Geometric-Arithmetic Downhill and Modified First Downhill Indices of Graphs

Kulli V R

Professor, Department of Mathematics, Gulbarga University, Kalaburgi, India
Email: *vrkulli[at]gmail.com*

Abstract: ***In this paper, we introduce the geometric-arithmetic downhill and the modified first downhill indices of a graph. Furthermore, we compute these newly defined downhill indices for some standard graphs, wheel graphs, gear graphs, helm graphs and honeycomb networks.***

Keywords: geometric-arithmetic downhill index, modified first downhill index, graph

1. Introduction

The simple graphs which are finite, undirected, connected graphs without loops and multiple edges are considered. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u.

A u-v path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \ldots v_{k+1}$ in G is a downhill path if for every i, $1 \leq i \leq k$, $d_G(v_i) \geq d_G(v_{i+1})$.

A vertex v is downhill dominates a vertex u if there exists a downhill path originated from u to v. The downhill neighborhood of a vertex v is denoted by $N_{dn}(v)$ and defined as: $N_{dn}(v) = \{u: v \text{ downhill dominates } u\}$. The downhill degree $d_{dn}(v)$ of a vertex v is the number of downhill neighbors of v [1].

In [2], Vukičević et al. introduced the geometric-arithmetic index and this index is defined as

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_G(u)d_G(v)}}{d_G(u) + d_G(v)}.$$

Motivated by the geometric-arithmetic index, the geometric-arithmetic downhill index of a graph G is defined as

$$GADW(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_{dn}(u)d_{dn}(v)}}{d_{dn}(u) + d_{dn}(v)}.$$

The first downhill index was introduced in [3] and it is defined as

$$DWM_1(G) = \sum_{uv \in E(G)} \left(d_{dn}(u) + d_{dn}(v)\right).$$

We define the modified first downhill index of a graph G as

$$^{m}DWM_1(G) = \sum_{uv \in E(G)} \frac{1}{d_{dn}(u) + d_{dn}(v)}.$$

Recently, some downhill indices were studied such as the downhill Sombor index [4], harmonic downhill index [5], hyper downhill indices [6], downhill Nirmala indices [7], downhill product connectivity indices [8], downhill Nirmala alpha Gourava indices [9], inverse sum indeg downhill index [10], Gourava downhill indices [11].

In this paper, the geometric-arithmetic downhill and the modified first downhill indices for some graphs and honeycomb networks are determined.

2. Results for Some Standard Graphs

Proposition 1. Let G be r-regular with n vertices and $r \geq 2$. Then

$$GADW(G) = \frac{nr}{4}.$$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v) = n-1$ for every v in G.

From definition,

$$GADW(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_{dn}(u)d_{dn}(v)}}{d_{dn}(u) + d_{dn}(v)}$$

$$= \frac{nr}{2} \frac{\sqrt{(n-1)(n-1)}}{(n-1)+(n-1)}$$

$$= \frac{nr}{4}.$$

Corollary 1.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$GADW(C_n) = \frac{n}{2}.$$

Corollary 1.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$GADW(K_n) = \frac{n(n-1)}{4}.$$

Proposition 2. Let P_n be a path with $n \geq 3$ vertices. Then

$$GADW(P_n) = n-3.$$

Proof: Let P_n be a path with $n \geq 3$ vertices. Clearly, P_n has two types of edges based on the downhill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(P_n) | \ d_{dn}(u)=0, \ d_{dn}(v) = n-1\}, |E_1| = 2.$
$E_2 = \{uv \in E(P_n) | \ d_{dn}(u)= d_{dn}(v) = n-1 \}, |E_2| = n-3.$
Then

$$GADW(P_n) = \sum_{uv \in E(P_n)} \frac{2\sqrt{d_{dn}(u) d_{dn}(v)}}{d_{dn}(u)+d_{dn}(v)}$$
$$= 2\frac{2\sqrt{0 \times (n-1)}}{0+(n-1)}$$
$$+(n-3)\frac{2\sqrt{(n-1)\times(n-1)}}{(n-1)+(n-1)}$$
$$= n-3.$$

Proposition 3. Let $K_{m,n}$ be a complete bipartite graph with $m < n$. Then
$$GADW(K_{m,n}) = 0.$$

Proof. Let $K_{m,n}$ be a complete bipartite graph with $m < n$. There are $m+n$ vertices and mn edges. Clearly, $K_{m,n}$ has one type of edges based on the downhill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(K_{m,n}) | \ d_{dn}(u)=0, \ d_{dn}(v) = n\}, |E_1| = mn.$

Then

$$GADW(K_{m,n}) = \sum_{uv \in E(K_{m,n})} \frac{2\sqrt{d_{dn}(u) d_{dn}(v)}}{d_{dn}(u)+d_{dn}(v)}$$
$$= mn\frac{2\sqrt{0 \times n}}{0+n} = 0.$$

3. Results for Wheel Graphs

Let W_n be a wheel with n+1vertices and $2n$ edges, $n \geq 4$. Then there are two types of edges based on the downhill degree of end vertices of each edge as follows:
$E_1 = \{uv \in E(W_n) \mid d_{dn}(u) = n, \ d_{dn}(v) = n-1\}, | E_1 | = n.$
$E_2 = \{uv \in E(W_n) \mid d_{dn}(u) = d_{dn}(v) = n-1\}, | E_2 | = n.$

Theorem 1. Let W_n be a wheel with n+1vertices and $2n$ edges, $n \geq 4$. Then
$$GADW(W_n) = \frac{2n\sqrt{n(n-1)}}{2n-1} + n.$$
Proof. From definition,
$$GADW(W_n) = \sum_{uv \in E(W_n)} \frac{2\sqrt{d_{dn}(u) d_{dn}(v)}}{d_{dn}(u)+d_{dn}(v)}$$
$$= \frac{n2\sqrt{n(n-1)}}{n+(n-1)} + \frac{n2\sqrt{(n-1)(n-1)}}{(n-1)+(n-1)}$$
$$= \frac{2n\sqrt{n(n-1)}}{2n-1} + n.$$

Theorem 2. Let W_n be a wheel with n+1vertices and $2n$ edges, $n \geq 4$. Then
$${}^{m}DWM_1(W_n) = \frac{n}{2n-1} + \frac{n}{2(n-1)}.$$
Proof. From definition,
$${}^{m}DWM_1(W_n) = \sum_{uv \in E(W_n)} \frac{1}{d_{dn}(u)+d_{dn}(v)}$$
$$= \frac{n}{n+(n-1)} + \frac{n}{(n-1)+(n-1)}$$
$$= \frac{n}{2n-1} + \frac{n}{2(n-1)}.$$

4. 4. Results for Gear Graphs

A bipartite wheel graph is a graph obtained from W_n with n+1 vertices adding a vertex between each pair of adjacent rim vertices and this graph is denoted by G_n and also called as a gear graph. Clearly, $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$. A gear graph G_n is depicted in Figure 1.

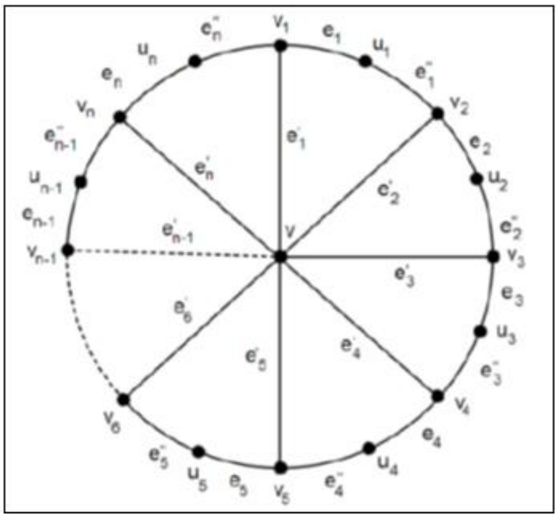

Figure 1

Let G_n be a gear graph with $2n$+1vertices, $3n$ edges, $n \geq 4$. Then G_n has two types of edges based on the downhill degree of the vertices of each edge as follows:

$E_1 = \{u \in E(G_n) \mid d_{dn}(u) = 2n, \ d_{dn}(v) = 2\}, | E_1 | = n.$
$E_2 = \{u \in E(G_n) \mid d_{dn}(u) = 2, , \ d_{dn}(v) = 0\}, | E_2 | = 2n.$

Theorem 3. Let G_n be a gear graph with $2n$+1vertices, $3n$ edges, $n \geq 4$. Then the inverse sum indeg downhill index of G_n is
$$GADW(G_n) = \frac{2n\sqrt{n}}{n+1}.$$

Proof: From definition,
$$GADW(G_n) = \sum_{uv \in E(G_n)} \frac{2\sqrt{d_{dn}(u) d_{dn}(v)}}{d_{dn}(u)+d_{dn}(v)}$$

Paper ID: SR25719182626 DOI: https://dx.doi.org/10.21275/SR25719182626

$$= \frac{n2\sqrt{2n\times 2}}{2n+2} + \frac{2n2\sqrt{2\times 0}}{2+0}$$
$$= \frac{2n\sqrt{n}}{n+1}.$$

Theorem 4. Let G_n be a gear graph with $2n+1$vertices, $3n$ edges, $n\geq4$. Then the inverse sum indeg downhill polynomial of G_n is

$${}^{m}DWM_{1}(G_n) = \frac{n}{2n+2} + n.$$

Proof: From definition,

$${}^{m}DWM_1(G_n) = \sum_{uv\in E(G_n)} \frac{1}{d_{dn}(u)+d_{dn}(v)}$$
$$= \frac{n}{2n+2} + \frac{2n}{2+0}$$
$$= \frac{n}{2n+2} + n.$$

5. Results for Helm Graphs

The helm graph H_n is a graph obtained from W_n (with $n+1$ vertices) by attaching an end edge to each rim vertex of W_n. Clearly, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. A graph H_n is shown in Figure 2.

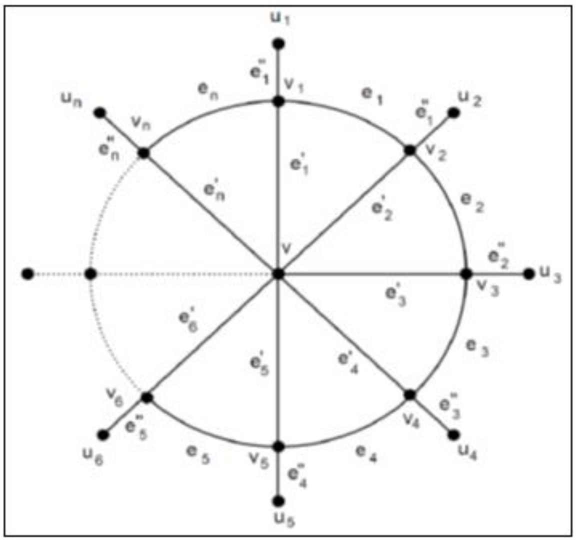

Figure 2

Let H_n be a helm graph with $3n$ edges, $n\geq5$. Then H_n has three types of edges based on the downhill degree of the vertices of each edge as follows:

$E_1 = \{uv \in E(H_n) \mid d_{dn}(u) = 2n,\ d_{dn}(v) = 2n - 1\}$, $|E_1| = n$.
$E_2 = \{uv \in E(H_n) \mid d_{dn}(u) = d_{dn}(v) = 2n - 1\}$, $|E_2| = n$.
$E_3 = \{uv \in E(H_n) \mid d_{dn}(u) = 2n - 1,\ d_{dn}(v) = 0\}$, $|E_3| = n$.

Theorem 5. Let H_n be a helm graph with $2n+1$ vertices, $n\geq5$. Then the inverse sum indeg downhill index of H_n is

$$GADW(H_n) = \frac{2n\sqrt{2n(2n-1)}}{4n-1} + n.$$

Proof: From definition,

$$GADW(H_n) = \sum_{uv\in E(H_n)} \frac{2\sqrt{d_{dn}(u)d_{dn}(v)}}{d_{dn}(u)+d_{dn}(v)}$$
$$= \frac{n\times 2\sqrt{2n\times(2n-1)}}{2n+(2n-1)}$$
$$+ \frac{n\times 2\sqrt{(2n-1)\times(2n-1)}}{(2n-1)+(2n-1)}$$
$$+ \frac{n\times 2\sqrt{(2n-1)\times 0}}{(2n-1)+0}$$
$$= \frac{2n\sqrt{2n(2n-1)}}{4n-1} + n.$$

Theorem 6. Let H_n be a helm graph with $2n+1$vertices, $3n$ edges, $n\geq5$. Then the inverse sum indeg downhill polynomial of H_n is

$${}^{m}DWM_1(H_n) = \frac{n}{4n-1} + \frac{3n}{2(2n-1)}.$$

Proof: From definition,

$${}^{m}DWM_1(H_n) = \sum_{uv\in E(H_n)} \frac{1}{d_{dn}(u)+d_{dn}(v)}$$
$$= \frac{n}{2n+(2n-1)} + \frac{n}{(2n-1)+(2n-1)} + \frac{n}{(2n-1)+0}$$
$$= \frac{n}{4n-1} + \frac{3n}{2(2n-1)}.$$

6. Results for Honeycomb Networks

Honeycomb networks are very useful in computer graphics and also in chemistry. A honeycomb network of dimension n is denoted by HCn where n is the number of hexagons between central and boundary hexagon.

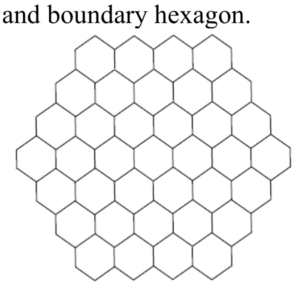

Figure 3: A 4-dimensional honeycomb network

Let H be the graph of honeycomb network HCn where $n \geq 3$. By calculation, we obtain that H has $6n^2$ vertices and $9n^2 - 3n$ edges. Then there are four types of edges based on the downhill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(H) \mid d_{dn}(u) = 1,\ d_{dn}(v) = 1\}$, $|E_1| = 6$.
$E_2 = \{uv \in E(H) \mid d_{dn}(u) = 1,\ d_{dn}(v) = 6\ n^2 - 1\}$, $|E_2| = 12$.
$E_3 = \{uv \in E(H) \mid d_{dn}(u) = 0,\ d_{dn}(v) = 6\ n^2 - 1\}$, $|E_2| = 12(n - 2)$.
$E_4 = \{uv \in E(H) \mid d_{dn}(u) = d_{dn}(v) = 6\ n^2 - 1\}$, $|E_4| = 9n^2 - 15n+6$.

Paper ID: SR25719182626 DOI: https://dx.doi.org/10.21275/SR25719182626

Theorem 7. Let H be a honeycomb network with $6n^2$ vertices, $n\geq4$. Then

$$GADW(H)=\frac{4\sqrt{(6n^2-1)}}{n^2}+\frac{(9n^2-15n+18)}{2}.$$

Proof. From definition,

$$GADW(H)=\sum_{uv\in E(H)}\frac{2\sqrt{d_{dn}(u)d_{dn}(v)}}{d_{dn}(u)+d_{dn}(v)}$$

$$=\frac{6\times2\sqrt{1\times1}}{1+1}+\frac{12\times2\sqrt{1\times(6n^2-1)}}{1+(6n^2-1)}$$

$$+\frac{12(n-2)\times2\sqrt{0\times(6n^2-1)}}{0+(6n^2-1)}$$

$$+\frac{(9n^2-15n+6)\sqrt{(6n^2-1)\times(6n^2-1)}}{(6n^2-1)+(6n^2-1)}$$

$$=\frac{4\sqrt{(6n^2-1)}}{n^2}+\frac{(9n^2-15n+18)}{2}.$$

Theorem 8. Let H be a honeycomb network with $6n^2$ vertices, $n\geq4$. Then

$${}^{m}DWM_1(H)=3+\frac{2}{n^2}+\frac{12(n-2)}{(6n^2-1)}$$

$$+\frac{(9n^2-15n+6)}{2(6n^2-1)}.$$

Proof. From definition,

$${}^{m}DWM_1(H)=\sum_{uv\in E(H)}\frac{1}{d_{dn}(u)+d_{dn}(v)}$$

$$=\frac{6}{1+1}+\frac{12}{1+(6n^2-1)}+\frac{12(n-2)}{0+(6n^2-1)}$$

$$+\frac{(9n^2-15n+6)}{(6n^2-1)+(6n^2-1)}$$

$$=3+\frac{2}{n^2}+\frac{12(n-2)}{(6n^2-1)}+\frac{(9n^2-15n+6)}{2(6n^2-1)}.$$

7. Conclusion

In this paper, the geometric-arithmetic downhill and modified first downhill indices are defined. Also the geometric-arithmetic downhill and modified first downhill indices of some standard graphs, wheel graphs, gear graphs, helm graphs and honeycomb networks are determined.

References

[1] Al-Ahmadi B., Saleh A., Al-Shammakh W., (2021). Downhill Zagreb topological indices of graphs, *International Journal of Analysis and Applications,* 19(2), 205-227.

[2] Vukičević D., Furtula B., (2009). Topological index based on the ratios of geometrical and arithmetical means of end vertex degrees of edges, J.Math. Chem., 46, 1369-1376.

[3] Al-Ahmadi B., Saleh A., Al-Shammakh W., (2021). Downhill Zagreb topological indices and Mdn-Polynomial of some chemical structures applied to the treatment of COVID-19 patients, *Open Journal of Applied Sciences,* 11(4), 395-413.

[4] Kulli VR., (2025). Downhill Sombor and modified downhill Sombor indices of graphs, Annals of Pure and Applied Mathematics, 31(2), 107-112.

[5] Kulli VR., (2025). Harmonic downhill indices of graphs, Journal of Mathematics and Informatics, 28, 33-37.

[6] Kulli VR., (2025). Hyper downhill indices and their polynomials of certain chemical drugs, *International Journal of Engineering Sciences & Research Technology,* 14(4), 23-30.

[7] Kulli VR., (2025). Downhill Nirmala indices of graphs, *International Journal of Mathematics and Computer Research,* 13(4), 5126-5131.

[8] Kulli VR., (2025). Downhill product connectivity indices of graphs, *International Journal of Mathematics and Computer Research,* 13(5), 5223-5226.

[9] Kulli VR., (2025). Downhill Nirmala alpha Gourava indices of chloroquine, hydroxychloroquine and remdesivir, *International Journal of Mathematics and Computer Research,* 13(6), 5276-5284.

[10] Kulli VR., (2025). Inverse sum indeg downhill index of graphs, *International Journal of Science and Research,* 14(6), 1159-1162.

[11] Kulli VR., (2025). Gourava downhill indices, *International Journal of Engineering Sciences & Research Technology,* 14(6), 19-29.