Stability of Partial Differential Equation

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Abstract: ***In this paper, we prove the Hyers-Ulam-Rassias stability of second order partial differential equation.***

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1. Introduction

In 1940, S. M. Ulam [17], delivered a notable presentation to the Mathematics Club at the University of Wisconsin, where he addressed several significant unsolved problems. One of them was concerned with the stability of group homomorphism and in 1941, D. H. Hyers [5] provided a partial solution to this problem. Thereafter number of authors have studied the stability of solutions of differential equations [3, 6, 7, 16] and partial differential equations [8, 9, 15]. This is now known as Hyers-Ulam (HU) stability and its various extensions has been named with additional word. One such extension is Hyers Ulam Rassias (HUR) stability. HURstability for linear differential operators of n^{th} order with non-constant coefficients was studied in [10] and [11]. HUR stability for special types of non-linear equations have been studied in [1, 2, 12,13]. HUR stability of second order partial differential equation have been studied in [14]. In 2011, Gordji et al. [4], proved the HUR stability of non-linear partial differential equations by using Banach's Contraction Principle. In this paper, by using the result of [4], we prove the HUR stability of second order partial differential equation:

$$p(x,t)u_{xx}(x,t)+p_x(x,t)u_x(x,t)+q(x,t)u_{tx}(x,t)+q(x,t)u_x(x,t)+q_x(x,t)u(x,t)=g(x,t,u(x,t)). \qquad (1.1)$$

Here $p, q : J \times J \to \mathbb{R}^+$ be a differentiable function at least once w. r. t. both the arguments and $p(x, t) \neq 0$, $q(x, t) \neq 0\ \forall x, t \in J$, $g : J \times J \times \mathbb{R} \to \mathbb{R}$ bea continuous function and $J = [a, b]$ be a closed interval.

Definition 1.1: A function $u: J\times J \to R$ is called a solution of equation (1.1) if $u \in C^2(J\times J)$ and satisfies the equation (1.1).

2. Preliminaries

Definition 2.1: The equation (1.1) is said to be HUR stable if the following holds:
Let$\varphi: J \times J \to (0, \infty)$ be a continuous function. Then there exists a continuous function
$\Psi: J \times J \to (0, \infty)$, which depends on φ such that whenever $u: J \times J \to \mathbb{R}$ is a continuous function with
$|p(x,t)u_{xx}(x,t)+p_x(x,t)u_x(x,t)+q(x,t)u_{tx}(x,t)+q(x,t)u_x(x,t)+q_x(x,t)u(x,t)-g(x,t,u(x,t))| \le \varphi(x,t)$,
(2.1)there exists a solution $u_0 : J \times J \to \mathbb{R}$ of (1.1) such that

$$|u(x,t)-u_0(x,t)| \le \Psi(x,t),\ \forall (x,t) \in J \times J.$$

We need the following.

Banach Contraction Principle:
Let (Y, d) be a complete metric space, then each contraction map $T: Y \to Y$ has aunique fixed point, that is, there exists $b \in Y$ such that $Tb = b$. Moreover,
$d(b,w) \le \frac{1}{(1-\alpha)} d(w,Tw),\ \forall\, w \in Y$ and $0 \le \alpha < 1$.

Following the results from Gordji et al. [4], we establish the following result.

3. Main Result

In this section we prove the HUR stability of first order partial differential equation (1.1).
Theorem 3.1: Let $c \in J$. Let p, q and g be as in (1.1) with additional conditions:
1) $p(x,t) \ge 1$, $\forall x, t \in J$.
2) $\varphi: J \times J \to (0, \infty)$ be a continuous function and $M: J \times J \to [1, \infty)$ be an integrable function.
3) Assume that there exists α, $0 < \alpha < 1$ such that

$$\int_c^x M(\tau,t)\varphi(\tau,t)d\tau \le \alpha\,\varphi(x,t). \qquad (3.1)$$

Let $h(c,t) = -\{p(c,t)u_x(c,t)+q(c,t)u_t(c,t)+q(c,t)u(c,t)\}$ (3.2)
And

$$K\big(x,t,u(x,t)\big) = -\{p(x,t)\}^{-1}\Big\{q(x,t)u_t(x,t)+q(x,t)u(x,t)+\mathrm{h}(c,t)-\int_c^x q_x(\tau,t)u_t(\tau,t)\,d\tau - \int_c^x g(\tau,t,u(\tau,t))\,d\tau\Big\}.$$

(3.3)

Suppose that the following holds:
C1: $|K(\tau, t, l(\tau, t)) - K(\tau, t, m(\tau, t))| \leq M(\tau, t)|l(\tau, t) - m(\tau, t)|$, $\forall \tau, t \in J$ and $l, m \in C(J \times J)$.
C2: $u : J \times J \rightarrow$ R be a function satisfying the inequality (2.1).

Then there exists a unique solution $u_0 : J \times J \rightarrow$ R of the equation (1.1) of the form

$$u_0(x,t) = u(c,t) + \int_c^x K\big(\tau, t, u_0(x,t)\big) d\tau$$

such that

$$|u(x,t) - u_0(x,t)| \leq \frac{\alpha}{(1-\alpha)} \varphi(x,t), \ \forall x, t \in J.$$

Proof: Consider

$$| p(x,t)u_{xx}(x,t) + p_x(x,t)u_x(x,t) + q(x,t)u_{tx}(x,t) + q(x,t)u_x(x,t) + q_x(x,t)u(x,t) - g(x,t,u(x,t))|$$
$$= | \{p(x,t)u_x(x,t) + q(x,t)u_t(x,t) + q(x,t)u(x,t)\}_x - q_x(x,t)u_t(x,t) - g(x,t,u(x,t))|$$

From the inequality (2.1), we get

$$| \{p(x,t)u_x(x,t) + q(x,t)u_t(x,t) + q(x,t)u(x,t)\}_x - q_x(x,t)u_t(x,t) - g(x,t,u(x,t))| \leq \varphi(x,t).$$
$$\Rightarrow -\varphi(x,t) \leq \{p(x,t)u_x(x,t) + q(x,t)u_t(x,t) + q(x,t)u(x,t)\}_x - q_x(x,t)u_t(x,t) - g(x,t,u(x,t)) \leq \varphi(x,t). \quad (3.4)$$
$$\Rightarrow \{p(x,t)u_x(x,t) + q(x,t)u_t(x,t) + q(x,t)u(x,t)\}_x - q_x(x,t)u_t(x,t) - g(x,t,u(x,t)) \leq \varphi(x,t).$$

Integrating from c to x we get,

$$[p(x,t)u_x(x,t) + q(x,t)u_t(x,t) + q(x,t)u(x,t)] - [p(c,t)u_x(c,t) + q(c,t)u_t(c,t) + q(c,t)u(c,t)] - \int_c^x q_x(\tau,t)u_t(\tau,t)\, d\tau - \int_c^x g(\tau,t,u(\tau,t))\, d\tau \leq \int_c^x \varphi(\tau,t) d\tau.$$

$$\Rightarrow [p(x,t)u_x(x,t) + q(x,t)u_t(x,t) + q(x,t)u(x,t)] + \mathrm{h(c,t)} - \int_c^x q_x(\tau,t)u_t(\tau,t)\, d\tau - \int_c^x g(\tau,t,u(\tau,t))\, d\tau \leq \int_c^x \varphi(\tau,t) d\tau.$$

where h(c,t) is given by (3.2).

$$\Rightarrow p(x,t)\Bigg[u_x(x,t) + \{p(x,t)\}^{-1}\Bigg\{q(x,t)u_t(x,t) + q(x,t)u(x,t) + h(c,t) - \int_c^x q_x(\tau,t)u_t(\tau,t)\, d\tau - \int_c^x g(\tau,t,u(\tau,t))\, d\tau\Bigg\}\Bigg] \leq \int_c^x \varphi(\tau,t) d\tau.$$

$$\Rightarrow \Bigg[u_x(x,t) + \{p(x,t)\}^{-1}\Bigg\{q(x,t)u_t(x,t) + q(x,t)u(x,t) + h(c,t) - \int_c^x q_x(\tau,t)u_t(\tau,t)\, d\tau - \int_c^x g(\tau,t,u(\tau,t))\, d\tau\Bigg\}\Bigg] \leq \{p(x,t)\}^{-1} \int_c^x \varphi(\tau,t) d\tau.$$

$$\Rightarrow \Bigg[u_x(x,t) + \{p(x,t)\}^{-1}\Bigg\{q(x,t)u_t(x,t) + q(x,t)u(x,t) + \mathrm{h(c,t)} - \int_c^x q_x(\tau,t)u_t(\tau,t)\, d\tau - \int_c^x g(\tau,t,u(\tau,t))\, d\tau\Bigg\}\Bigg] \leq \int_c^x \varphi(\tau,t) d\tau,$$

$(\because\ p(x,t) \geq 1)$.

$$\Rightarrow \{u_x(x,t) - K\big(x,t,u(x,t)\big)\} \leq \int_c^x \varphi(\tau,t) d\tau.$$

where $K\big(x,t,u(x\ t)\big)$ is given by equation (3.3).

Since $M: J \times J \to [1, \infty)$ be an integrable function, we have

$$\Rightarrow \{u_x(x,t) - K(x,t,u(x\,t))\} \le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$\{u_x(x,t) - K(x,t,u(x\,t))\} \le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \le \alpha\varphi(x,t).$$

$$\{u_x(x,t) - K(x,t,u(x\,t))\} \le \alpha\varphi(x,t).$$

$$\{u_x(x,t) - K(x,t,u(x\,t))\} \le \varphi(x,t). \quad (3.5)$$

Again, integrating from c to x we get,

$$u(x,t) - u(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \le \int_c^x \varphi(\tau,t)d\tau.$$

Since $M: J \times J \to [1, \infty)$ be an integrable function, we have

$$u(x,t) - u(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$u(x,t) - u(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \le \alpha\varphi(x,t).$$

$$\Rightarrow u(x,t) - u(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \le \alpha\varphi(x,t). \quad (3.6)$$

In a similar way, from the left inequality of (3.4), we obtain

$$-\left[u(x,t) - \left[u(c,t) + \int_c^x K(\tau,t,u(\tau,t))d\tau\right]\right] \le \alpha\varphi(x,t). \quad (3.7)$$

From the inequalities (3.6) and (3.7) we get,

$$\left|u(x,t) - \left[u(c,t) + \int_c^x K(\tau,t,u(x\,t))d\tau\right]\right| \le \alpha\varphi(x,t). \quad (3.8)$$

Let Y be the set of all continuously differentiable functions $\gamma : J \times J \to$ R. We definea metric d and an operator T on Y as follows: For $l, m \in Y$

$$d(l,m) = sup_{x,t\in J} \left|\frac{l(x,t) - m(x,t)}{\varphi(x,t)}\right|$$

and the operator

$$(Tm)(x,t) = \left[u(c,t) + \int_c^x K(\tau,t,m(\tau,t))d\tau\right]. \quad (3.9)$$

Consider,

$$d(Tl,Tm) = sup_{x,t\in J} \left\{\frac{Tl(x,t) - Tm(x,t)}{\varphi(x,t)}\right\}.$$

$$= sup_{x,t\in J} \left\{\frac{\int_c^x K(\tau,t,l(\tau,t))\,d\tau - \int_c^x K(\tau,t,m(\tau,t))\,d\tau}{\varphi(x,t)}\right\}.$$

$$\le sup_{x,t\in J} \left\{\frac{\int_c^x |K(\tau,t,l(\tau,t)) - K(\tau,t,m(\tau,t)|\,d\tau}{\varphi(x,t)}\right\}.$$

By using condition C1 we get,

$$d(Tl,Tm) \le sup_{x,t\in J} \left\{\frac{\int_c^x \{M(\tau,t)|l(\tau,t) - m(\tau,t)|\}\,d\tau}{\varphi(x,t)}\right\}.$$

$$= sup_{x,t\in J} \left\{\frac{\int_c^x \left\{M(\tau,t)\varphi(\tau,t)\left(\frac{|l(\tau,t) - m(\tau,t)|}{\varphi(\tau,t)}\right)\right\} d\tau}{\varphi(x,t)}\right\}.$$

$$\le sup_{x,t\in J} \left\{\frac{\int_c^x \left\{M(\tau,t)\varphi(\tau,t)\times sup_{\tau,t\in J}\left(\frac{|l(\tau,t) - m(\tau,t)|}{\varphi(\tau,t)}\right)\right\} d\tau}{\varphi(x,t)}\right\}.$$

$$d(Tl,Tm) \le d(l,m) \times sup_{x,t\in J} \left\{\frac{\int_c^x \{M(\tau,t)\varphi(\tau,t)\}\,d\tau}{\varphi(x,t)}\right\}.$$

By using inequality (3.1) we get,

$$d(Tl,Tm) \le \alpha d(l,m).$$

By using Banach contraction principle, there exists a unique $u_0 \in X$ such that
$Tu_0 = u_0$, that is

$$\left[u(c,t) + \int_c^x K(\tau,t,u_0(\tau,t))d\tau\right] = u_0(x,t),$$

(By using equation (3.9))
and

$$d(u_0,u) \le \frac{1}{(1-\alpha)} d(u,Tu). \quad (3.10)$$

Now by using inequality (3.8) we get,

$$|u(x,t) - (Tu)(x,t)| \le \alpha\,\varphi(x,t).$$

$$\Rightarrow \frac{|u(x,t) - (Tu)(x,t)|}{\varphi(x,t)} \le \alpha.$$

$$\Rightarrow sup_{x,t\in J} \frac{|u(x,t) - (Tu)(x,t)|}{\varphi(x,t)} \le \alpha.$$

Thus

$$d(u,Tu) \le \alpha. \quad (3.11)$$

Again

$$d(u_0,u) = sup_{x,t\in J} \left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right|.$$

From equation (3.10) we get,

$$d(u_0,u) \le \frac{1}{(1-\alpha)} d(u,Tu).$$

$$sup_{x,t\in J} \left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \le \frac{1}{(1-\alpha)} d(u,Tu).$$

$$\left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \le sup_{x,t\in J} \left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \le \frac{1}{(1-\alpha)} d(u,Tu).$$

$$\left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \le \frac{1}{(1-\alpha)} d(u,Tu).$$

From equation (3.11) we get,

$$\left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \le \frac{1}{(1-\alpha)} \alpha.$$

$$|u(x,t) - u_0(x,t)| \le \frac{\alpha}{(1-\alpha)} \varphi(x,t), \qquad \forall\, x,t \in J.$$

Hence the result.

4. Conclusion

In this paper we have proved the HUR stability of the second order partial differential equation(1.1) by employing Banach's contraction principle.

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