International Journal of Science and Research (IJSR)
ISSN: 2319-7064
Impact Factor 2024: 7.101

A Generalized Extended b2 Metric Space and a Fixed - Point Result for φ-contraction on a Generalized Extended b2 Metric Space

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Abstract: ***Z. Mustafa [8] introduced generalized metric space called b2-metric space. Kamran et al. [7], have dealt with an extended b-metric space. The aim of this paper is to establish a notion of a generalized extended b2 metric space which extends and generalizes metric space due to Z. Mustafa [8], Khan et al [10] and Kamran et al. [7]. Also we prove a fixed point theorem on a generalized extended b2 metric space.***

Keywords: metric space, b metric space, 2 metric space, generalized 2 metric space, extended 2 metric space.

2020 AMS Subject Classification: 47H10, 54H25

1. Introduction

The notion of a 2-metric space was introduced by Gahler, in [4]. Several fixed-point results were obtained in [1,2,3,4,5 6], as a generalization of the concept of a metric space. A 2-metric is not a continuous function of its variables, whereas an ordinary metric is. The basic philosophy is that since a 2-metric measures area, a contraction should send the space towards a configuration of zero area, which is to say a line.

Z. Mustafa introduced a new type of generalized metric space called b2-metric space, as a generalization of the 2-metric space, [8].

Recently, Kamran et al., have dealt with an extended b-metric space and obtained unique fixed-point results, [7].

Definition 1.1. [4, 9] Let X be a non-empty set and d :X×X×X → R_+ be a map satisfying the following properties
(i) d(x,y,z) = 0 if at least two of the three points are the same .
(ii) For x, y ∈ X such that x ≠y there exists a point z ∈X such that d(x,y,z) ≠0.
(iii) symmetry property: for x, y, z ∈ X,
$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$
(iv) rectangle inequality:
$$d(x,y,z) \le d(x,y,t) + d(y,z,t) + d(z,x,t)$$
for x,y,z,t ∈ X.
Then d is a 2-metric and (X, d) is a 2-metric space.

Definition 1.2. [8] Let X be a non-empty set and d :X×X×X → R_+ be a map satisfying the following properties
(i) d(x,y,z) = 0if at least two of the three points are the same.
(ii) For x,y∈X such that x ≠y there exists a point z ∈X such that d(x,y,z) ≠0.
(iii) symmetry property: for x,y,z∈X,
$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$
(iv) s-rectangle inequality:there exists s ≥1such that
$$d(x,y,z)\le s[d(x,y,t) + d(y,z,t) + d(z,x,t)]$$
for x,y,z,t ∈ X.
Then d is a b2-metric and (X, d) is a b2-metric space
If s=1, the b2-metric reduces to the 2-metric.

Definition 1.3. [10] Let X be a non-empty set and d :X×X×X → R_+ be a map satisfying the following properties:
(i) d(x,y,z) = 0 if at least two of the three points are the same.
(ii) For x,y∈X such that x ≠y there exists a point z ∈X such that d(x,y,z) ≠0.
(iii) symmetry property: for x,y,z∈X,
$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).$$
(iv) modified rectangle inequality: there exists α, β, γ≥1such that
$$d(x,y,z) \le \alpha d(x,y,t) + \beta d(y,z,t) + \gamma d(z,x,t)]$$
for x,y,z,t ∈ X.

Then d is a generalized b2-metric and (X,d)is a generalized b2- metric space.

If α = β = γ = s then a generalized b2-metric is a b2-metric. If α = β = γ = 1 then the b2- metric is a 2-metric. The example that follows provides a motivation for the generalization of the concept of a b2-metric.

In recent times, Kamran et al. [19] introduced an expansion of b-metric space known as extended b-metric space.

Definition 1.4. [19] Consider a nonempty set S and a mapping $\varphi : S \times S \to [1, +\infty)$. A mapping $d_\varphi : S \times S \to [0, +\infty)$ is known to be an extended b-metric space if it satisfies the succeeding assumptions:
(d_φ1) $d_\varphi(x, y) =0$ if x = y,
(d_φ 2) $d_\varphi(x, y) = d_\varphi(x, y)$,
(d_φ 3) $d_\varphi(x, y) \le \varphi(x, y) \{ d_\varphi(x, z) + d_\varphi(z, y)\}$

for every x, y, z ∈ S. (S, d_φ) is known as an extended b-metric space.

Inspired from Definition 1.3 and Definition 1.4 given above, here in this paper we introduce the concept of generalized extended b2-metric space and we prove a fixed point result in this space.

2. Main Result

In this section, we introduce the following generalized extended b2 metric space and then prove a fixed point result on it.

Definition 2.1 Let X be a non-empty set Let ρ, τ, σ: X×X×X → [1, +∞) and d :X×X×X → R_+ be a map satisfying the following properties:

(i) d(x,y,z) = 0 if at least two of the three points are the same.

(ii) For x,y∈X such that x ≠y there exists a point z ∈X such that d(x,y,z) ≠0.

(iii) symmetry property: for x,y,z ∈ X,

d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x).

(iv) modified rectangle inequality:

d(x,y,z) ≤ ρ(x,y,z) d(x,y,t) + τ(x,y,z)d(y,z,t) + σ(x,y,z)d(z,x,t)]

for x,y,z,t ∈ X.

If ρ(x,y,z) = α, τ(x,y,z) = β, σ(x,y,z) = γ then a generalized extended b2-metric is a generalized b2-metric ρ(x,y,z) = τ(x,y,z) = σ(x,y,z) =s then a generalized extended b2-metric is a b2-metric. If ρ(x,y,z) = τ(x,y,z) = σ(x,y,z) = 1 then a generalized extended b2-metric is a 2-metric.

Definition 2.2. Let $\{x_n\}$n ∈ N be a sequence in a generalized b2-metric space (X, d).

a) the sequence $\{x_n\}$n ∈ N is convergent to x ∈X iff for all z ∈ X , $\lim_{n\to\infty} d(x_n,x,z) = 0$.

b) the sequence $\{x_n\}$n ∈ N is a Cauchy sequence in X iff for all z ∈ X, $\lim_{n,m\to\infty} d(x_n,x_m,z) = 0$.

Definition 2.3: Let φ : $R_+ \to R_+$ be a function satisfying

i) φ is continuous

ii) $\varphi(t) < t$

iii) $\sum_1^n \varphi^i(t) < \infty$

iv) $\lim_{n\to\infty} \sum_1^n \varphi^i(t) \to 0$ as $n \to \infty$.

Theorem 2.5: Let (X,d)be a complete generalized extended b2-metric space and T :X→X be a self mapping

(1) $d(Tx, Ty, z) \le \varphi(d(x, y, z))$ (φ- contraction)

for all x,y,z ∈ X. Then T has a fixed point in X. Also assume that max {ρ(x,y,z), τ(x,y,z) σ(x,y,z)} < 1/k where k ∈ (0, 1).

Proof: Let x_0 ∈ X and define a sequence $\{x_n\}$ n ∈ N in X by $x_n=Tx_{n-1}$, for all n ∈ N. We shall show that the sequence $\{x_n\}$ n ∈ N is a Cauchy sequence of real. Using (1), we get

$d(x_n, x_{n+1}, z) = d(Tx_{n-1}, Tx_n, z) \le \varphi(d(x_{n-1}, x_n, z))$

which on repeating application implies that

$d(x_n, x_{n+1}, z) \le \varphi^n(d(x_0, x_1, z))$

Let n, m ∈ N so that n < m,

$d(x_n, x_m, z) \le \rho(x_n, x_m, z)\, d(x_n, x_{n+1}, z) + \tau(x_n, x_m, z) d(x_n, x_m, x_{n+1}) + \sigma(x_n, x_m, z) d(x_{n+1}, x_m, z)$

$\le \rho(x_n, x_m, z)\, \varphi^n(d(x_0, x_1, z)) + \tau(x_n, x_m, z)\, \varphi^n(d(x_0, x_1, x_m)) + \sigma(x_n, x_m, z) d(x_{n+1}, x_m, z)$

$\le \rho(x_n, x_m, z)\, \varphi^n(d(x_0, x_1, z)) + \tau(x_n, x_m, z)\, \varphi^n(d(x_0, x_1, x_m)) + \sigma(x_n, x_m, z)\,(\rho(x_{n+1}, x_m, z)\, \varphi^{n+1}(d(x_0, x_1, z)) + \tau(x_{n+1}, x_m, z)\, \varphi^{n+1}(d(x_0, x_1, x_m)) + \sigma(x_{n+1}, x_m, z) d(x_{n+2}, x_m, z))$

continuing we get

$$d(x_n, x_m, z) \le \sum_{i=0}^{m-n-1} \sigma(x_{n+i}, x_m, z)\,(\rho(x_{n+i+1}, x_m, z)\, \varphi^{n+i+1}(d(x_0, x_1, z)) + \sum_{i=0}^{m-n-1} \sigma(x_{n+i}, x_m, z)\,(\tau(x_{n+i+1}, x_m, z)\, \varphi^{n+i+1}(d(x_0, x_1, x_m)) + \sum_{i=0}^{m-n-1} \sigma(x_{n+i}, x_m, z)\,(\sigma(x_{n+i+1}, x_m, z)\, \varphi^{n+i+1}(d(x_0, x_1, z)) \le \sum_{i=0}^{m-n-1} \frac{1}{k^2}\,(2\varphi^{n+i+1}(d(x_0, x_1, z)) + \varphi^{n+i+1}(d(x_0, x_1, x_m)) \to 0 \text{ as } n \to \infty$$

by condition iv) of Definition 2.3.

And so $\lim_{n,m\to\infty} d(x_n, x_m, z) = 0$ which state that $\{x_n\}$ is a Cauchy sequence in complete generalized extended b2-metric space X so it is convergent in X i,e, $\{x_n\}$ converges to some x ∈ X. Now we prove that x is a fixed point of T.

$d(x_n, Tx, z) = d(Tx_{n-1}, Tx, z)$

$\le \varphi(d(x_{n-1}, x, z))$

$< d(x_{n-1}, x, z)$

using condition i) of Definition 2.3

Which on letting $n \to \infty$ gives d(x, Tx, z) =0 so that Tx = x i.e. x is a fixed point of the mapping T.

For the uniqueness of x, let x ≠ y ∈ X be such that Ty = y. Then

$$d(x, y, z) = d(Tx, Ty, z) \le \varphi(d(x, y, z)) < d(x, y, z)$$

which is a contradiction. So x = y.

If we take $\varphi(t) = kt$ where k ∈ (0, 1), then φ -contraction is a Banach type contraction.

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