycle with $n\geq 3$ vertices. Then

$$ABSUII(C_n)=\left(\sqrt{\frac{(n-2)}{(n-1)}}\right)^{n}.$$

Corollary 2.2. Let K_n be a complete graph with $n\geq 3$ vertices. Then

$$ABSUII(K_n)=\left(\sqrt{\frac{(n-2)}{(n-1)}}\right)^{\frac{n(n-1)}{2}}.$$

Proposition 3. Let P_n be a path with $n\geq 3$ vertices. Then

$$ABSU(P_n)=2\sqrt{\frac{(2n-7)}{(2n-5)}}+\sqrt{(n-3)(n-4)}.$$

Proof: Let P_n be a path with $n\geq 3$ vertices. Clearly, P_n has two types of edges based on the uphill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(P_n) \mid d_{up}(u)=n-2,\ d_{up}(v)=n-3\}$, $|E_1| = 2$.
$E_2 = \{uv \in E(P_n) \mid d_{up}(u)= d_{up}(v)=n-3\}$, $|E_2| = n-3$.

$$ABSU(P_n)=\sum_{uv\in E(P_n)}\sqrt{\frac{d_{up}(u)+d_{up}(v)-2}{d_{up}(u)+d_{up}(v)}}$$

$$=2\sqrt{\frac{(n-2)+(n-3)-2}{(n-2)+(n-3)}}$$

$$+(n-3)\sqrt{\frac{(n-3)+(n-3)-2}{(n-3)+(n-3)}}$$

$$=2\sqrt{\frac{(2n-7)}{(2n-5)}}+\sqrt{(n-3)(n-4)}.$$

Proposition 4. Let P_n be a path with $n\geq 3$ vertices. Then

$$ABSUII(P_n)=\left(\sqrt{\frac{2n-7}{2n-5}}\right)^{2}\times\left(\sqrt{\frac{n-4}{n-3}}\right)^{n-3}.$$

Proof: We obtain

$$ABSUII(P_n)=\prod_{uv\in E(P_n)}\sqrt{\frac{d_{up}(u)+d_{up}(v)-2}{d_{up}(u)+d_{up}(v)}}$$

$$=\left(\sqrt{\frac{(n-2)+(n-3)-2}{(n-2)+(n-3)}}\right)^{2}$$

$$\times\left(\sqrt{\frac{(n-3)+(n-3)-2}{(n-3)+(n-3)}}\right)^{n-3}$$

$$=\left(\sqrt{\frac{2n-7}{2n-5}}\right)^{2}\times\left(\sqrt{\frac{n-4}{n-3}}\right)^{n-3}.$$

3. Results for Wheel Graphs

Let W_n be a wheel with n+1vertices and $2n$ edges, $n\geq 4$. Then there are two types of edges based on the uphill degree of end vertices of each edge as follows:

$E_1 = \{uv\in E(W_n) \mid d_{up}(u) = 0,\ d_{up}(v) = n\}$, $|E_1| = n$.
$E_2 = \{uv\in E(W_n) \mid d_{up}(u) = d_{up}(v) = n\}$, $|E_2| = n$.

Theorem 1. Let W_n be a wheel with n+1vertices and $2n$ edges, $n\geq 4$. Then

$$ABSU(W_n)=n\sqrt{\frac{n-2}{n}}+n\sqrt{\frac{n-1}{n}}.$$

Proof. We deduce

$$ABSU(W_n)=\sum_{uv\in E(W_n)}\sqrt{\frac{d_{up}(u)+d_{up}(v)-2}{d_{up}(u)+d_{up}(v)}}$$

$$=n\sqrt{\frac{0+n-2}{0+n}}+n\sqrt{\frac{n+n-2}{n+n}}$$

$$=n\sqrt{\frac{n-2}{n}}+n\sqrt{\frac{n-1}{n}}.$$

Theorem 2. Let W_n be a wheel with n+1vertices and $2n$ edges, $n\geq 4$. Then

$$ABSUII(W_n)=\left(\sqrt{\frac{n-2}{n}}\right)^{n}\times\left(\sqrt{\frac{n-1}{n}}\right)^{n}.$$

Proof. We obtain

$$ABSUII(W_n)=\prod_{uv\in E(W_n)}\sqrt{\frac{d_{up}(u)+d_{up}(v)-2}{d_{up}(u)+d_{up}(v)}}$$

$$=\left(\sqrt{\frac{0+n-2}{0+n}}\right)^{n}\times\left(\sqrt{\frac{n+n-2}{n+n}}\right)^{n}$$

$$=\left(\sqrt{\frac{n-2}{n}}\right)^{n}\times\left(\sqrt{\frac{n-1}{n}}\right)^{n}.$$

4. Results for Helm Graphs

The helm graph H_n is a graph obtained from W_n (with n+1 vertices) by attaching an end edge to each rim vertex of W_n. Clearly, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. A graph H_n is shown in Figure 1.

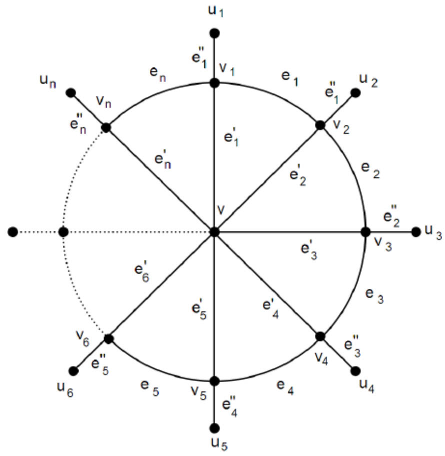

Figure 1: Helm graph H_n

Let H_n be a helm graph with $3n$ edges, $n\geq 3$. Then H_n has three types of the uphill degree of edges as follows:

$E_1 = \{uv\in E(H_n) \mid d_{up}(u) = n+1, d_{up}(v) = n\}, |E_1| = n.$
$E_2 = \{uv\in E(H_n) \mid d_{up}(u) = d_{up}(v) = n\}, |E_2| = n.$
$E_3 = \{uv\in E(H_n) \mid d_{up}(u) = n, d_{up}(v) = 0\}, |E_3| = n.$

Theorem 3. Let H_n be a helm graph with $2n+1$ vertices, $n\geq 3$. Then

$$ABSU(H_n)=n\sqrt{\frac{2n-1}{2n+1}}+\sqrt{n(n-1)}+\sqrt{n(n-2)}.$$

Proof: We obtain

$$ABSU(H_n)=\sum_{uv\in E(H_n)}\sqrt{\frac{d_{up}(u)+d_{up}(v)-2}{d_{up}(u)+d_{up}(v)}}$$

$$=n\sqrt{\frac{(n+1)+n-2}{(n+1)+n}}+n\sqrt{\frac{n+n-2}{n+n}}$$

$$+n\sqrt{\frac{n+0-2}{n+0}}$$

$$=n\sqrt{\frac{2n-1}{2n+1}}+\sqrt{n(n-1)}+\sqrt{n(n-2)}.$$

Theorem 4. Let H_n be a helm graph with $2n$+1vertices, $n\geq 3$. Then

$$ABSU(H_n)=\left(\sqrt{\frac{2n-1}{2n+1}}\right)^n\times\left(\sqrt{\frac{n-1}{n}}\right)^n$$

$$\times\left(\sqrt{\frac{n-2}{n}}\right)^n.$$

Proof: We deduce

$$ABSUII(H_n)=\prod_{uv\in E(H_n)}\sqrt{\frac{d_{up}(u)+d_{up}(v)-2}{d_{up}(u)+d_{up}(v)}}$$

$$=\left(\sqrt{\frac{(n+1)+n-2}{(n+1)+n}}\right)^n\times\left(\sqrt{\frac{n+n-2}{n+n}}\right)^n$$

$$\times\left(\sqrt{\frac{n+0-2}{n+0}}\right)^n$$

$$=\left(\sqrt{\frac{2n-1}{2n+1}}\right)^n\times\left(\sqrt{\frac{n-1}{n}}\right)^n\times\left(\sqrt{\frac{n-2}{n}}\right)^n.$$

5. Results for Tadpole Graphs

Let $G=T_{n,m}$ be the tadpole graph with $n+m$ vertices, where $n, m\geq 3$.Then G has five types of the uphill degree of edges as follows:

Table 1

$d_{up}(u), d_{up}(v)$	Number of edges
$(m-1, 0)$	2
$(m-1, m-1)$	$m-2$
$(0, n-1)$	1
$(n-1, n-1)$	$n-2$
$(n-1, n)$	1

Theorem 5. Let $G=T_{n,m}$ be a tadpole graph with $n+m$ vertices. Then

$$ABSU(T_{n,m})=2\sqrt{\frac{m-3}{m-1}}+(m-2)\sqrt{\frac{m-2}{m-1}}$$

$$+1\sqrt{\frac{n-3}{n-1}}+(n-2)\sqrt{\frac{n-2}{n-1}}+1\sqrt{\frac{2n-3}{2n-1}}.$$

Proof: We obtain

$$ABSU(T_{n,m})=\sum_{uv\in E(T_{n,m})}\sqrt{\frac{d_{up}(u)+d_{up}(v)-2}{d_{up}(u)+d_{up}(v)}}$$

$$=2\sqrt{\frac{m-1+0-2}{m-1+0}}+(m-2)\sqrt{\frac{m-1+m-1-2}{m-1+m-1}}$$

$$+1\sqrt{\frac{0+n-1-2}{0+n-1}}+(n-2)\sqrt{\frac{n-1+n-1-2}{n-1+n-1}}$$

$$+1\sqrt{\frac{n-1+n-2}{n-1+n}}$$

$$=2\sqrt{\frac{m-3}{m-1}}+(m-2)\sqrt{\frac{m-2}{m-1}}+1\sqrt{\frac{n-3}{n-1}}$$

$$+(n-2)\sqrt{\frac{n-2}{n-1}}+1\sqrt{\frac{2n-3}{2n-1}}.$$

Theorem 6. Let $G=T_{n,m}$ be a tadpole graph with $n+m$ vertices. Then

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$$ABSUII(T_{n,m}) = \left(\sqrt{\frac{m-3}{m-1}}\right)^2 \times \left(\sqrt{\frac{m-2}{m-1}}\right)^{m-2}$$
$$\times \left(\sqrt{\frac{n-3}{n-1}}\right)^1 \times \left(\sqrt{\frac{n-2}{n-1}}\right)^{n-2} \times \left(\sqrt{\frac{2n-3}{2n-1}}\right)^1.$$

Proof: We obtain

$$ABSUII(T_{n,m}) = \prod_{uv \in E(T_{n,m})} \sqrt{\frac{d_{up}(u)+d_{up}(v)-2}{d_{up}(u)+d_{up}(v)}}$$
$$= \left(\sqrt{\frac{m-1+0-2}{m-1+0}}\right)^2 \times \left(\sqrt{\frac{m-1+m-1-2}{m-1+m-1}}\right)^{m-2}$$
$$\times \left(\sqrt{\frac{0+n-1-2}{0+n-1}}\right)^1 \times \left(\sqrt{\frac{n-1+n-1-2}{n-1+n-1}}\right)^{n-2}$$
$$\times \left(\sqrt{\frac{n-1+n-2}{n-1+n}}\right)^1$$
$$= \left(\sqrt{\frac{m-3}{m-1}}\right)^2 \times \left(\sqrt{\frac{m-2}{m-1}}\right)^{m-2} \times \left(\sqrt{\frac{n-3}{n-1}}\right)^1$$
$$\times \left(\sqrt{\frac{n-2}{n-1}}\right)^{n-2} \times \left(\sqrt{\frac{2n-3}{2n-1}}\right)^1.$$

6. Conclusion

In this paper, theatom bond sum connectivity uphill and multiplicative atom bond sum connectivity uphill indices of a graph are defined. Also these newly defined uphill indices for some standard graphs, wheel graphs,helm graphs and tadpole graphsare determined.

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