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Inverse Sum Indeg Downhill Index of Graphs

Kulli V R

Professor, Department of Mathematics, Gulbarga University, Kalaburgi, India
Email: *vrkulli[at]gmail.com*

Abstract: ***In this paper, we introduce the inverse sum indeg downhill index and the inverse sum indeg downhill polynomial of a graph. Furthermore, we compute the inverse sum indeg downhill index and its polynomial for some standard graphs, wheel graphs, gear graphs helm graphs and honeycomb networks.***

Keywords: inverse sum indeg downhill index, inverse sum indeg downhill polynomial, graph

1. Introduction

The simple graphs which are finite, undirected, connected graphs without loops and multiple edges are considered. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u.

In [1], Vukičević et al. observed that many graph indices are defined simply as the sum of individual bound contributions. They have proposed a class of discrete Adriatic indices to study whether there other possibly significant graph indices of this form. One of these discrete Adriatic indices is the inverse sum indeg index, and this index is defined as

$$ISI(G)=\sum_{uv\in E(G)}\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}.$$

Some inverse sum indices were studied in [2, 3].

Motivated by the inverse sum indeg index, the inverse sum indeg downhill index of a graph G is defined as

$$ISIDW(G)=\sum_{uv\in E(G)}\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}.$$

In view of the inverse sum indeg downhill index, the inverse sum indeg downhill polynomial of a graph G is defined as

$$ISIDW(G,x)=\sum_{uv\in E(G)}x^{\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}}.$$

Recently, some downhill indices were studied such as the downhill Sombor index [4], harmonic downhill index [5].

In this paper, the inverse sum indeg downhill index and its polynomial for some graphs and honeycomb networks are determined.

2. Results for Some Standard Graphs

Proposition 1. Let G be r-regular with n vertices and $r\geq 2$. Then

$$ISIDW(G)=\frac{nr(n-1)}{4}.$$

Proof: Let G be an r-regular graph with n vertices and $r\geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v)=n-1$ for every v in G.
From definition,

$$ISIDW(G)=\sum_{uv\in E(G)}\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}=\frac{nr}{2}\frac{(n-1)(n-1)}{(n-1)+(n-1)}=\frac{nr(n-1)}{4}.$$

Corollary 1.1. Let C_n be a cycle with $n\geq 3$ vertices. Then

$$ISIDW(C_n)=\frac{n(n-1)}{2}.$$

Corollary 1.2. Let K_n be a complete graph with $n\geq 3$ vertices. Then

$$ISIDW(K_n)=\frac{n(n-1)^2}{4}.$$

Proposition 2. Let P_n be a path with $n\geq 3$ vertices. Then

$$ISIDW(P_n)=\frac{(n-3)(n-1)}{2}.$$

Proof: Let P_n be a path with $n\geq 3$ vertices. Clearly, P_n has two types of edges based on the downhill degree of end vertices of each edge as follows:

$E_1=\{uv\in E(P_n)\mid d_{dn}(u)=0,\ d_{dn}(v)=n-1\}$, $|E_1|=2$.
$E_2=\{uv\in E(P_n)\mid d_{dn}(u)=d_{dn}(v)=n-1\}$, $|E_2|=n-3$.
Then

$$ISIDW(P_n)=\sum_{uv\in E(P_n)}\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}$$

$$=2\frac{0\times(n-1)}{0+(n-1)}+(n-3)\frac{(n-1)(n-1)}{(n-1)+(n-1)}=\frac{(n-3)(n-1)}{2}$$

Proposition 3. Let $K_{m,n}$ be a complete bipartite graph with $m< n$. Then $ISIDW(K_{m,n})=0$.

Proof. Let $K_{m,n}$ be a complete bipartite graph with $m< n$. There are $m+n$ vertices and mn edges. Clearly, $K_{m,n}$ has one type of edges based on the downhill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(K_{m,n}) \mid d_{dn}(u)=0,\ d_{dn}(v) = n\}, \qquad |E_1| = mn.$

Then

$$ISIDW\left(K_{m,n}\right)=\sum_{uv\in E\left(K_{m,n}\right)}\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}=mn\frac{0\times n}{0+n}=0.$$

3. Results for Wheel Graphs

Let W_n be a wheel with n+1vertices and $2n$ edges, n≥4. Then there are two types of edges based on the downhill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(W_n) \mid d_{dn}(u) = n,\ d_{dn}(v) = n - 1\},\ |E_1| = n.$

$E_2 = \{uv \in E(W_n) \mid d_{dn}(u) = d_{dn}(v) = n - 1\},\ |E_2| = n.$

Theorem 1: Let W_n be a wheel with n+1vertices and $2n$ edges, n≥4. Then

$$ISIDW\left(W_n\right)=\frac{n^2(n-1)}{2n-1}+\frac{n(n-1)}{2}.$$

Proof: From definition,

$$ISIDW\left(W_n\right)=\sum_{uv\in E\left(W_n\right)}\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}$$

$$=\frac{nn(n-1)}{n+(n-1)}+\frac{n(n-1)(n-1)}{(n-1)+(n-1)}$$

$$=\frac{n^2(n-1)}{2n-1}+\frac{n(n-1)}{2}.$$

Theorem 2: Let W_n be a wheel with n+1vertices and $2n$ edges, n≥4. Then

$$ISIDW\left(W_n,x\right)=nx^{\frac{n(n-1)}{2n-1}}+nx^{\frac{(n-1)}{2}}.$$

Proof: From definition,

$$ISIDW\left(W_n,x\right)=\sum_{uv\in E\left(W_n\right)}x^{\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}}$$

$$=nx^{\frac{n(n-1)}{n+(n-1)}}+nx^{\frac{(n-1)(n-1)}{(n-1)+(n-1)}}$$

$$=nx^{\frac{n(n-1)}{2n-1}}+nx^{\frac{(n-1)}{2}}.$$

4. Results for Gear Graphs

A bipartite wheel graph is a graph obtained from W_n with n+1 vertices adding a vertex between each pair of adjacent rim vertices and this graph is denoted by G_n and also called as a gear graph. Clearly, $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$. A gear graph G_n is depicted in Figure 1.

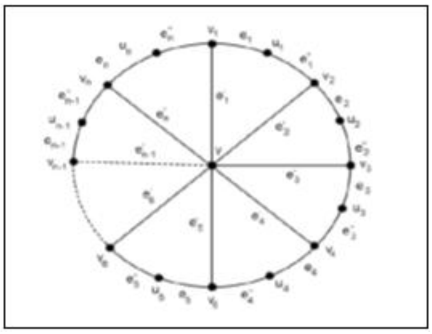

Figure 1

Let G_n be a gear graph with $2n$+1vertices, $3n$ edges, n≥4. Then G_n has two types of edges based on the downhill degree of the vertices of each edge as follows:

$E_1 = \{u \in E(G_n) \mid d_{dn}(u) =2n,\ d_{dn}(v) = 2\},\ |E_1| = n.$

$E_2 = \{u \in E(G_n) \mid d_{dn}(u) = 2, ,\ d_{dn}(v) = 0\},\ |E_2| = 2n.$

Theorem 3. Let G_n be a gear graph with $2n$+1vertices, $3n$ edges, n≥4. Then the inverse sum indeg downhill index of G_n is

$$ISIDW\left(G_n\right)=\frac{2n^2}{n+1}.$$

Proof: From definition,

$$ISIDW\left(G_n\right)=\sum_{uv\in E\left(G_n\right)}\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}$$

$$=n\left(\frac{2n\times 2}{2n+2}\right)+2n\left(\frac{2\times 0}{2+0}\right)$$

$$=\frac{2n^2}{n+1}.$$

Theorem 4. Let G_n be a gear graph with $2n$+1vertices, $3n$ edges, n≥4. Then the inverse sum indeg downhill polynomial of G_n is

$$ISIDW\left(G_n,x\right)=nx^{\frac{2n}{n+1}}+2nx^0.$$

Proof: From definition,

$$ISIDW\left(G_n,x\right)=\sum_{uv\in E\left(G_n\right)}x^{\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}}$$

$$=nx^{\frac{2n\times 2}{2n+2}}+2nx^{\frac{2\times 0}{2+0}}$$

$$=nx^{\frac{2n}{n+1}}+2nx^0.$$

5. Results for Helm Graphs

The helm graph H_n is a graph obtained from W_n (with n+1 vertices) by attaching an end edge to each rim vertex of W_n. Clearly, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. A graph H_n is shown in Figure 2.

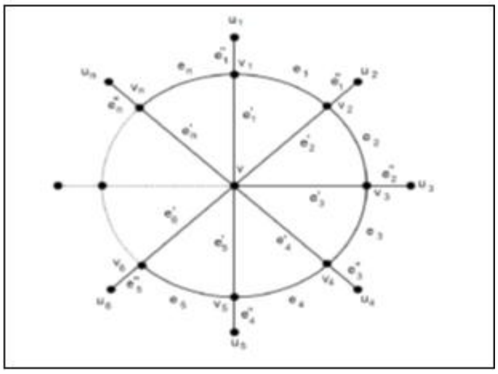

Figure 2

Let H_n be a helm graph with $3n$ edges, $n\geq 5$. Then H_n has three types of edges based on the downhill degree of the vertices of each edge as follows:

$E_1 = \{uv \in E(H_n) \mid d_{dn}(u) = 2n,\ d_{dn}(v) = 2n - 1\},\ |E_1| = n.$

$E_2 = \{uv \in E(H_n) \mid d_{dn}(u) = d_{dn}(v) = 2n - 1\},\ |E_2| = n.$

$E_3 = \{uv \in E(H_n) \mid d_{dn}(u) = 2n - 1,\ d_{dn}(v) = 0\},\ |E_3| = n.$

Theorem 5. Let H_n be a helm graph with $2n+1$ vertices, $n\geq 5$. Then the inverse sum indeg downhill index of H_n is

$$ISIDW(H_n) = \frac{2n^2(2n-1)}{4n-1} + \frac{n(2n-1)}{2}.$$

Proof: From definition,

$$ISIDW(H_n) = \sum_{uv\in E(H_n)} \frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}$$

$$= \frac{n\times 2n\times(2n-1)}{2n+(2n-1)} + \frac{n\times(2n-1)\times(2n-1)}{(2n-1)+(2n-1)} + \frac{n\times(2n-1)\times 0}{(2n-1)+0}$$

$$= \frac{2n^2(2n-1)}{4n-1} + \frac{n(2n-1)}{2}.$$

Theorem 6. Let H_n be a helm graph with $2n+1$vertices, $3n$ edges, $n\geq 5$. Then the inverse sum indeg downhill polynomial of H_n is

$$ISIDW(H_n,x) = nx^{\frac{2n(2n-1)}{4n-1}} + nx^{\frac{(2n-1)}{2}} + nx^0.$$

Proof: From definition,

$$ISIDW(H_n,x) = \sum_{uv\in E(H_n)} x^{\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}}$$

$$= nx^{\frac{2n(2n-1)}{2n+(2n-1)}} + nx^{\frac{(2n-1)(2n-1)}{(2n-1)+(2n-1)}} + nx^{\frac{(2n-1)\times 0}{(2n-1)+0}}$$

$$= nx^{\frac{2n(2n-1)}{4n-1}} + nx^{\frac{(2n-1)}{2}} + nx^0.$$

6. Results for Honeycomb Networks

Honeycomb networks are very useful in computer graphics and also in chemistry. A honeycomb network of dimension n is denoted by HCn where n is the number of hexagons between central and boundary hexagon.

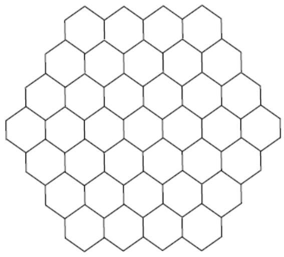

Figure 3: A 4-dimensional honeycomb network

Let H be the graph of honeycomb network HCn where $n \geq 3$. By calculation, we obtain that H has $6n^2$ vertices and $9n^2 - 3n$ edges. Then there are four types of edges based on the downhill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(H) \mid d_{dn}(u) = 1,\ d_{dn}(v) = 1\},\quad |E_1| = 6.$

$E_2 = \{uv \in E(H) \mid d_{dn}(u) = 1,\ d_{dn}(v) = 6n^2 - 1\},\ |E_2| = 12.$

$E_3 = \{uv \in E(H) \mid d_{dn}(u) = 0,\ d_{dn}(v) = 6n^2 - 1\},\ |E_2| = 12(n-2).$

$E_4 = \{uv \in E(H) \mid d_{dn}(u) = d_{dn}(v) = 6n^2 - 1\},\quad |E_4| = 9n^2 - 15n+6.$

Theorem 7: Let H be a honeycomb network with $6n^2$ vertices, $n\geq 4$. Then

$$ISIDW(H) = 24n^4 - 45n^3 + \frac{27}{2}n^2 + \frac{15}{2}n - \frac{2}{n^2} + 12.$$

Proof: From definition,

$$ISIDW(H) = \sum_{uv\in E(H)} \frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}$$

$$= \frac{6}{1+1} + \frac{12(6n^2-1)}{1+(6n^2-1)} + \frac{12(n-2)\times 0\times(6n^2-1)}{0+(6n^2-1)} + \frac{(9n^2-15n+6)(6n^2-1)(6n^2-1)}{(6n^2-1)+(6n^2-1)}$$

$$= 24n^4 - 45n^3 + \frac{27}{2}n^2 + \frac{15}{2}n - \frac{2}{n^2} + 12.$$

Theorem 8: Let H be a honeycomb network with $6n^2$ vertices, $n\geq 4$. Then

$$ISIDW(H,x) = 6x^{\frac{1}{2}} + 12x^{1-\frac{1}{6n^2}} + 12(n-2)x^0 + (9n^2-15n+6)x^{3n^2-\frac{1}{2}}.$$

Proof: From definition,

$$ISIDW(H,x)=\sum_{uv\in E(H)} x^{\frac{d_{dn}(u)d_{dn}(v)}{d_{dn}(u)+d_{dn}(v)}}$$

$$=6x^{\frac{1}{1+1}}+12x^{\frac{1(6n^2-1)}{1+(6n^2-1)}}+12(n-2)x^{\frac{0\times(6n^2-1)}{0+(6n^2-1)}}+(9n^2-15n+6)x^{\frac{(6n^2-1)(6n^2-1)}{(6n^2-1)+(6n^2-1)}}$$

$$=6x^{\frac{1}{2}}+12x^{1-\frac{1}{6n^2}}+12(n-2)x^{0}+(9n^2-15n+6)x^{3n^2-\frac{1}{2}}.$$

7. Conclusion

In this paper, the inverse sum indeg downhill index and its corresponding polynomial of some standard graphs, wheel graphs, gear graphs, helm graphs and honeycomb networks are determined.

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