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Fixed Point Results for Expansive Mappings in Dislocated Metric Spaces

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Abstract: ***In this paper we explore the existence and uniqueness of fixed points in a dislocated quasi-metric space and present some fixed-point theorems in dislocated quasi metric space for expansive type mappings which serve to generalize, extend, and unify numerous related results found in the literature. The purpose of this paper is to present some fixed-point theorem in dislocated quasi metric space for expansive type mappings.***

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1. Introduction

It is well known that Banach Contraction mappings principle is one of the pivotal results of analysis. Generalizations of this principle have been obtained in several directions. Dass and Gupta [5] generalized Banach's Contraction principle in metric space. Also, Rhoades [13] established a partial ordering for various definitions of contractive mappings. In 2005, Zeyada Salunke [17] proved some results on fixed point in dislocated quasimetric spaces. In 2005, Zeyada et al. [17] established a fixed point theorem in dislocated quasimetric spaces. In 2008, Aage and Salunke [1] proved some results on fixed point in dislocated quasimetric spaces. Recently, Isufati [7], proved fixed point theorem for contractive type condition with rational expression in dislocated quasimetric spaces. The following definitions will be needed in the sequel.

2. Preliminaries

Definition 2.1[17]. Let Xbe a nonempty set, and let d: X × X → [0, ∞) be a function, called a distance
function. If it satisfies the following conditions:
(M1): d(x, x) = 0
(M2): d(x, y) = d(y, x) = 0 then x = y
(M3): d(x, y) = d(y, x)
(M4): d(x, y) ≤ d(x, z) + d(z, y)
(M5): d(x, y) ≤ max{d(x, z), d(z, y)} ∀ x, y, z ∈X

If d satisfies conditions (M1)-(M4), then it is called a metric on X . If d satisfies conditions (M1), (M2), (M3),and (M4), it is called a quasimetric on X . If it satisfies conditions (M2)-(M4) ((M2) and (M4)), it is called a dislocated metric (or simply d-metric) (a dislocated quasi metric (or simply dq-metric)) on X , respectively. If a metric d satisfies the strong triangle inequality (M5), then it is called an ultra-metric.

Definition 2.2 [17]. A sequence $\{x_n\}$ in dq-metric space (dislocated quasimetric space) d(X, d) is called a Cauchy sequence if , for given ε > 0, there exists $n_0 \in$ N such that $d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$, i. e. $\min\{d(x_m, x_n), d(x_m, x_n)\} < \varepsilon$ for all $m, \quad n \le x_0$.

Definition 2.3 [17]. A sequence $\{x_n\}$in dq-metric space [d-metric space] is said to be d-converge to $X \in X$ provided that $\lim d(x_n, x) = \lim d(x, x_n) = 0$

In this case, x is called a dq-limit (d-limit) of $\{x_n\}$ and we write $x_n \to x$.

Definition 2.4 [17]. A dq-metric space (X, d) is called complete if every Cauchy sequence in it is a dq-convergent.

3. Main Result

Theorem 3.1: Let (X, d) be a complete dislocated metric space and P be a continuous mapping satisfying the condition:

$$d(Px, Py) + \alpha\left[\frac{d(x,Py)+d(y,Px)}{1+d(x,Py)d(y,Px)}\right] \ge \beta\frac{d(x,Px)[1+d(y,Py)]}{1+d(x,y)} + \gamma d(x, y) \quad (3.1)$$

for all x, y ∈ X, x ≠ y, where α, β, γ ≥ 0 are real constants and β + γ >1 + 2α, γ > 1 + α. Then T has a fixed point in X. Then P has a fixed point in X.

Proof: Choose $x_0 \in$ X be arbitrary to define the iterative sequence $\{x_n\}$ as follows and $Px_n = x_{n+1}$ for n=1,2,3,...
Using (3.1) we obtain

$d(Px_{n+1}, Px_{n+2}) + \alpha\left[\frac{d(x_{n+1}, Px_{n+2})+d(x_{n+2}, Px_{n+1})}{1+d(x_{n+1}, Px_{n+2})d(x_{n+2}, Px_{n+1})}\right] \geq \beta\frac{d(x_{n+1}, Px_{n+1})[1+d(x_{n+2}, Px_{n+2})]}{1+d(x_{n+1}, x_{n+2})} + \gamma d(x_{n+1}, x_{n+2})$

$\Rightarrow d(x_n, x_{n+1}) + \alpha\left[\frac{d(x_{n+1}, x_{n+1})+d(x_{n+2}, x_n)}{1+d(x_{n+1}, x_{n+1}).d(x_{n+2}, x_n)}\right]$

$\geq \beta\frac{d(x_{n+1}, x_n)[1+d(x_{n+2}, x_{n+1})]}{1+d(x_{n+1}, x_{n+2})} + \gamma d(x_{n+1}, x_{n+2})$

$\Rightarrow d(x_n, x_{n+1}) + \alpha\, d(x_{n+2}, x_n) \geq \beta\, d(x_n, x_{n+1}) + \gamma d(x_{n+1}, x_{n+2})$

$\Rightarrow d(x_n, x_{n+1}) + \alpha\, d(x_n, x_{n+1}) + \alpha\, d(x_{n+1}, x_{n+2}) \geq \beta\, d(x_n, x_{n+1}) + \gamma d(x_{n+1}, x_{n+2})$

$\Rightarrow (1 + \alpha + \beta)\, d(x_n, x_{n+1}) \geq (\gamma - \alpha)\, d(x_{n+1}, x_{n+2})$

$\Rightarrow d(x_{n+1}, x_{n+2}) \leq \frac{(1+\alpha+\beta)}{(\gamma-\alpha)} d(x_n, x_{n+1}) \leq c\, d(x_n, x_{n+1})$ (3.2)

Where $c = \frac{(1+\alpha+\beta)}{(\gamma-\alpha)} < 1$

Therefore, by induction

$d(x_{n+1}, x_{n+2}) \leq c\, d(x_n, x_{n+1}) \leq c^2 d(x_{n-1}, x_n) \leq . . . \leq c^{n+!} d(x_0, x_1)$

Now for m, n ∈ N with m > n, we have

$d(x_m, x_n) \leq d(x_m, x_{m-1}) \leq d(x_{m-1}, x_{m-2}) \leq . . . \leq d(x_{n+1}, x_n)$

$\Rightarrow d(x_m, x_n) \leq (c^{m-1} + c^{m-2} + . . . + c^n)\, d(x_0, x_1)$

$\Rightarrow d(x_m, x_n) \leq c^n(1 + c + c^2 + . . . + c^{m-n-1})\, d(x_0, x_1)$

$\Rightarrow d(x_m, x_n) \leq c^n(1 + c + c^2 + . . . + c^{m-n-1})\, d(x_0, x_1)$

$\Rightarrow d(x_m, x_n) \leq c^n \frac{1-c^{m-n}}{1-c} d(x_0, x_1) < \frac{c^n}{1-c} d(x_0, x_1)$ (3.3)

Since, $0 \leq c < 1$, $\frac{c^n}{1-c} \to 0$ as $n \to \infty$.

Hence, $d(x_m, x_n) \to 0$ as $m, n \to \infty$.

So that $\{x_n\}$ be a Cauchy sequence in X. Since X is complete dislocated metric space hence $\{x_n\}$ d-converges to u (say) on X.

i.e. $x_n \to u$ as $n \to \infty$.

Continuity of P implies

$Pu = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} x_{n-1} = u$ (3.4)

Thus, u be a fixed point of P in X.

Uniqueness

Let v be another fixed point of P in X, then Pv = v.

From (3.1), we have

$d(Pu, Pv) + \alpha\left[\frac{d(u,Pv)+d(v,Pu)}{1+d(u,Pv)d(v,Pu)}\right] \geq \beta\frac{d(u,Pu)[1+d(v,Pv)]}{1+d(u,v)} + \gamma d(u, v)$

$\Rightarrow d(Pu, Pv) + \alpha\left[\frac{d(u,Pv)+d(v,Pu)}{1+d(u,Pv)d(v,Pu)}\right] \geq \beta\frac{d(u,Pu)[1+d(v,Pv)]}{1+d(u,v)} + \gamma d(u, v)$

$\Rightarrow d(u, v) + \alpha\left[\frac{d(u,v)+d(v,u)}{1+d(u,v)d(v,u)}\right] \geq \beta\frac{d(u,u)[1+d(v,v)]}{1+d(u,v)} + \gamma d(u, v)$

$\Rightarrow d(u, v) + 2\alpha\frac{d(u,v)}{1+[d(u,v)]^2} \geq \gamma d(u, v)$

$\Rightarrow d(u, v) + [d(u,v)]^3 + 2\alpha d(u, v) \geq \gamma d(u, v) + \gamma[d(u,v)]^3$

$\Rightarrow (1 + 2\alpha - \gamma)\, d(u, v) \geq (\gamma-1)[d(u,v)]^3$

$\Rightarrow (1 + 2\alpha - \gamma)\, d(u, v) \leq \left[\frac{(1+2\alpha-\gamma)}{(\gamma-1)}\right]^{\frac{1}{3}} d(u, v)$

Which is true only when d(u, v) = 0. Similarly d(v, u) = 0.

Hence, d(u, v) = d(v, u) = 0, which implies that u = v.

This completes the proof.

Theorem 3.2: Let (X, d) be a complete dislocated metric space and P be a surjective mapping satisfying the condition (3.1), for all x, y ∈ X, x ≠ y, where α, β, γ ≥ 0 are real constants and β + γ >1 + 2α, γ > 1 + α. Then T has a fixed point in X.

Proof: Choose x_n ∈ X (arbitrarily)and define the sequence $\{x_n\}$ as follows:

$Px_n = x_{n-1}$ for all n ∈ N.

From (3.1) we have the sequence $\{x_n\}$ be a Cauchy sequence in X.

Since X is complete dislocated metric space hence $\{x_n\}$ d-converges to u (say) on X.

i.e. $x_n \to u$ as $n \to \infty$.

Existence of fixed point

Since P is an expansive mapping, so there exists a point y in X such that x = Py.

From (3.1)

$\mathbf{d}(x_n, x) = d(Px_{n-1}, Py)$

$\geq \beta\frac{d(x_{n+1}, Px_{n+1})[1+d(y, Py)]}{1+d(x_{n+1}, y)} + \gamma d(x_{n+1}, y) - \alpha\left[\frac{d(x_{n+1}, Py)+d(y, Px_{n+1})}{1+d(x_{n+1}, Py)d(y, Px_{n+1})}\right]$

Or

$d(x_n, x) \geq \beta\frac{d(x_{n+1}, x_n)[1+d(y, Py)]}{1+d(x_{n+1}, y)} + \gamma d(x_{n+1}, y) - \alpha\left[\frac{d(x_{n+1}, Py)+d(y, x_n)}{1+d(x_{n+1}, Py)d(y, x_n)}\right]$

On taking limit $n \to \infty$, we have

$d(x, x) \geq \beta\frac{d(x,x).[1+d(y,x)]}{1+d(x,y)} + \gamma d(x, y) - \alpha\left[\frac{d(x,x)+d(y,x)}{1+d(x,x)d(y,x)}\right]$

$\Rightarrow (\gamma - \alpha)\, d(x, y) \leq 0.$

Which implies that d(x, y) = 0, as γ > α.

Similarly d(y,x) = 0.

Hence, d(x,y) = d(y,x) = 0

Thus, x =y and hence Px = x, i.e. x be a fixed point of P.

Uniqueness

Let v be another fixed point of P in X, then Pv = v and Pu = u.
From (3.1), we have

$$d(Pu, Pv) + \alpha\left[\frac{d(u,Pv)+d(v,Pu)}{1+d(u,Pv)d(v,Pu)}\right] \geq \beta\frac{d(u,Pu)[1+d(v,Pv)]}{1+d(u,v)} + \gamma d(u, v)$$

$$\Rightarrow d(Pu, Pv) + \alpha\left[\frac{d(u,Pv)+d(v,Pu)}{1+d(u,Pv)d(v,Pu)}\right] \geq \beta\frac{d(u,Pu)[1+d(v,Pv)]}{1+d(u,v)} + \gamma d(u, v)$$

$$\Rightarrow d(u, v) + \alpha\left[\frac{d(u,v)+d(v,u)}{1+d(u,v)d(v,u)}\right] \geq \beta\frac{d(u,u)[1+d(v,v)]}{1+d(u,v)} + \gamma d(u, v)$$

$$\Rightarrow d(u, v) + 2\alpha\frac{d(u,v)}{1+[d(u,v)]^2} \geq \gamma d(u, v)$$

$$\Rightarrow d(u, v) + [d(u,v)]^3 + 2\alpha d(u, v) \geq \gamma d(u, v) + \gamma[d(u,v)]^3$$

$$\Rightarrow (1 + 2\alpha - \gamma)\, d(u, v) \geq (\gamma-1)[d(u,v)]^3$$

$$\Rightarrow (1 + 2\alpha - \gamma)\, d(u, v) \leq \left[\frac{(1 + 2\alpha - \gamma)}{(\gamma-1)}\right]^{\frac{1}{3}} d(u, v)$$

Which is true only when d(u, v) = 0. Similarly d(v, u) = 0.

Hence, d(u, v) = d(v, u) = 0, which implies that u =v. This completes the proof.

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