# Stochastic Processes for finding Prediction of Weight Status and their Predictive Lipid Makers using $\alpha$ - Brownian Motion

**Dr. N. Umamaheswari[1], R. Guna Sundhari[2]**

[1]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy, Tamil Nadu, India.
Email: *maheshreji11[at]gmail.com*

[2]M. Sc Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy, Tamil Nadu, India.
Email: *rgunasundhari26[at]gmail.com*

**Abstract:** ***The study aims to forecast an individual's weight category e.g., underweight, normal, overweight, obese. A generalization of Brownian motion, allowing for more complex and realistic modelling of dynamic systems. Computational experiments on the presented methods are reported.***

**Keywords:** Brownian Motion, It's Formula: The Essence, The Girsanov Theorem, Gartner – Ellis Theorem, α-Brownian Motion, Large Deviation for Energy, Sharp Large Deviation for Energy (SLDE), High-Density Lipoprotein (HDL), Low-Density Lipoprotein (LDL)

## 1. Introduction

Obesity is well recognized as a risk factor for cardiometabolic diseases. The development of obesity is a dynamic process that can be described as a multistate process with an emphasis on transitions between weight states. However, it is still unclear what convenient biomarkers predict transitions between weight states. The aim of this study was to show the dynamic nature of weight status in adults stratified by age and sex and to explore blood markers of metabolic syndrome (MetS) that predict transitions between weight states. $\alpha$ −Brownian motion is a stochastic process that generalizes the standard Brownian motion by introducing a parameter α, which controls the roughness or smoothness of the paths. $\alpha$ −Brownian motion is a type of fractional Brownian motion that uses the parameter $\alpha$ to control the scale of the motion. It is a non-Markovian and non-stable self-similar process.

### 1.1 Brownian Motion

Definition of Brownian motion: Brownian motion is closely linked to the normal distribution. Recall that a random variable X is normally distributed with mean μ and variance $\sigma^2$ if

$$\mathbb{P}\{X > x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{\frac{-(u-\mu)^2}{2\sigma^2}}\, du, \qquad for\ all\ x \in \mathbb{R}\,.$$

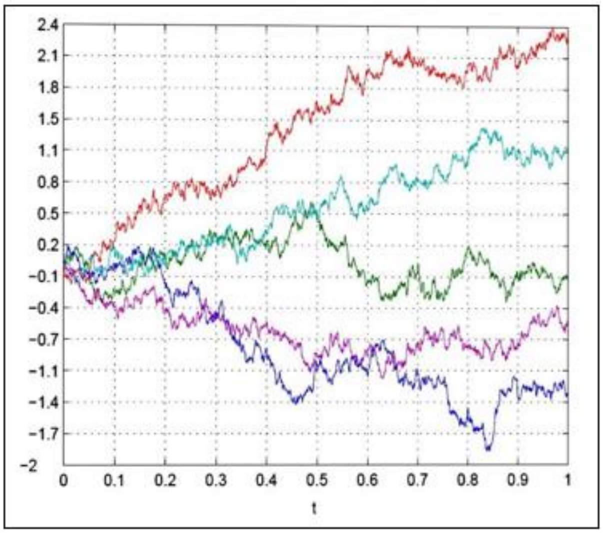
**Definition 1.1**. A *real-valued stochastic process* {B(t) : t ≥ 0} is called a (linear) Brownian motion with start in $x \in \mathbb{R}$ if the following holds:

- B(0) = $x$,
- The process has *independent increments*, i.e. for all times $0 \le t_1 \le t_2 \le \cdots \le t_n$ the increments B($t_n$)−B($t_{n-1}$), B($t_{n-1}$)−B($t_{n-2}$), . . . , B($t_2$)−B($t_1$) are independent random variables.
- For all t ≥ 0 and h > 0, the increments B(t + h) − B(t) are normally distributed with expectation zero and variance h.
- Almost surely, the function t → B(t) is continuous.

We say that {B(t): t ≥ 0} is a *standard Brownian motion* if x = 0.

Let us step back and look at some technical points.

We have defined Brownian motion as a stochastic process {B(t) : t ≥ 0} which is just a family of (uncountably many) random variables ω → B(t, ω) defined on a single probability space (Ω, A, P). At the same time, a stochastic process can also be interpreted as a random function with the sample functions defined by t → B(t, ω).




Graphs of five sampled Brownian motion

By the marginal distributions of a stochastic process {B(t): t ≥ 0} we mean the laws of all the finite dimensional random vectors (B ($t_1$), B($t_2$), . . . , B($t_n$) ) , for all $0 \le t_1 \le t_2 \le \cdots \le t_n$.

To describe these joint laws it suffices to describe the joint law of B (0) and all the increments

$B(t_1) - B(0), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ ), for all $0 \le t_1 \le t_2 \le \cdots \le t_n$.

This is what we have done in the first three items of the definition, which specify the marginal distributions of Brownian motion. However, the last item, almost sure continuity, is also crucial, and this is information which goes beyond the marginal distributions of the process in the sense above, technically because the set,
$\{\omega \in \Omega : t \to B(t, \omega) \text{ continuous}\}$

is in general not in the σ-algebra generated by the random vectors
$(B(t_1), B(t_2), \ldots, B(t_n))$ n ∈ N.

**Example 1.1**. Suppose that B is a Brownian motion and U is an independent random variable, which is uniformly distributed on [0, 1]. Then the process $\{\tilde{B}(t) : t \ge 0\}$ defined by

$$\tilde{B}(t) = \begin{cases} B(t), & if\ t \neq U \\ 0, & if\ t = U \end{cases}$$

has the same marginal distributions as a Brownian motion, but is discontinuous if $B(U) \neq 0$, i.e. with probability one, and hence this process is not a Brownian motion.

## 2. Itô's Formula: The Essence

At its core, Itô's formula is a generalization of the chain rule from ordinary calculus to the realm of stochastic processes. It allows us to find the differential of a function of a stochastic process.

**Theorem 2.1 (Itô's formula).**
*Suppose $f$ is a $C^2$ function and $B_t$ is a standard Brownian motion. Then for every $t$,*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds, \text{ or, written in differential form,}$$

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt.$$

***Proof.*** Let $\{\Pi_n\}$ *be a* sequence of partitions

$$0 = t_{0,n} < t_{1,n} < \cdots < t_{k_n,n} = t,$$

Such that

$$\sum_{n=1}^{\infty} ||\Pi_n|| < \infty, \quad ||\Pi_n|| = \max_{1\le j\le k_n} \{t_{j,n} - t_{j-1,n}\}.$$

For any n, we can write the telescoping sum (denoting $B_t$ *as* $B(t)$ *and* $t_{j,n}$ *as* $t_j$),

$$f((B_t)) - f((B_0)) = \sum_{j=1}^{k_n} [f\left(B(t_j)\right) - f(B(t_{j-1}))]$$

Let m (j, n) and M (j, n) be the minimum and maximum, respectively, of $f''(x)$ for $B(t_{j-1}) \le x \le B(tj)$.
By Taylor's theorem,

$$f\left(B(t_j)\right) - f\left(B(t_{j-1})\right) = f'\left(B(t_{j-1})\right)[B(t_j) - B(t_{j-1})] + \xi_n,$$

Where,

$$\frac{m(j,n)}{2}[B(t_j) - B(t_{j-1})]^2 \le \xi_{j,n} \le \frac{M(j,n)}{2}\{B(t_j) - B(t_{j-1})]^2$$

Hence if we let

$$Q^1(\Pi_n) = \sum_{j=1}^{k_n} f'\left(B(t_{j-1})\right)[B(t_j) - B(t_{j-1})],$$

$$Q^2 - (\Pi_n) = \sum_{j=1}^{k_n} \frac{M(j,n)}{2}\{B(t_j) - B(t_{j-1})]^2,$$

$$Q^2 + (\Pi_n) = \sum_{j=1}^{k_n} \frac{M(j,n)}{2}\{B(t_j) - B(t_{j-1})]^2,$$

Then we have

$$Q^2 - (\Pi_n) \le f(B(t)) - f(B(0)) - Q^1(\Pi_n) \le Q^2 + (\Pi_n) \tag{2.1}$$

First, we will try to understand $Q^2$.

We know that
" *Suppose $B(t)$ is a standard Brownian motion and $\{\Pi_n\}$ is a sequence of partitions on $[0,t]$ with $||\Pi_n|| \to 0$. Then $Q(t;\Pi_n) \to t$ in probability that is, for all $\epsilon > 0$* "

With Probability one, for all $0 < q < r < t$,

$$\lim_{n\to\infty} \sum_{q \le t_{j,n} < r} [B(t_{j,n}) - B(t_{j-1,n})]^2 = q - r$$

On the event that this is true, by the continuity of $B_t$ and $f''$,

We have,
$\lim_{n\to\infty} Q^2 - (\Pi_n) = \lim_{n\to\infty} Q^2 + (\Pi_n) = \frac{1}{2}\int_0^t f''(B(s))ds$

Now we will try to understand $Q^1$. We will prove the theorem under the additional assumption that there exits $K > \infty$ such that $|f'''(x)| \le K \text{ for all } x$.

By the Mean value theorem,
$|f'(B(s)) - f''\left(B(t_{j-1,n})\right)| \le K|B(s) - B(t_{j-1,n})|$.

Let the simple process
$A^{(n)}(s) = f'B(t_{j-1,n}), \quad (t_{j-1,n}) < s < B(t_{j,n})$.

For s∈ $[t_{j-1,n}, t_{j,n}]$,
$\mathbb{E}[f'(B(s)) - A^{(n)}(s)|^2] \le K^2\mathbb{E}| \le K[|B(s) - B(t_{j-1,n})|^2]$

$= K^2 Var[B(s) - B(t_{j-1,n})]$ $= K^2[s - t_{j-1,n}] \le K^2||\Pi_n||$

Therefore,

$$0 \le \lim_{n\to\infty} \int_0^t \mathbb{E}[f'(B(s)) - A^{(n)}(s)|^2 ds \le \lim_{n\to\infty} t K^2||\Pi_n|| = 0$$

So

$$\int_0^t f'(B(s))ds = \lim_{n\to\infty} A^{(n)}(s)|^2 ds = \lim_{n\to\infty} Q^1(\Pi_n)$$

So, by taking limits in equation (2.1), we see that
$f(B(t)) - f(B(0)) = \int_0^t f'(B(s))ds = \frac{1}{2}\int_0^t f''(B(s))ds.$

**Example 2.2**
*We can use* Itô's formula to solve stochastic integrals

$$\text{such as } \int_0^t B_s dB_s.$$

*If we let* $f(x) = x^2$, *then we find that*

$$B_t^2 = f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B(s))ds$$

$$=B_0^2 + 2\int_0^t B_s dB_s + t.$$

Therefore,

$$\int_0^t B_s dB_s = \frac{1}{2}[B_t^2 - t].$$

## 3. The Girsanov Theorem

Let $\{\theta_t\}$ be an adapted process satisfying the hypotheses of Novikov's Proposition.

Let

$$Z(t)=\exp\left\{\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right\}$$

For each T>0 the random variable Z(t) is a likelihood ratio: That is, the formula

$$Q(F) = E_p(Z(T)1_F)$$

Defines a new probability measure on $(\Omega, \mathcal{F})$

Girsanov's theorem describes the distribution of the stochastic process $\{W(t)\}_{t\geq 0}$ under this new probability measure.
Define

$$\widetilde{w}(t) = W(t) - \int_0^t \theta_s ds$$

***Theorem* 3.1 (*Girsanov*)** *Under the probability measure Q, the stochastic process* $\{\widetilde{w}(t)\}_{0\leq t\leq T}$ *is a standard Wiener process.*
We know that *the Novikov theorem* ,
" *If for each* $t \geq 0, E\exp\{\frac{1}{2}\int_0^t \theta_s^2\, ds/2\} < \infty$, *then for each* $t \geq 0, EZ(t) = 1$
*If this is the case then the process* $\{Z(t)\}_{t\geq 0}$ *isa positive martingale* "

The Girsanov theorem is nothing more than a routine calculation. To show that the process $\widetilde{w}_t$, under Q, is a standard Wiener process, it suffices to show that it has independent, normally distributed increments with the correct variances. For this, it suffices to show that the joint moment generating function (under Q) of the increments

$$\widetilde{w}(t_1), \widetilde{w}(t_2) - \widetilde{w}(t_1), \dots\dots, \widetilde{w}(t_n) - \widetilde{w}(t_{n-1})$$

Where $0 < t_1 < t_2 < \cdot\cdot\cdot < t_n$, is the same as that of n independent, normally distributed random

variables with expectations 0 and variances $t_1, t_2 - t_1, \ldots,$ that is,

$$E_Q\exp\{\textstyle\sum_{k=1}^n \alpha_k(\widetilde{w}(t_n) - \widetilde{w}(t_{n-1}))\} = \prod_{k=1}^n \exp\{\alpha_k^2(t_k - t_{k-1})\}. \quad (3.1)$$

We shall do only the case n = 1, leaving the rest to the industrious reader as an exercise. To evaluate the expectation $E_Q$ on the left side of (25), we rewrite as an expectation under using the basic likelihood ratio identity relating the two expectation operators:

$$\mathrm{E}_Q \exp\{\alpha\widetilde{w}(t)\} = \mathrm{E}_Q \exp\left\{\alpha w(t) - \alpha\int_0^t \theta_s ds\right\}$$

$$= \mathrm{E}_Q \exp\left\{\alpha\, w(t) - \alpha\int_0^t \theta_s ds\right\}\exp\left\{\int_0^t \theta_s dW_s - \frac{\int_0^t \theta_s^2 ds}{2}\right\}$$

$$= \mathrm{E}_P \exp\left(\int(\alpha+\theta_n)\mathrm{d}W_s - \int_0^t (2\alpha\,\theta_s + \theta_s^2)ds/2\right\}$$

$$= e^{\alpha^2 t/2}\mathrm{E}_P \exp\left(\int(\alpha+\theta_n)\mathrm{d}W_s - \int_0^t (\alpha+\theta_s)^2 ds/2\right\}$$

$$= e^{\alpha^2 t}$$

$$\mathrm{E}_Q \exp\{\alpha\widetilde{w}(t)\} = e^{\alpha^2 t} \qquad (3.2)$$

as desired. Notice that in the last step we used the fact that the exponential integrates to one, a consequence of Novikov's theorem, and that in the second to last step we merely completed a square.

## 4. Gartner – Ellis Theorem

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$.

Assume that for all t ∈ $\mathbb{R}^d$ a possibly infinite limit
$A_n(t) := \lim_{n\to\infty}\frac{1}{n} ln A_n(nt) := \lim_{n\to\infty}\frac{1}{n} ln \int_{\mathbb{R}^d} e^{(nt,x)}d\mu_n \in [-\infty, \infty]$ is exists fot all t ∈ $\mathbb{R}^d$ (4.1)

The convexity of

$$A_n(t) = ln\int_{\mathbb{R}^d} e^{(nt,x)}d\mu_n$$

for each n ∈ **N** and the limit definition of $A$ immediately imply that $A$ is convex.

Similarly to the assumption in Cramér's theorem, we shall assume throughout this note that $0 \in D_A^0$ . This will ensure, in particular, that $A > -\infty$.

Indeed, note that as $A_n(0) = 0$ for all n, so $A(0) = 0$. If for some t we had $A(t) = -\infty$ then by convexity we would have for all $a \in (0, 1]$
$A(\alpha t) = A(\alpha t + (1-\alpha)0) \leq \alpha A(t) + (1-\alpha)A(0) = -\infty$

But then
$0 = A(0) = A\left(\frac{1}{2}(\alpha t) + \frac{1}{2}(-\alpha t)\right) \leq \frac{1}{2}A(\alpha t) + \frac{1}{2}A(-\alpha t)$
we would also have $A(-\alpha t) = \infty$ for all $\alpha \in (0, 1]$. This contradicts the assumption $0 \in D_A^0$. We shall also need the following definition

***Definition* 4.1** *Let* $\mathcal{T}$ *be the Legendre* − *Fenchel transform of A i.e* $\mathcal{T}(x) = \sup_{x\in\mathbb{R}^r}(< t, x > -A(t))$.
A point $x \in D_{\mathcal{T}} = \{x \in \mathbb{R}^d : \mathcal{T}(x) < \infty\}$ is said to be exposed for $\mathcal{T}$ if there is a $\eta \in \mathbb{R}^d$ such that

$\mathcal{T}(y) - \mathcal{T}(x) > \prec \eta, y - x \succ$ is called an exposing hyperplane the graph of $\mathcal{T}$ at $x$
For a given $(x, \mathcal{T}(x))$, it is characterized by its normal $\eta$.

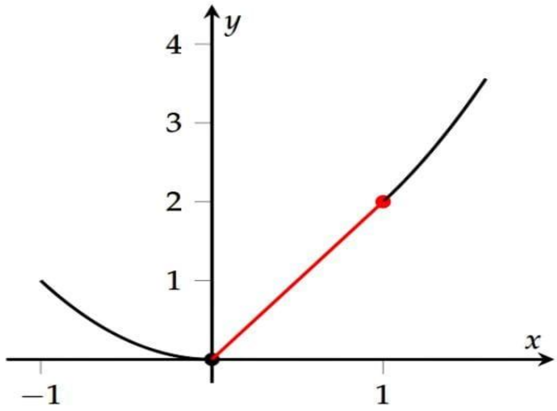



Figure 1: Points $x \in (0,1]$ are not exposed for the pictured function.

With a slight abuse of terminology, $\eta$ itself will be referred to as an exposing hyperplane.

Since I is the Legendre-Fenchel transform of $A, \mathcal{T}$ is convex and satisfies all conditions of a rate function, i.e. it is non-negative and has compact sub-level sets.

**Theorem 4.2** ***Gartner-Ellies***
*Let $(\mu_n)_{n\in N}$ be a sequence of probability measures on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$*
*Assume that for all $t \in \mathbb{R}^d$*
*a possibly infinite limit $A(t)$ in equation 4.1 exists and that $0 \in D_A^0$.*

***Proof***
(i) For every closed set $C \subset \mathbb{R}^d$,
$$\limsup_{n\to\infty} \frac{1}{2} ln\mu_n(C) \leq -\inf_{x\in C} \mathcal{T}(x).$$
(ii) For every open set $O \subset \mathbb{R}^d$
$$\limsup_{n\to\infty} \frac{1}{2} ln\mu_n(O) \leq -\inf_{x\in O\cap E} \mathcal{T}(x),$$
Where E is the set of those exposed points for which have an exposing
Hyperplane in $D_A^0$.

Suppose in addition that $A$ is lower semi-continuous on $\mathbb{R}^d$, differentiable on $D_A^0$ and either $D_A = \mathbb{R}^d$ or $A$ is steep
$i.e. \lim_{n\to\infty} |\nabla A(t_n)| = \infty$
Whenever $t_n \in D_A^0, t_n \to t \in \partial D_A^0$ as $n \to \infty$. Then $(\mu_n)_{n\in N}$ satisfies the LDP with rate function $\mathcal{T}$.

**Example 4.3**
$$Let\ \mu_n\big((-\infty, x]\big) = (1 - e^{-nx})\mathbb{I}_{[0,\infty)}(x)$$
(exponential distribution with parameter n) $x \in \mathbb{R}$. Then
$$A(t) = \lim_{n\to\infty} \frac{1}{2} \ln\left(\int_0^\infty ne^{ntx-nx} dx\right) = \begin{cases} 0, if\ t < 1 \\ \infty, if\ t \geq 1 \end{cases} \quad (4.2)$$
$$\mathcal{T}(x) = \sup_{t\in\mathbb{R}} \big(tx - A(t)\big) = \begin{cases} x, if\ t \geq 1 \\ \infty, if\ t < 1 \end{cases} \quad (4.3)$$

We see that $E = \{0\}$ while $D_{\mathcal{T}} = [0, \infty)$, and for each open set O with $O \cap E = \emptyset$ Gartner-Ellies theorem gives only a trivial lower bound $-\infty$.

But it is easy to see directly that for every open set O for which $O \cap D_{\mathcal{T}} \neq \emptyset$
$$\lim_{n\to\infty} \frac{1}{2} \ln\mu_n(O) = \lim_{n\to\infty} \frac{1}{2} \ln \int_{O\cap[0,\infty)} ne^{-nx} dx$$
$$\geq -\inf\{x, x \in O \cap [0, \infty)\}$$
$$= -\inf_{x\in O} \mathcal{T}(x)$$
This says that $(\mu_n)_{n\in N}$ satisfy a LDP with rate $\mathcal{T}$.

## 5. α-Brownian Motion

*"α-Brownian motion, denoted as {B(t), t ≥ 0}, is a type of stochastic process that generalizes standard Brownian motion"*

We consider the following α-Brownian bridge:
$dX_i = -\frac{\alpha}{T-t} X_t \mathrm{d}t + \mathrm{d}W_t$ , $X_0 = 0$, (5.1)
where $W$ is a standard Brownian motion, $t \in [0, T)$, $T \in (0, \infty)$, and the constant $\alpha > 1/2$.

Let $P_\alpha$ denote the probability distribution of the solution $\{X_t, t \in [0, T)\}$ of (5.1). The $\alpha$-Brownian bridge is first used to study the arbitrage profit associated with a given future contract in the absence of trans-action costs.

$\alpha$-Brownian bridge is a time inhomogeneous diffusion process which has been studied by. They studied the central limit theorem and the large deviations for parameter estimators and hypothesis testing problem of $\alpha$-Brownian bridge. While the large deviation is not so helpful in some statistics problems since it only gives a logarithmic equivalent for the deviation probability, overcame this difficulty by the sharp large deviation principle for the empirical mean. Recently, the sharp large deviation principle is widely used in the study of Gaussian quadratic forms.

In this paper we consider the Sharp Large Deviation Principle (SLDP) of energy $S_t$, where
$S_t = \int_o^t \frac{X_s^2}{(s-T)^2} \mathrm{d}s$ (5.2)
Our main results are the following.

**Theorem 5. 1** Let $\{X_t, t \in [0, T)\}$ be the process given by the stochastic differential equation (1). Then $\{S_t / \lambda_t, t \in [0, T)\}$ satisfies the large deviation principle with speed $\lambda t$ and good rate function $I(\cdot)$ defined by the following:
$$I\ (x) = \begin{cases} \frac{1}{8x}((2a_0 - 1)^2, if\ x > 0; \\ +\infty,, if\ x \geq 0; \end{cases} \quad (5.3)$$
where $\lambda_t = \log(T/(T - t))$.

**Theorem 5.2** $\{S_t / \lambda_t, t \in [0, T)\}$ satisfies SLDP; that is, for any c > 1/ (2α -1), there exists a sequence $b_{c,k}$ such that, for any p > 0, when t approaches T enough,
$$\mathrm{P}(S_t \geq c\lambda_t) = \frac{\exp\{-I(c)\lambda_t + H(a_c)\}}{\sqrt{2\pi} a_c \beta_t} \times \left(1 + \sum_{k=1}^{p} \frac{b_{c,k}}{\lambda_t} + O\left(\frac{1}{\lambda_t^{p+1}}\right)\right), \quad (5.4)$$
where
$\sigma_c^2 = 4c^2, \beta_t = \sigma_c\sqrt{\lambda_t}$,
$a_c = \frac{(1-2\alpha)^2 c^2 - 1}{8c^2}$ (5.5)

$H(a_c) = -\frac{1}{2} log \frac{(1-(1-2\alpha)c}{2}$

The coefficients $b_{c,k}$ may be explicitly computed as function of the derivatives of L and H (defined in lemma 6.1) at point $a_c$.

For example,
$b_{c,1}$ is given by
$b_{c,1} = \frac{1}{\sigma_c^2}(-\frac{h_2}{2} - \frac{h_1^2}{2} + \frac{l_4}{8\sigma_c^2} + \frac{l_3 h_1}{2\sigma_c^2} - \frac{5l_3^2}{24\sigma_c^4} + \frac{h_1}{2a_c} - \frac{l_3}{2a_c\sigma_c^2} - \frac{1}{a_c^2})$, (5.6)
with $l_k = L^{(k)}(a_c)$, and $h_k = H^{(k)}(a_c)$.

## 6. Large Deviation for Energy

Given $\alpha > 1/2$, we first consider the following logarithmic moment generating function of $S_t$;that is,
$l_t(u) := log\mathbb{E}_\alpha\, exp\, \left\{u \int_0^t \frac{X_s^2}{(s-T)^2} ds\right\}, \forall\lambda \in \mathbb{R}$. (6.1)
And let
$\mathfrak{D}_{L_t} := \{u \in \mathbb{R}, L_t(u) < +\infty\}$ (6.2)
be the effective domain of $L_t$. By the same method as in Zhao and Liu , we have the following lemma.

**Lemma 6.1**. *Let $\mathfrak{D}_L$ be the effective domain of the limit L of $L_t$; then for all $u \in \mathfrak{D}_L$, one has*
$\frac{L_t(u)}{\lambda_t} = L(u) + \frac{H(u)}{\lambda_t} + \frac{R(u)}{\lambda_t}$, *(6.3)*
*with*
$L(u) = -\frac{1-2\alpha-\varphi(u)}{4}$,
$H(\lambda) = -\frac{1}{2} log\, \{\frac{1}{2}(1 + h(u))\}$, *(6.4)*
$R(u) = -\frac{1}{2} log\{1+ \frac{1-h(u)}{1+h(u)} exp\, \{2\varphi(u)\lambda_t\}\}$,
*Where*
$\varphi(u) = -\sqrt{(1-2\alpha)^2 - 8u}$ *and*
$h(u) = (1-2\alpha)/\varphi(u)$.
*Furthermore, the remainder $R(u)$ satisfies*
$R(u) = O_{t-T}(exp\{2\, \varphi(u)\lambda_t\})$. (6.5)

***Proof.*** By 2.1 Itô's formula and 3.1 Girsanov's formula, for all $u \in \mathfrak{D}_L$ and $t \in [0,T]$,
$log \frac{dP_\alpha}{dP_\beta}|_{[0,t]} = (\alpha-\beta)\int_0^t \frac{X_s}{s-T} dX_s - \frac{\alpha^2-\beta^2}{2} \int_0^t \frac{X_s^2}{(s-T)^2} ds$, (6.6)
$\int_0^t \frac{X_s}{s-T} dX_s = \frac{1}{2}\left(\frac{X_t^2}{(t-T)} + \int_0^t \frac{X_s^2}{(s-T)^2} ds - \log\left(1-\frac{t}{T}\right)\right)$.

Therefore
$L_t(u) = log\mathbb{E}_\beta(\exp\left\{u\int_0^t \frac{X_s^2}{(s-T)^2} ds\right\}\frac{dP_\alpha}{dP_\beta}|_{[0,t]})$
$= log\mathbb{E}_\beta \exp\, \{\frac{\alpha-\beta}{2(t-T)}X_t^2 - \frac{\alpha-\beta}{2}\log\left(1-\frac{t}{T}\right) + \frac{1}{2}(\beta^2 - \alpha^2 + \alpha - \beta + 2u) \times \int_0^t \frac{X_s^2}{(s-T)^2} ds$ (6.7)

If $4u \leq c(1-2\alpha)^2$, we can choose $\beta$ such that $\left(\beta - \frac{1}{2}\right)^2 - \left(\alpha - \frac{1}{2}\right)^2 + 2u = 0$.
Then,
$L_t(u) = -\frac{1-2\alpha-\varphi(u)}{4}\lambda_t - \frac{1}{2}\log\left\{\frac{1}{2}(1+h(u))\right\} - \frac{1}{2}\log\{1+\frac{1-h(u)}{1+h(u)}\exp\{2\varphi(u)\lambda_t\}\}$ (6.8)
Where,
$\varphi(u) = -\sqrt{(1-2\alpha)^2 - 8u}$ , $h(u) = \frac{(1-2\alpha)}{\varphi(u)}$.
Therefore,
$\frac{L_t(u)}{\lambda_t} = -\frac{1-2\alpha-\varphi(u)}{4} - \frac{1}{2\lambda_t}\log\left\{\frac{1}{2}(1+h(u))\right\} - \frac{1}{2\lambda_t}\log\{1+\frac{1-h(u)}{1+h(u)}\exp\{2\varphi(u)\lambda_t\}\}$ (6.9)
$= L(u) + \frac{H(u)}{\lambda_t} + \frac{R(u)}{\lambda_t}$

***Proof of Theorem*** 5.1. From Lemma 6.1, we have
$L(u) = \lim_{t\to T} \frac{L_t(u)}{\lambda_t} = \frac{1-2\alpha-\varphi(u)}{4}$, (6.10)
and $L(\cdot)$ is steep; by 4.1 the Gartner-Ellis theorem
$\text{I}(x) = \begin{cases} \frac{1}{8x}((2a_0-1)^2, if\ x > 0; \\ +\infty, if\ +x \leq 0; \end{cases}$ (6.11)

## 7. Sharp Large Deviation Energy

For $c > 1/(2\alpha - 1)$,
Let
$a_c = \frac{(1-2\alpha)^2c^2-1}{8c^2}$, $\sigma_c^2 = L''(a_c) = 4c^3$, (7.1)
$\text{H}(a_c) = -\frac{1}{2}\log(1-(1-2\alpha)c)$.
Then
$\text{P}(S_t \geq c\lambda_t) = \int_{S_t \geq c\lambda_t} \exp\, \{L(a_c) - ca_c\lambda_t + ca_c\lambda_t - a_cS_t\}dQ_t$
$= \exp\{L(a_c) - ca_c\lambda_t\}\mathbb{E}_Q \exp\{-a_c\beta_tU_TI_{\{U_T\geq 0\}}\} = A_tB_t$, (7.2)
where $\mathbb{E}_Q$ is the expectation after the change of measure
$\frac{dQ_t}{dP} = \exp\, \{a_cS_t - L_t(a_c)\}$,
$U_t = \frac{S_t - c\lambda_t}{\beta_t}$, $\beta_t = \sigma_c\sqrt{\lambda_t}$. (7.3)
By Lemma 6.1, we have the following expression of $A_t$.

**Lemma 7.1.** *For all* $c > 1/(2\alpha - 1)$, *when t approaches T enough* ,
$A_t = \exp\{-I(c)\lambda_t + H(a_c)\}(1 + O((T-t)^2))$. (7.4)
For $B_t$, one gets the following.

**Lemma 7.2**. *For all $c >$* $1/(2\alpha-1)$, the distribution of *$U_t$ under $Q_t$ converges to $N(0,1)$ distribution. Furthermore, there exists a sequence $\psi_k$ such that, for $p > 0$ when t approaches T enough,*
$B_t = \frac{1}{a_c\sigma_c\sqrt{2\pi\lambda_t}}\left(1 + \sum_{k=1}^{\psi_k} \frac{\psi_k}{\lambda_t^k} + O\left(\lambda_t^{-(p+1)}\right)\right)$. (7.5)

*Proof of Theorem* 5.2. The theorem follows from Lemma 7.1 and Lemma 7.2.

It only remains to prove Lemma 7.2.

Let $\phi_t(.)$ be the characteristic function of $U_t$ *under* $Q_t$; then we have the following.

Lemma 7.3.
*When t approaches T, $\phi_t$ belongs to $L^2(\mathbb{R})$ and for all $u \in \mathbb{R}$,*
$\phi_t(u) = \exp\left\{-\frac{iu\sqrt{\lambda_t c}}{\sigma_c}\right\} \times \exp\left\{\left(L_t\left(a_c + \frac{iu}{\beta_t}\right) - L_t(a_c)\right)\right\}$. (7.6)

Moreover,
$B_t = \mathbb{E}_Q \exp\{-a_c\beta_tU_tI_{\{U_T\geq 0\}}\} = C_t + D_t$, (7.7)

with

$C_t = \frac{1}{2\pi a_c \beta_t} \int_{|u| \le s_t} (1 + \frac{iu}{a_c \beta_t})^{-1} \phi_t(u) du,$

$D_t = \frac{1}{2\pi a_c \beta_t} \int_{|u| > s_t} (1 + \frac{iu}{a_c \beta_t})^{-1} \phi_t(u) du,$ (7.8)

$|D_t| = O\left(\exp\left\{-D\lambda_t^{\frac{1}{3}}\right\}\right),$

where

$s_t = s(log(\frac{T}{T-t}))^{\frac{1}{6}},$ (7.9)

$for\ some\ positive\ constant\ s, and\ D\ is\ some\ positive\ constant.$

***Proof.*** For any $u \epsilon \mathbb{R}$,

$\phi_t(u) = \mathbb{E}(\exp\{iuU_t\} \exp\{a_c S_t - L_t(a_c)\})$ (7.10)

$= \exp\left\{-\frac{iu\sqrt{\lambda_t c}}{\sigma_c}\right\} \times \exp\left\{\left(L_t\left(a_c + \frac{iu}{\beta_t}\right) - L_t(a_c)\right)\right\}$

Then, there exist two positive constants $\tau$ and $\kappa$ such that

$|\phi_t(u)|^2 \le (1 + \frac{\tau u^2}{\lambda_t})^{-\left(\frac{\kappa}{2}\right)\lambda_t};$ (7.11)

Therefore,

$\phi_t(.)$ belongs to $L^2(\mathbb{R})$, and by Parseval's formula, for some positive constant s,

Let

$s_t = s(log(\frac{T}{T-t}))^{\frac{1}{6}};$ (7.12)

We get

$B_t = \frac{1}{2\pi a_c \beta_t} \int_{|u| \le s_t} (1 + \frac{iu}{a_c \beta_t})^{-1} \phi_t(u) du + \frac{1}{2\pi a_c \beta_t} \int_{|u| > s_t} (1 + \frac{iu}{a_c \beta_t})^{-1} \phi_t(u) du$ (7.13)

$=: C_t + D_t,$ (7.14)

$|D_t| = O\left(\exp\left\{-D\lambda_t^{\frac{1}{3}}\right\}\right),$ (7.15)

where D is some positive constant.

***Proof of 7.2.*** By lemma 6.1, we have

$\frac{L_t^{(k)} a_c}{\lambda_t} = L^{(k)}(a_c) + \frac{H^{(k)}(a_c)}{\lambda_t} + \frac{O(\lambda_t^k (T-t)^{-2c})}{\lambda_t}.$ (7.16)

Noting that L' $(a_c) = 0, L''(a_c) = \sigma_c^2$ and $\frac{L''(a_c)}{2} (\frac{iu}{\beta_t})^2 \lambda_t = -\frac{u^2}{2},$ (7.17)

For any p > 0, by Taylor expansion, we obtain

$log\phi_t(u) = -\frac{u^2}{2} + \lambda_t \sum_{k=3}^{2p+3} (\frac{iu}{\beta_t})^k \frac{L^{(k)}(a_c)}{k!} + \sum_{k=1}^{2p+1} (\frac{iu}{\beta_t})^k \frac{H^{(k)}(a_c)}{k!} + \mathrm{O}\left(\frac{\max(1,|u|^{2p+4})}{\lambda_t^{p+1}}\right);$ (7.18)

Therefore, there exist integers q(p), r(p) and a sequence $\psi_{k,l}$ independent of p;

when t approaches T,

we get

$\phi_t(u) = \exp\left\{-\frac{u^2}{2}\right\} + \left(1 + \frac{1}{\sqrt{\lambda_t}} \sum_{k=0}^{2p} \sum_{l=k+1}^{q(p)} \frac{\psi_{k,l} u^l}{\lambda_t^{\frac{k}{2}}} + O\left(\frac{\max(1,|u|^{r(p)})}{\lambda_t^{p+1}}\right)\right),$ (7.19)

where O is uniform as soon as $|u| \le s_t$.

Finally, we get the proof of lemma 7.2 by lemma 7.3 together with standard calculations on the N(0,1) distribution.

**Example**

The results regarding the progression from healthy weight to obesity in each group are presented in Figure 1. The rate of obesity in individuals beginning with healthy weight increased with time. Young groups were more likely to develop obesity than middle-aged groups, and males were more likely to develop obesity than females. The predicted rates of obesity varied greatly between groups. After 7.5 years, obesity developed in an estimated 13.7% of young males and 4.5% of middle-aged females. Table displays the effects of covariates on each transition in females. The 95% confidence interval (CI) indicated that age group and Glu, TC, TG, HDL and LDL levels were significant factors for some particular transitions in the univariate models. Compared with young females, middle-aged females tended to maintain their preceding weight state. Increases in TC, TG and LDL levels predicted the transition from healthy weight to over-weight, and increases in Glu and LDL levels also made females less likely to recover from obesity.

| | | | | | |
|---|---|---|---|---|---|
| Young males | 0.0001 | 1.9338 | 4.9133 | 7.0265 | 8.5403 |
| Middle-aged males | 0.5625 | 4.136 | 6.923 | 9.1003 | 11.28125 |
| Young female | 0.5878 | 5.2139 | 10.3206 | 16.4058 | 21.2146 |
| Middle-aged females | 1.9765 | 8.64 | 16.8576 | 25.086 | 31.6714 |

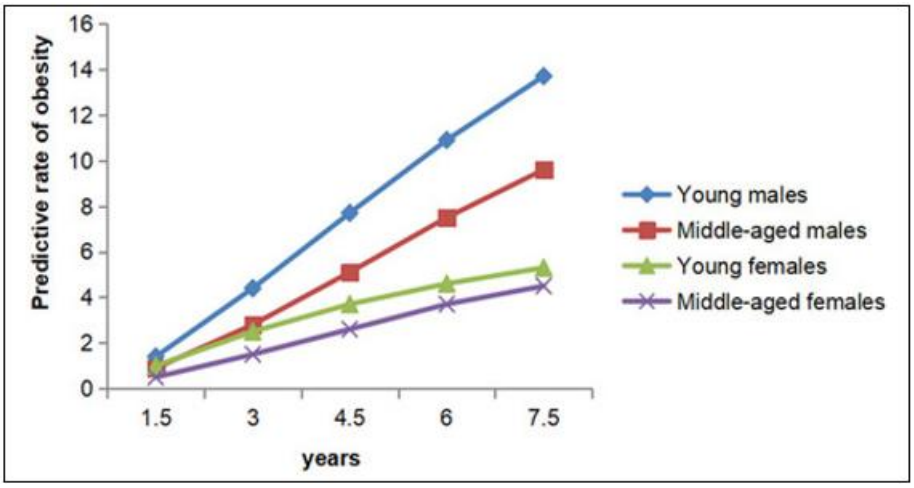



**Figure 1**

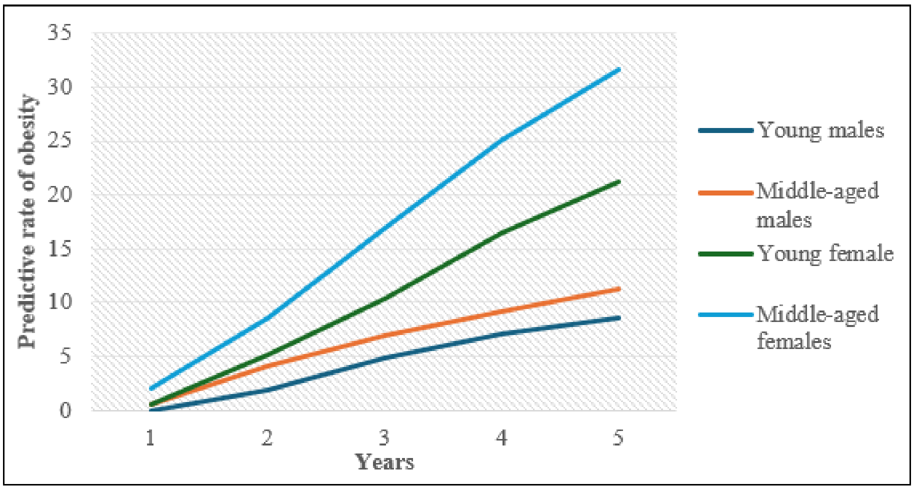



**Figure 2**

## 8. Conclusion

The Present study investigated the dynamic evolution of obesity by sex and age and explores the blood lipids that can predict weight transitions based on a large longitudinal dataset. Our results indicated that males were more likely to transit from healthy weight to overweight and more resistant to recover from worse states than females. Moreover, males had a higher prevalence and more sojourn time in overweight and obesity than females. We analysed using α-Brownian Motion methods. Finally, from fig (2), We conclude that the results correlate with the Mathematical and Medical report.

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