Pythagorean Fuzzy Sublattices and Ideals

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Abstract

In this research paper, we introduce the concepts of Pythagorean fuzzy sublattices and Pythagorean fuzzy ideals within the context of a lattice. We proceed to establish various properties associated with these Pythagorean fuzzy sublattices and ideals. Moreover, we introduce the notion of (γ, δ) level set and (γ, δ) strong level sets, accompanied by relevant theorems. Lastly, we define Pythagorean fuzzy convex sublattices and engage in a discussion concerning their properties..
Keywords: Lattice, Pythagorean fuzzy sets, Fuzzy Sublattice, Fuzzy ideals.

1 Introduction

The foundational notion of fuzzy sets was initially introduced by [3] .This concept found applications in various algebraic structures, such as groups, rings, semigroups, and lattices, as explored by multiple authors .Within the lattice theory, [1] pioneered the development of fuzzy sets and delved into the realm of fuzzy sublattices . Additionally, [2] introduced the concept of intuitionistic fuzzy sets as a potent tool for addressing uncertainty, extending the idea of fuzzy sets. Subsequently, [6] expanded upon this foundation, introducing the theory of intuitionistic fuzzy sublattices and ideals. In intuitionistic fuzzy sets, the sum of an element's membership grade and non-membership grade always remains within the unit interval [0, 1]. However, there exist scenarios where the sum of the membership function and non-membership function surpasses one. To tackle such situations, a more comprehensive model was needed, leading [4, 7] to introduce the concept of Pythagorean fuzzy sets (PFS), which serves as a generalization of intuitionistic fuzzy sets (IFS). The principal distinction between IFS and PFS lies in the fact that, in PFS, the sum of the membership function and non-membership function cannot exceed one. Instead, the sum of the squares of the membership function and non-membership function is constrained to be less than or equal to one. This paper explores the concepts of Pythagorean fuzzy sublattices (PFL) and Pythagorean fuzzy ideals (PFI) within a lattice framework, establishing their fundamental properties.

2 preliminaries

In this section some elementary notions such as Intuitionistic Fuzzy set(IFS), Pythagorean fuzzy set (PFS) are given

2.1 Definition

[2] Let M be a nonempty set. An Intuitionistic fuzzy set (IFS) N in M is an object of the form $\{< m, \varphi_N(m), \omega_N(m) > /m \in M\}$ where $\varphi_N(m) : m \to [0,1]$ and $\omega_N(m) : m \to [0,1]$ represents degree of the membership function and non-membership function resp, with the condition that $\varphi_N(m) + \omega_N(m) \leq 1$

2.2 Definition

[2] Let $\{< m, \varphi_{N_1}(m), \omega_{N_1}(m) > /m \in M\}$ and $\{< m, \varphi_{N_2}(m), \omega_{N_2}(m) > /m \in M\}$ are two intuitionistic fuzzy sets on M, then for all $m, n \in M$

1. $N_1 \subseteq N_2 \Longrightarrow \varphi_{N_1}(m) \leq \varphi_{N_2}(m)$ and $\omega_{N_1}(m) \geq \omega_{N_2}(m)$
2. $N_1 = N_2 \Longrightarrow \varphi_{N_1}(m) = \varphi_{N_2}(m)$ and $\omega_{N_1}(m) = \omega_{N_2}(m)$
3. ${N_1}^c = \{< m, \omega_{N_1}(m), \varphi_{N_1}(m) > /m \in M\}$
4. $N_1 \cap N_2 = \{< m, \varphi_{N_1 \cap N_2}(m), \omega_{N_1 \cap N_2}(m) > /m \in M\}$ where $\varphi_{N_1 \cap N_2}(m) = min\{\varphi_{N_1}(m), \varphi_{N_2}(m)\}$ and $\omega_{N_1 \cap N_2}(m) = max\{\omega_{N_1}(m), \omega_{N_2}(m)\}$
5. $N_1 \cup N_2 = \{< m, \varphi_{N_1 \cup N_2}(m), \omega_{N_1 \cup N_2}(m) > /m \in M\}$ where $\varphi_{N_1 \cup N_2}(m) = max\{\varphi_{N_1}(m), \varphi_{N_2}(m)\}$ and $\omega_{N_1 \cup N_2}(m) = min\{\omega_{N_1}(m), \omega_{N_2}(m)\}$
6. $[N] = \{< m, \varphi_N(m), 1 - \varphi_N(m) > m \in M\}$
7. $< N >= \{< m, 1 - \omega_N(m), \omega_N(m) > m \in M\}$

2.3 Definition

[6] Let T be a lattice and $A = \{< m, \varphi_A, \omega_A > /m \in T\}$ be a IFS of T. Then A is called an intuitionistic fuzzy sublattice if the following conditions are satisfied for all $m,\ n \in T$

1. $\varphi_A(m \vee n) \geq min\{\varphi_A(m), \varphi_A(n)\}$
2. $\varphi_A(m \wedge n) \geq min\{\varphi_A(m), \varphi_A(n)\}$

3. $\omega_A(m \vee n) \leq max\{\omega_A(m), \omega_A(n)\}$
4. $\omega_A(m \wedge n) \leq max\{\omega_A(m), \omega_A(n)\}$

2.4 Definition

[6] Let T be a lattice and $A = \{< m, \varphi_A, \omega_A > /m \in T\}$ be a IFS of T. Then A is called an intuitionistic fuzzy ideal if the following conditions are satisfied for all $m,\ n \in T$

1. $\varphi_A(m \vee n) \geq min\{\varphi_A(m), \varphi_A(n)\}$
2. $\varphi_A(m \wedge n) \geq max\{\varphi_A(m), \varphi_A(n)\}$
3. $\omega_A(m \vee n) \leq max\{\omega_A(m), \omega_A(n)\}$
4. $\omega_A(m \wedge n) \leq min\{\omega_A(m), \omega_A(n)\}$

2.5 Definition

[7, 4] Let M be a nonempty set. A Pythagorean fuzzy set (PFS) G in M is an object of the form $\{< m, \varphi_G(m), \omega_G(m) > /m \in M\}$ where $\varphi_G(m) : M \to [0, 1]$ and $\omega_G(m) : M \to [0, 1]$ represents degree of the membership function and non-membership function resp, with the condition that $\varphi_G^2(m) + \omega_G^2(m) \leq 1$

2.6 Definition

[4, 7, 5] Let $\{< m, \varphi_{G_1}(m), \omega_{G_1}(m) > /m \in M\}$ and $\{< m, \varphi_{G_2}(m), \omega_{G_2}(m) > /m \in M\}$ are two Pythagorean fuzzy sets on M, then for all $m, n \in M$

1. $G_1 \subseteq G_2 \Longrightarrow \varphi_{G_1}(m) \leq \varphi_{G_2}(m)$ and $\omega_{G_1}(m) \leq \omega_{G_2}(m)$
2. $G_1 = G_2 \Longrightarrow \varphi_{G_1}(m) = \varphi_{G_2}(m)$ and $\omega_{G_1}(m) = \omega_{G_2}(m)$
3. ${G_1}^c = \{< m, \omega_G(m), \varphi_G(m) > /m \in M\}$
4. $G_1 \cap G_2 = \{< m, \varphi_{G_1 \cap G_2}(m), \omega_{G_1 \cap G_2}(m) > /m \in M\}$ where $\varphi_{G_1 \cap G_2}(m) = min\{\varphi_{G_1}(m), \varphi_{G_2}(m)\}$ and $\omega_{G_1 \cap G_2}(m) = max\{\omega_{G_1}(m), \omega_{G_2}(m)\}$
5. $G_1 \cup N_2 = \{< m, \varphi_{G_1 \cup G_2}(m), \omega_{G_1 \cup G_2}(m) > /m \in M\}$ where $\varphi_{G_1 \cup G_2}(m) = max\{\varphi_{G_1}(m), \varphi_{G_2}(m)\}$ and $\omega_{G_1 \cup G_2}(m) = min\{\omega_{G_1}(m), \omega_{G_2}(m)\}$
6. $[G] = \{< m, \varphi_G(m), (1 - \varphi_G^2(m))^{0.5} > /m \in M\}$
7. $< G >= \{< m, (1 - \omega_G^2(m))^{0.5}, \omega_G(m) > /m \in M\}$

3 pythagorean fuzzy lattice and ideals

In this section we define Pythagorean fuzzy sublattices (PFL) and Pythagorean fuzzy ideals (PFI) and study their characterizations. Through out this paper T stands for the lattice $(T, \vee, \wedge)$, where $\vee$, $\wedge$ represents join and meet.

3.1 Definition

Let T be a lattice and $G = \{< m, \varphi_G, \omega_G > /m \in T\}$ be a PFS of T. Then G is called an Pythagorean fuzzy sublattice if the following conditions are satisfied for all $m,\ n \in T$

1. $\varphi_G^2(m \vee n) \geq min\{\varphi_G^2(m), \varphi_G^2(n)\}$
2. $\varphi_G^2(m \wedge n) \geq min\{\varphi_G^2(m), \varphi_G^2(n)\}$
3. $\omega_G^2(m \vee n) \leq max\{\omega_G^2(m), \omega_G^2(n)\}$
4. $\omega_G^2(m \wedge n) \leq max\{\omega_G^2(m), \omega_G^2(n)\}$

3.2 Example

Consider the lattice "divisors of 12" $T = \{1, 2, 3, 4, 6, 12\}$ and $G = \{< 1, 0.9, 0.3 >, < 2, 0.8, 0.4 >, < 3, 0.9, 0.3 >, < 4, 0.7, 0.6 >, < 6, 0.8, 0.4 >, < 12, 0.8, 0.4 >\}$ be a PFS of T. G satisfies the four conditions and G is a PFL

3.3 Definition

Let T be a lattice and $G = \{< m, \varphi_G, \omega_G > /m \in T\}$ be a PFS of T. Then G is called an Pythagorean fuzzy ideal (PFI) if the following conditions are satisfied for all $m,\ n \in T$

1. $\varphi_G^2(m \vee n) \geq min\{\varphi_G^2(m), \varphi_G^2(n)\}$
2. $\varphi_G^2(m \wedge n) \geq max\{\varphi_G^2(m), \varphi_G^2(n)\}$
3. $\omega_G^2(m \vee n) \leq max\{\omega_G^2(m), \omega_G^2(n)\}$
4. $\omega_G^2(m \wedge n) \leq min\{\omega_G^2(m), \omega_G^2(n)\}$

3.4 Example

Consider the lattice "divisors of 10" $T = \{1, 2, 5, 10\}$ and $G = \{< 1, 0.9, 0.4 >, < 2, 0.8, 0.4 >, < 5, 0.7, 0.4 >, < 10, 0.7, 0.4 >\}$ be a PFS of T. It is easy to verify that G is a PFI of T

3.5 Theorem

If $G = \{< m, \varphi_G, \omega_G > /m \in T\}$ is an IFL, then it is a PFL
Proof: Consider five cases
Case 1 : $\varphi_G(m) > \varphi_G(n)$ and $\omega_G(m) > \omega_G(n)\ \forall m, n \in T$

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$$
\begin{aligned}
&\Rightarrow \varphi_G^2(m) > \varphi_G^2(n) and \omega_G^2(m) > \omega_G^2(n)\\
&\varphi_G(m \vee n) \geq \min\{(\varphi_G(m), \varphi_G(n)\} = \varphi_G(n)\\
&\Rightarrow \varphi_G^2(m \vee n) \geq \varphi_G^2(n) = \min\{(\varphi_G^2(m), \varphi_G^2(n)\}\\
&\varphi_G(m \wedge n) \geq \min\{(\varphi_G(m), \varphi_G(n)\} = \varphi_G(n)\\
&\Rightarrow \varphi_G^2(m \wedge n) \geq \varphi_G^2(n) = \min\{(\varphi_G^2(m), \varphi_G^2(n)\}\\
&\omega_G(m \vee n) \leq \max\{(\omega_G(m), \omega_G(n)\} = \omega_G(m)\\
&\Rightarrow \omega_G^2(m \vee n) \leq \omega_G^2(n) = \max\{(\omega_G^2(m), \omega_G^2(n)\}\\
&\omega_G(m \wedge n) \leq \max\{(\omega_G(m), \omega_G(n)\} = \omega_G(m)\\
&\Rightarrow \omega_G^2(m \wedge n) \leq \omega_G^2(m) = \max\{(\omega_G^2(m), \omega_G^2(n)\}
\end{aligned}
$$

Case 2: $\varphi_G(m) < \varphi_G(n)$ and $\omega_G(x) < \omega_G(n)$ $\forall m, n \in T$

$$
\begin{aligned}
&\Rightarrow \varphi_G^2(m) < \varphi_G^2(n) \ and\ \omega_G^2(m) < \omega_G^2(n)\\
&\varphi_G(m \vee n) \geq \min\{(\varphi_G(m), \varphi_G(n)\} = \varphi_G(m)\\
&\Rightarrow \varphi_G^2(m \vee n) \geq \varphi_G^2(m) = \min\{(\varphi_G^2(m), \varphi_G^2(n)\}\\
&\varphi_G(m \wedge n) \geq \min\{(\varphi_G(m), \varphi_G(n)\} = \varphi_G(m)\\
&\Rightarrow \varphi_G^2(m \wedge n) \geq \varphi_G^2(m) = \min\{(\varphi_G^2(m), \varphi_G^2(n)\}\\
&\omega_G(m \vee n) \leq \max\{(\omega_G(m), \omega_G(n)\} = \omega_G(n)\\
&\Rightarrow \omega_G^2(m \vee n) \leq \omega_G^2(n) = \max\{(\omega_G^2(m), \omega_G^2(n)\}\\
&\omega_G(m \wedge n) \leq \max\{(\omega_G(m), \omega_G(n)\} = \omega_G(n)\\
&\Rightarrow \omega_G^2(m \wedge n) \leq \omega_G^2(n) = \max\{(\omega_G^2(m), \omega_G^2(n)\}
\end{aligned}
$$

case 3 $\varphi_G(m) = \varphi_G(n)$ and $\omega_G(m) = \omega_G(n)$ $\forall m, n \in T$

$$
\begin{aligned}
&\Rightarrow \varphi_G^2(m) = \varphi_G^2(n) \ and \omega_G^2(m) = \omega_G^2(n)\\
&\varphi_G(m \vee n) \geq \min\{(\varphi_G(m), \varphi_G(n)\} = \varphi_G(m) = \varphi_G(n)\\
&\Rightarrow \varphi_G^2(m \vee n) \geq \varphi_G^2(m) = \varphi_G^2(n) = \min\{(\varphi_G^2(m), \varphi_G^2(n)\}\\
&\varphi_G(m \wedge n) \geq \max\{(\varphi_G(m), \varphi_G(n)\} = \varphi_G(m) = \varphi_G(n)\\
&\Rightarrow \varphi_G^2(m \wedge n) \geq \varphi_G^2(m) = \varphi_G^2(n) = \max\{(\varphi_G^2(m), \varphi_G^2(n)\}\\
&\omega_G(m \vee n) \leq \max\{(\omega_G(m), \omega_G(n)\} = \omega_G(m) = \omega_G(n)\\
&\Rightarrow \omega_G^2(m \vee n) \leq \omega_G^2(m) = \omega_G^2(n) = \max\{(\omega_G^2(m), \omega_G^2(n)\}\\
&\omega_G(m \wedge n) \leq \max\{(\omega_G(m), \omega_G(y)\} = \omega_G(m) = \varphi_G(y)\\
&\Rightarrow \omega_G^2(m \wedge y) \leq \omega_G^2(m) = \omega_G^2(y) = \max\{(\omega_G^2(m), \omega_G^2(y)\}
\end{aligned}
$$

Similar is the proof for other cases. In all the cases the conditions for G to be a PFL is satisfied and hence it is a PFL of T

Remark

Every IFL of T is a PFL, but the converse need not be true.

3.6 Example

Consider the lattice "divisors of 12" $T = \{1, 2, 3, 4, 6, 12\}$and $G = \{< 1, 0.9, 0.3 >, < 2, 0.8, 0.4 >, < 3, 0.9, 0.3 >, < 4, 0.7, 0.6 >, < 6, 0.8, 0.4 >, < 12, 0.8, 0.4 >\}$ be a PFS of T. G satisfies the four conditions and G is a PFL Here 0.9+0.3 = 1.2, greater than one, so G is not an IFS and hence it is not a IFL of T. This example shows that PFL of T may not be IFL of T

3.7 Theorem

If G_1, G_2 are two PFL of T, then $G_1 \cap G_2$ is also a PFL of T (PFI of T)
Proof: Let $\{< m, \varphi_{G_1}(m), \omega_{G_1}(m) > /m \in M\}$ and $\{< m, \varphi_{G_2}(m), \omega_{G_2}(m) > /m \in M\}$ are two Pythagorean fuzzy sets on M, then by definition 2.4,
$G_1 \cap G_2 = \{< m, \varphi_{G_1 \cap G_2}(m), \omega_{G_1 \cap G_2}(m) > /m \in M\}$
where $\varphi_{G_1 \cap G_2}(m) = min\{\varphi_{G_1}(m), \varphi_{G_2}(m)\}$ and $\omega_{G_1 \cap G_2}(m) = max\{\omega_{G_1}(m), \omega_{G_2}(m)\}$
Note that
$\varphi^2_{G_1 \cap G_2}(m) = min\{\varphi^2_{G_1}(m), \varphi^2_{G_2}(m)\}$ and $\omega^2_{G_1 \cap G_2}(m) = max\{\omega^2_{G_1}(m), \omega^2_{G_2}(m)\}$

$$\begin{aligned}
\varphi^2_{G_1 \cap G_2}(m \vee n) &= \min\{\varphi^2_{G_1}(m \vee n),\ \varphi^2_{G_2}(m \vee n)\} \\
&\geq \min\{\min\{\varphi^2_{G_1}(m), \varphi^2_{G_1}(n)\}, \min\{\varphi^2_{G_2}(m), \varphi^2_{G_2}(n)\} \\
&= \min\{\min\{\varphi^2_{G_1}(m), \varphi^2_{G_2}(m)\}, \min\{\varphi^2_{G_1}(n), \varphi^2_{G_2}(n)\} \\
&= \min\left\{\varphi^2_{G_1 \cap G_2}(m), \varphi^2_{G_1 \cap G_2}(n)\right\}
\end{aligned}$$

therefore $\varphi^2_{G_1 \cap G_2}(m \vee n) \geq \min\left\{\varphi^2_{G_1 \cap G_2}(m), \varphi^2_{G_1 \cap G_2}(n)\right\}$
Similarly, we can prove that
$\varphi^2_{G_1 \cap G_2}(m \wedge n) \geq \min\left\{\varphi^2_{G_1 \cap G_2}(m), \varphi^2_{G_1 \cap G_2}(n)\right\}$

$$\begin{aligned}
\omega^2_{G_1 \cap G_2}(m \vee n) &= \max\left\{\ \omega^2_{G_1}(m \vee n),\ \omega^2_{G_2}\ (m \vee n)\right\} \\
&\leq \max\left\{\max\{\omega^2_{G_1}(m),\ \omega^2_{G_1}(n)\}, \max\{\omega^2_{G_2}(m),\ \omega^2_{G_2}(n)\right\} \\
&= \max\{\max\{\omega^2_{G_1}(m), \omega^2_{G_2}(m)\}, \max\{\omega^2_{G_1}(n), \omega^2_{G_2}(n)\} \\
&= \max\left\{\omega^2_{G_1 \cap G_2}(m), \omega^2_{G_1 \cap G_2}(n)\right\}
\end{aligned}$$

therefore $\omega^2_{G_1 \cap G_2}(m \vee n) \leq \max\left\{\omega^2_{G_1 \cap G_2}(m),\ \omega^2_{G_1 \cap G_2}(n)\right\}$
Similiarly we can prove that

$\omega^2_{G_1 \cap G_2}(m \wedge n) \leq \max\left\{\omega^2_{G_1 \cap G_2}(m),\ \omega^2_{G_1 \cap G_2}(n)\right\}$
All the conditions of PFL are satisfied for $G_1 \cap G2$ and hence it is a PFL.

Remark 1

Union of two PFL need not be a PFL.
$G_1 = \{<1, 0.9, 0.3>, <2, 0.8, 0.4>, <3, 0.9, 0.3>, <4, 0.7, 0.6>, <6, 0.8, 0.4>, <12, 0.8, 0.4>\}$ and
$G_2 = \{<1, 0.8, 0.4>, <2, 0.9, 0.3>, <3, 0.8, 0.4>, <4, 0.9, 0.3>,\ <6, 0.9, 0.3>, <12, 0.8, 0.4>\}$
$G_1 \cup G_2 = \{<1, 0.9, 0.3>, <2, 0.9, 0.3>, <3, 0.9, 0.3>, <4, 0.9, 0.3>, <6, 0.9, 0.3>, <12, 0.8, 0.4>\}$
here $\varphi^2(4 \vee 3) = \varphi^2(12) = (0.8)^2 = 0.64$,
$\min\{\{\varphi^2(4),\ \varphi^2(3)\} = \{0.81, 0.81\} = 0.81$
$\varphi^2(4 \vee 3) = 0.64 \leq \min\{\varphi(4), \varphi(3)\} = 0.81$ and hence $G_1 \cup G_2$ is not a PFL

Remark 2

Every PFI of T is a PFL, but the converse need not be true.
$G = \{<m, \varphi_G, \omega_G> /m \in T\}$ be a PFI of T,
then

1. $\varphi^2_G(m \vee n) \geq min\{\varphi^2_G(m), \varphi^2_G(n)\}$
2. $\varphi^2_G(m \wedge n) \geq max\{\varphi^2_G(m), \varphi^2_G(n)\} \geq min\{\varphi^2_G(m), \varphi^2_G(n)\}$
3. $\omega^2_G(m \vee n) \leq max\{\omega^2_G(m), \omega^2_G(n)\}$
4. $\omega^2_G(m \wedge n) \leq min\{\omega^2_G(m), \omega^2_G(n)\} \leq max\{\omega^2_G(m), \omega^2_G(n)\}$

hence G is a PFL.
Converse need not be true, for example Consider the lattice "divisors of 12" T={ 1, 2, 3, 4, 6, 12 } and
$G = \{<1, 0.9, 0.3>, <2, 0.8, 0.4>, <3, 0.9, 0.3>, <4, 0.7, 0.6>, <6, 0.8, 0.4>, <12, 0.8, 0.4>\}$
But $\varphi^2_G(4 \wedge 12) = \varphi^2_G(4) = (0.7)^2 = 0.49, \max(\varphi^2_G(4), \varphi^2_G(12)) = \max(0.49, 0.64) = 0.64$
Therefore $\varphi^2_G(4 \wedge 12) \leq \max(\varphi^2_G(4), \varphi^2_G(12))$ and hence it is not a PFI.

Remark 3

Union of two PFI of T need not be a PFI. For example, consider
$G_1 = \{\ <1,\ 0.9,\ 0.4>,\ <2,\ 0.8,\ 0.4>,\ <5,\ 0.8,\ 0.5>,\ <10,\ 0.7,\ 0.4>\}$
$G_2 = \{\ <1,\ 0.8,\ 0.4>,\ <2,\ 0.7,\ 0.4>,\ <5,\ 0.7,\ 0.4>,\ <10,\ 0.6,\ 0.5>\}$
$G_1 \cup G_2 = \{<1,\ 0.9,\ 0.4>,\ <2,\ 0.8,\ 0.4>,\ <5,\ 0.8,\ 0.4>,\ <10,\ 0.7,\ 0.4>\}$
which is not a PFI ,
since $0.49 = \varphi^2_{G_1 \cup G_2}(10) =\ \varphi^2_{G_1 \cup G_2}(2 \vee 5) \leq \min\left\{\ \varphi^2_{G_1 \cup G_2}(2),\ \ \varphi^2_{G_1 \cup G_2}(5)\right\} = \min\{0.64,\ 0.64\} = 0.64$.

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3.8 Proposition

G is a PFL(PFI) of T if and only if $[G]$ and $< G >$ are PFL(PFI) of T
proof:Assume that G is a PFL of T.
$[G] = \{< m, \varphi_G(m), (1 - \varphi_G^2(m))^{0.5} > /m \in M\}$

$$\begin{aligned}
\varphi_G^2\,(m \vee n) &\geq \min\{(\varphi_G^2\,(m)\,,\ \varphi_G^2\,(n)\} \\
\varphi_G^2\,(m \wedge n) &\geq \min\{(\varphi_G^2\,(m)\,,\ \varphi_G^2\,(n)\} \\
1 - \varphi_G^2\,(m \vee y) &\leq 1 - \min\{(\varphi_G^2\,(m)\,,\ \varphi_G^2\,(n)\} \\
&= max(1 - \varphi_G^2\,(m)\,,\ 1 - \varphi_G^2\,(n) \\
\Longrightarrow 1 - \varphi_G^2\,(m \vee n) &\leq \left\{\left(1 - \varphi_G^2(m)\right),\ 1 - \varphi_G^2(n)\right)\right\}
\end{aligned}$$

Similarly, we can prove that

$$1 - \varphi_G^2\,(m \wedge n) \leq \max\left\{\left(1 - \varphi_G^2(m)\right),\ \left(1 - \varphi_G^2(n)\right)\right\}$$

Hence $[G] = \{< m, \varphi_G(m), (1 - \varphi_G^2(m))^{0.5} > /m \in M\}$ is a PFL.
$< G >= \{< m, (1 - \omega_G^2(m))^{0.5}, \omega_G(m) > /m \in M\}$

$$\begin{aligned}
\omega_G^2(m \vee n) &\leq \max\{(\omega_G^2(m),\ \omega_G^2(n)\} \\
\omega_G^2(m \wedge n) &\leq \max\{(\omega_G^2(m),\ \omega_G^2(n)\} \\
1 - \omega_G^2(m \vee n) &\geq 1 - \max\{(\omega_G^2(m),\ \omega_G^2(n)\} \\
&= \min\{1 - \omega_G^2(m),\ 1 - \omega_G^2(n)\} \\
\Longrightarrow (1 - \omega_G^2(m \vee n) &\geq \min\{(1 - \omega_G^2(m), (1 - \omega_G^2(n)\}
\end{aligned}$$

Similarly, we can show that

$$\left(1 - \omega_G^2\,(m \wedge n)\right) \geq \min\left\{\left(1 - \omega_G^2\,(m)\right),\ \left(1 -\ \omega_G^2\,(n)\right)\right\}$$

Therefore $< G >= \{< m, (1 - \omega_G^2(m))^{0.5}, \omega_G(m) > /m \in M\}$ is a PFL of T.
Conversely assume $[G] = \{< m, \varphi_G(m), (1 - \varphi_G^2(m))^{0.5} > /m \in M\}$ is a PFL of T,
then

$$\begin{aligned}
\varphi_G^2(m \vee n) &\geq \min\{(\varphi_G^2(m),\ \varphi_G^2(n)\} \\
\varphi_G^2(m \wedge n) &\geq \min\{(\varphi_G^2(m),\ \varphi_G^2(n)\} \\
\omega_G^2\,(m \vee n) &\leq \max\{\ \omega_G^2\,(m)\,,\ \omega_G^2\,(n)\} \\
\omega_G^2\,(m \wedge n) &\leq \max\{\ \omega_G^2\,(m)\,,\ \omega_G^2\,(n)\}
\end{aligned}$$

Hence G is a PFL(PFI)

3.9 Definition

Let G is PFS of any set M, then (γ, δ) level subset of G is defined as

$$G^{[\gamma,\delta]} = \{m \in M / \ \ \varphi_G(m) \geq \gamma, \omega_G(m) \leq \delta\}$$

where $(\gamma, \delta) \in [0,1] \times [0,1]$ and $\gamma^2 + \delta^2 \leq 1$ and (γ, δ) strong level set of G is defined as

$$G^{(\gamma,\delta)} = \{m \in M / \ \ \varphi_G(m) > \gamma, \omega_G(m) < \delta\}$$

where $(\gamma, \delta) \in [0,1] \times [0,1]$ and $\gamma^2 + \delta^2 \leq 1$
Upper $(\gamma, \ \delta)$ level subset of G is defined as

$$U(\varphi_G, \gamma) = \{m \in M \ / \ \ \varphi_G(m) \geq \gamma\}$$

and lower $(\gamma, \ \delta)$ level subset of G is defined as

$$T(\omega_G, \delta) = \{m \in M / \ \ \omega_G(m) \leq \delta\}$$

It is clear from the definition that $G^{[\gamma,\delta]} = U(\varphi_G, \gamma) \cap T(\omega_G, \delta)$

3.10 Theorem

A PFS, G of a lattice T is a PFL of T(PFI of T) if and only if each non empty $(\gamma, \ \delta)$ level subsets $G^{[\gamma,\delta]}$ is a sublattice (ideal) of T.
Proof: Assume that G is a PFL of T. Let $m, n \in G^{[\gamma,\delta]}$
then $\varphi_G(m) \geq \gamma, \omega_G(m) \leq \delta$ and $\varphi_G(n) \geq \gamma, \omega_G(m) \leq \delta$
$\varphi_G^2(m \vee n) \geq \min\{\varphi_G^2(m), \varphi_G^2(n)\} \geq \gamma^2$
$\Rightarrow \varphi_G(m \vee n) \geq \gamma$
and $\omega_G^2(m \vee n) \leq \max\{(\omega_G^2(m), \omega_G^2(n)\} \leq \delta^2$
$\Rightarrow \omega_G(m \vee n) \leq \delta$ and therefore $m \vee n \in G^[\gamma, \delta]$
Also
$\varphi_G^2(m \wedge n) \geq \min\{(\varphi_G^2(m), \ \varphi_G^2(n)\} \geq \gamma^2$
$\Rightarrow \varphi_G(m \wedge n) \geq \gamma$
and $\omega_G^2(m \wedge n) \leq \max\{(\omega_G^2(m), \omega_G^2(n)\} \leq \delta^2$
$\Rightarrow \omega_G(m \wedge n) \leq \delta$ and therefore $m \wedge n \in G^{[\gamma,\delta]}$
Hence $G^{[\gamma,\delta]}$ is a sublattice of T.
Similiarly assume that G is PFI of T.Let $m, n \in G^{[\gamma,\delta]}$ and $l \in T$
then as above $\varphi_G(m \vee n) \geq \gamma$ and $\omega_G(m \vee n) \leq \delta$ and therefore $m \vee n \in G^[\gamma, \delta]$
$\varphi_G^2(m \wedge l) \geq \min\{(\varphi_G^2(m), \ \varphi_G^2(l)\} \geq \gamma^2$
$\Rightarrow \varphi_G(m \wedge l) \geq \gamma$
and $\omega_G^2(m \wedge l) \leq \max\{(\omega_G^2(m), \omega_G^2(l)\} \leq \delta^2$
$\Rightarrow \omega_G(m \wedge l) \leq \delta$ and therefore $m \wedge l \in G^{[\gamma,\delta]}$ and hence $G^{[\gamma,\delta]}$ is an ideal of T
Conversely assume that each nonempty (γ, δ) level subsets $G^{[\gamma,\delta]}$ is a sublattice of T. Let $m, n \in G^{[\gamma,\delta]}$
Set $\varphi_G(m) = \gamma_1, \varphi_G(n) = \gamma_2 \omega_G(m) = \delta_1$, and $\omega_G(n) = \delta_2$
With out the loss of generality assume that $\gamma_2 \leq \gamma_1, \ \delta_2 \geq \delta_1$
then $\varphi_G(m) = \gamma_1 \geq \gamma_2$ and $\omega_G(m) = \delta_1 \leq \delta_2 \Rightarrow m \in G^{[\gamma_2,\delta_2]}$

Hence $m, n \in G^{[\gamma_2,\delta_2]}$ and since $G^{[\gamma_2,\delta_2]}$ is a sublattice $m \vee n \in G^{[\gamma_2,\delta_2]}$ and $m \wedge n \in G^{[\gamma_2,\delta_2]}$ and therefore $\varphi_G(m \vee n) \geq \gamma_2$ and $\omega_G(m \vee n) \leq \delta_2$. Also $\varphi_G(m \wedge n) \geq \gamma_2$ and $\omega_G(m \wedge n) \leq \delta_2$ and hence

$$\varphi_G^2(m \vee n) \geq \gamma_2^2 = \varphi_G^2(n) = \min\{(\varphi_G^2(m),\ \varphi_G^2(n)\}$$

$$\varphi_G^2(m \wedge n) \geq \gamma_2^2 = \varphi_G^2(n) = \min\{(\varphi_G^2(m),\ \varphi_G^2(n)\}$$
$$\omega_G^2(m \vee n) \leq \delta_2^2 = \omega_G^2(n) = \max\{(\omega_G^2(m),\ \omega_G^2(n)\}$$
$$\omega_G^2(m \wedge n) \leq \delta_2^2 = \omega_G^2(n) = \max\{(\omega_G^2(m),\ \omega_G^2(n)\}$$

Hence G is a PFL of T. Proof is similiar for PFI.

3.11 Proposition

A PFS, G of a lattice T is a PFL of T if and only if all the non empty upper and lower level subsets $U(\varphi_G, \gamma)$ and $T(\omega_G, \delta)$ are subattices of T for each $(\gamma, \delta) \in [0,1] \times [0,1]$ with $\gamma^2 + \delta^2 \leq 1$

3.12 Definition

Let G be a pythagorean fuzzy sublattice of T. Then G is said to be a Pythagorean fuzzy convex sublattice if for every interval $[a,b] \subset T, \forall m \in [a,b]$

$$\varphi_G^2(m) \geq \min\{\varphi_G^2(a), \varphi_G^2(b)\}$$
$$\omega_G^2(m) \leq \max\{\omega_G^2(a), \omega_G^2(b)\}$$

3.13 Theorem

Let G be a Pythagorean fuzzy lattice of T. Then G is a Pythagorean fuzzy convex sublattice of T if and only if each level set $G^{[\gamma,\delta]}$ for $\gamma \in image(\varphi_G)$ and $\delta \in image(\omega_G)$ is a convex sublattice of T

Proof:
Assume that G is a pythagorean fuzzy convex sublattice of T, by theorem 3.11 $G^{[\gamma,\delta]}$ is a sublattice of T. To show that $G^{[\gamma,\delta]}$ is convex,
let $\gamma \in image(\varphi_G)$ and $\delta \in image(\omega_G)$ then for any $[a,b] \subset G^{[\gamma,\delta]}$,
$\varphi_G(a) \geq \gamma$ and $\varphi_G(b) \geq \gamma$
also $\omega_G(a) \leq \delta$and $\omega_G(b) \leq \delta$
$\Rightarrow \min\{\varphi_G(a), \varphi_G(b)\} \geq \gamma$ and $\max\{\omega_G(a), \omega_G(b)\} \leq \delta$
Since G is a Pythagorean fuzzy convex sublattice then $\forall m \in [a,b]$

$$\varphi_G^2(m) \geq \min\{\varphi_G^2(a), \varphi_G^2(b)\}$$

$$\omega_G^2(m) \leq \max\{\omega_G^2(a), \omega_G^2(b)\}$$

By definition 3.13. Therefore

$$\varphi_G^2(m) \geq \gamma^2 \Rightarrow \varphi_G(m) \geq \gamma$$

$$\omega_G^2(m) \leq \delta^2 \Rightarrow \omega_G(m) \leq \delta$$

and therefore $m \in G^{[\gamma,\delta]}$ and hence $G^{[\gamma,\delta]}$ is a convex sublattice of T
Conversly assume that $G^{[\gamma,\delta]}$ is a convex sublattice of T, for $\gamma \in image(\varphi_G)$ and $\delta \in image(\omega_G)$
Let $[a, b]$ be any interval in T
Set $\min\{\varphi_G(a), \varphi_G(b)\} = \gamma$ and $\max\{\omega_G(a), \omega_G(b)\} = \delta$
$\Rightarrow \varphi_G(a) \geq \gamma$ and $\varphi_G(b) \geq \gamma$
also $\omega_G(a) \leq \delta$and $\omega_G(b) \leq \delta$
$\Rightarrow a \in G^{[\gamma,\delta]}$ and $b \in G^{[\gamma,\delta]}$
Since $G^{[\gamma,\delta]}$ is a convex subattice, we have $m \in G^{[\gamma,\delta]}, \quad \forall m \in [a, b]$
$\Rightarrow \varphi_G(m) \geq \gamma$ and $\omega_G(m) \leq \delta$
$\Rightarrow \varphi_G^2(m) \geq \gamma^2 = \min\{\varphi_G^2(a), \varphi_G^2(b)\}$ and $\omega_G^2(m) \leq \delta^2 = \max\{\omega_G^2(a), \omega_G^2(b)\}, \forall m \in [a, b]$ and hence G is a Pythagorean fuzzy convex sulattice of T

4 Conclusion

In this paper, we introduced the concepts of Pythagorean fuzzy sublattices (PFL) and ideals (PFI) and explored their characteristics. We also provided counterexamples to illustrate specific properties related to these concepts. Additionally, we examined the (γ, δ) level set and (γ, δ) strong level set, along with associated theorems. Finally, we defined Pythagorean fuzzy convex sublattices and delved one of its key property.

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