Theorems of Fixed Points in Metric and 2-Metric Spaces

N. Sreenivasa Reddy

Lecturer in Mathematics, Government Degree College, Uravakonda, A.P, India
Email: *nsr.bpl9[at]gmail.com*

Abstract: ***We present the concept of compatibility for a pair of self-maps on a 2-metric space. We also give fixed point theorems for pairs and quadruples of self-maps on a 2-metric space that meet specific generalized contraction criteria. Versions of the same have also been obtained in metric space.***

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The following result was determined by Brian Fisher [1].

Keywords: Fixed point, self-maps, compatibility, metric space, 2-metric space.

Theorem 1: In a complete metric space (M, ρ), let f be a self-map such that

$$\rho^2(f x, f y) < \alpha\rho (x, f x)\, \rho (y, f y) + \beta\rho (x, f y).\, \rho (y, f x) \quad (1)$$

for all x, y in M for some nonnegative constants α, β with α < 1. Then f has a fixed point in M. If further β < 1, then f has a unique fixed point in M.

In this work, we first construct generalizations of Theorem 1's existence portion for a pair of self-maps on a 2-metric space and its uniqueness part for four self-maps on a 2-metric space.

We then present the metric space versions of some of these findings without providing any supporting evidence. To shed more light on the findings covered and the notion of compatibility of a pair of self-maps on a 2-metric space presented here, we also provide a number of examples.

For the purpose of completeness, remember a few fundamental concepts and details.

Definition 1: Assume that the set X is nonempty. The definition of a 2-metric on X is as follows:

1) given distinct elements x, y of X, there exists an element z of X such that d (x y, z) ≠ 0,
2) d (x, y, z) = 0 when at least two of x, y, z are equal,
3) d (x, y, z) = d (x, z, y) = d (y, z, x) for all x, y, z in X, and
4) d (x, y, z) ≤ d (x, y, w) +d (x, w, z) + d (w, y, z) for all x, y, z, w in X.

It is known as a 2-metric space when the ordered pair (X, d) has d as a 2-metric on X.

Definition 2: In a 2-metric space (X, d), a sequence $\{x_n\}$ is considered

1) Convergent with limit x in X if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all a in X,
2) Cauchy if limit x in X if $\lim_{n\to\infty} d(x_n, x_m, a) = 0$ for all a in X.

Definition 3: If each Cauchy sequence in a 2-metric space is convergent, the space is considered complete.

Definition 4: If two of its arguments are sequentially continuous, then A2-metriconasetX is said to be continuous on X.

Two metrics are known to be nonnegative real-valued functions that are sequentially continuous in any one of their arguments. If they are sequentially continuous in two of their arguments, then they are sequentially continuous in all three. Naidu and Prasad noted that (i) a convergent sequence in a 2-metric space does not always have to be Cauchy (see [4, Remark 0.1 and Example 0.1]). (ii) any convergent sequence in a 2-metric space (X, d) is Cauchy if d is continuous on X [4, Remark 0.2], and (iii) the opposite of (ii) is untrue [4, Remark 0.2 and Example 0.2].

Throughout this paper, unless otherwise stated, (X, d) is a 2-metric space; (M, ρ) is a metric space; R is the set of all real numbers; R^+ is the set of all nonnegative real numbers; for a self-map θ on R^+, θ^1 stands for θ and for a positive integer n, θ^{n+1} is the composite of θ and θ^n; ϕ is a monotonically increasing map from R^+ to R^+ with $\Sigma_{(n=1)}^{\infty}\sqrt{\varphi^n (t)} < \infty$ for all t in R^+; ψ is a map from R^+ to R^+ with ψ(0) = 0; K is an absolute non-negative real constant; and, depending upon the context, f , g, S, T are self-maps on X or M. We note that φ(t) < t for all t in (0, ∞) and that φ (0) = φ (0+) = 0.

Remark 1: For a monotonically increasing nonnegative real-valued function θ on R^+ the condition "$\Sigma_{(n=1)}^{\infty}\sqrt{\theta^n (t)} < +\infty$ for all t in R^+" neither implies nor is implied by the condition "θ(t+) < t for all t in (0, ∞)." Examples 1 and 2 illustrate this.

Example 1: Define θ: $R^+ \to R^+$ as $\theta(t) = t^2$ if $0 \le t \le 3/4$ and θ(t) = 3/4 if t > 3/4. Then θ is monotonically increasing on R^+. For a positive integer n, we have $\theta^n(t) = (3/4)^{2n-1}$ if t > 3/4 and $\theta^n(t) = t^{2^n}$ if t ≤ 3/4. Hence $\Sigma_{n=1}^{\infty}\sqrt{\theta^n (t)} < +\infty$ for all t in R^+. We note that θ ((3/4) +) = 3/4.

Example 2: Define θ: $R^+ \to R^+$ as θ(t) = t/(1+t) for all t in R+. Then θ is a strictly increasing continuous function on R^+ with θ(t) < t for all t in (0, ∞). We have θ(1/n) = 1/(n+1) for all n = 1,2, 3, ... Hence $\theta^n(1)$ = 1/(n+1) for all n = 1,2,3,.... Hence $\Sigma_{(n=1)}^{\infty}\sqrt{\theta^n(t)}$ is divergent.

The following Naidu lemma is required [3].

Lemma 1: (see [3]) Consider the sequence $\{y_n\}_{n=0}^{\infty}$ in (X, d). Let $d_n(a) = d(y_n, y_{n+1}, a)$, for a ∈ X. Assume that for any nonnegative integers m, n with n > m, $d_n(y_m) = 0$. Then, for any nonnegative integers i, j, and k, $d(y_i, y_j, y_k) = 0$.

Preposition 1: Assume that
$d^2(fx, gy, a) \leq \varphi$ (Kd (fx, Ty, a) d (Sx, gy, a) + max d^2(Sx, Ty, a), d^2(Sx, fx, a), d^2(Ty, gy, a) + Ψ d (fx, Ty, a) d (Sx, gy, a) (1)

for all x, y, a in X. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in X such that $fx_{2n} = Tx_{2n+1}(= y_{2n}$, say), $gx_{2n+1} = Sx_{2n+2}$ (= y_{2n+1}, say) (n=0,1, 2, ...). (2)

Then $\{y_n\}_{n=0}^{\infty}$ is Cauchy.

Proof. Let $d_n(a) = d(y_n, y_{n+1}, a)$. By taking x = x_{2n+2} and y = x_{2n+1} in inequality (1), We obtain

$$d_{2n+1}^2(a) \leq \varphi(\max\{d_{2n}^2(a), d_{2n+1}^2(a)\}). \quad (3)$$

By taking x = x_{2n} and y = x_{2n+1} in inequality (1) we obtain

$$d_{2n}^2(a) \leq \varphi(\max\{d_{2n-1}^2(a), d_{2n}^2(a)\}). \quad (4)$$

From the above two inequalities, we have

$$d_{n+1}^2(a) \leq \varphi(\max\{d_{2n}^2(a), d_{n+1}^2(a) \quad (n=0,1,2,...). \quad (5)$$

Given that φ is nonnegative and φ (t) < t for any t in (0, ∞), the inequality above gives

$$d_{n+1}^2(a) \leq \varphi(d_n^2(a)) \qquad (n=0,1,2,...). \quad (6)$$

Using inequality (6) and the monotonically growing character of φ repeatedly, we arrive at

$$d_n^2(a) \leq \varphi(d_0^2(a)) \qquad (n=0,1,2,...). \quad (7)$$

From inequality (6) we see that $d_{n+1}(a) = 0$ and if $d_n(a) = 0$. Since $d_m(y_m) = 0$ for every non-negative integer m, it follows that $d_m(y_m) = 0$ for any nonnegative integers m, n with n > m.

Hence from Lemma 1 we have $d(y_i, y_j, y_k) = 0$ for all nonnegative integers i, j, k. Hence for any nonnegative integers m and n with n < m.

$$d(y_n, y_m, a) \leq \sum_{k=1}^{m-1} d_k(a) \quad (8)$$

Hence from inequality (7) we have,

$$d(y_n, y_m, a) \leq \sum_{k=1}^{m-1}\sqrt{\varphi^k(t_0)} \quad (9)$$

where $t_0 = d_0^2(a)$. Since $\Sigma_{(n=1)}^{\infty}\sqrt{\varphi^k(t)} < +\infty$ ∀ t in R^+, $\sum_{k=1}^{m-1}\varphi^k(t_0) \to 0$ as both m and n → + ∞. Hence d (y_n, y_m, a) → 0, as both m and n →+ ∞. Since this is true for any a in X, it follows that $\{y_n\}$ is Cauchy.

Theorem 2: Assume that ψ is right continuous at Zero and

$$d^2(fx, gy, a) \leq \varphi\left(\max\begin{Bmatrix} d^2(x,y,a), d^2(x,y,a), d^2(x, fx.a), d^2(y, gy, a), \\ kd(fx,y,a)d(x,gy,a)\end{Bmatrix}\right) + \psi\left(d(fx,y,a)d(x.gy,a)\right) \quad (1a)$$

for all x, y, a in X. For any x_0 in X, let $\{x_n\}_{n=1}^{\infty}$ defined iterativity as

$$x_{2n+1} = fx_{2n}, \ x_{2n+2} = gx_{2n+1} \ (n = 0,1.2.......) \quad (1b)$$

Then
$\{x_n\}$is Cauchy. If $\{x_n\}$ converges to an element z of X, then z is a common fixed point of f and g. Further the fixed point sets of f and g are the same.

Proof: By taking S = T = I, the identity map on X, in Proposition 1, we can conclude that $\{x_n\}$isa Cauchy sequence in X. Suppose that it converges to an element z of X. By taking x = x_{2n} and y = z in inequality (1a) we obtain

$$d^2(x_{2n+1}, gz.a) \leq \varphi\left(\max\begin{Bmatrix} d^2(x_{2n}, z, a), d^2(x_{2n}, x_{2n+1}, a), d^2(z, gz, a), \\ kd(x_{2n+1}, z, a)d(x_{2n}, gz, a)\end{Bmatrix}\right) \psi(d(x_{2n+1}, z, a)d(x_{2n}, gz, a))$$

(1c)

The limit of the first term on the right-hand side of inequality (1c) as n → + ∞ is if $\varphi(d^2(z, gz, a))$ d(z, gz, a) is poisitive. Otherwise, it is φ(0) or φ(0 +). Since φ(0 +) = φ(0) = 0, in either case it can be written as $\varphi(d^2(z, gz, a))$. Since Ψ(0 +) = Ψ (0) = 0. by taking limits on both sides of the above inequality as n → +∞ we obtain

$$d^2(z, gz, a) \leq \varphi(d^2(z, gz, a)). \quad (1d)$$

Since φ(t) < t for all t in (0, ∞), we have $d^2(z, gz, a) = 0$. Since this is true for all a in X, gz = z. Similarly, it can be shown that fz = z. if x is a fixed point of f, then by taking y = x inequality (1a) we obtain $d^2(x, gx, a) \leq \varphi(d^2(x, gx, a))$. Hence gx = x. Similarly, it can be shown that any fixed point of g is also a fixed point of f. Hence f and g have the same fixed point sets.

Remark 2: The uniqueness of the common fixed point for f and g is not guaranteed by the hypothesis of Theorem 2. This can be seen by taking f and g as identity maps on X, K = 2, φ (t)= (1/2) t and Ψ(t) = 0 for all t in R^+. We can also take K =0 and φ (t)=Ψ(t) = (1/2) t for all t in R^+ or ϕ(t)=0 and Ψ(t)=t for all t in R^+. Theorem 2 is an improvement over the existence part of Theorem 3 of Naidu [3] in which the first Ψ occurring in the governing inequality is to be read as φ. Proposition 1 is also an improvement over that of Naidu [3]

Corollary 1: Assume that (X, d) is complete and
$d^2(fx, fy, a) \leq$
$\alpha d(x, fx, a)d(y, fy, a) + \beta d(x, fy, a)d(fx, y, a)$for all x, y in X.
(1e)

for some nonnegative constants α, β with $\alpha < 1$. Consequently, f has a fixed point in X. The fixed point of f in X is unique if $\beta < 1$.

Proof: The corollary's existence component is derived from Theorem 2 by using g = f and k = 0. $\varphi(t) = \alpha t$, and $\psi(t) = \beta t$ for all t in R^+. The rest of it is evident.

Remark 3: Corollary 1 is the version of Theorem 1 in 2-metric space. A review of the Theorem 2 proof yields the following variation.

Theorem 3: Assume that $\varphi(t+) < t$ ∀ t in (0, ∞), ψ is right continuous at zero and

$$d^2(fx, gy, a) \le \varphi(kd(fx, y, a)d(x, gy, a) + \max\{d^2(x, y, a), d^2(x, fx, a), d^2(y, gy. a) + \psi(d(fx, y, a)d(x, gy, a) \quad (1f)$$

for all x, y, a in X. for any x_0 in X, let $\{x_n\}_{n=1}^{\infty}$ defined iterativity as in Theorem 2. Then $\{x_n\}$ is Cauchy. Z is a common fixed point of x and g if $\{x_n\}$ converges to an element z of X. Additionally, f and g's fixed point sets are identical.

Remark 4: The uniqueness of a common fixed point for f and g is not guaranteed by the hypothesis of Theorem 3. This can be seen by taking f and g as identity maps on X, K=1, φ (t) = (1/2) t, and $\Psi(t)=0$ ∀ t in R^+.

The notions of weak continuity of a 2-metric and weak commutativity for a pair of self-maps on a 2-metric space were first presented by Naidu and Prasad [4]. In Jeong and Rhoades [2], the concepts of weak compatibility for a pair of self-maps on an arbitrary set and compatibility for a pair of self-maps on a metric space are introduced. For the interest of thoroughness, we list them below.

Definition 5: (see [4]). A convergent sequence in X with limit z is said to be weakly continuous if it is Cauchy at z ∈ X.

Definition 6:(see [4]). A pair of (f_1, f_2) of self-maps on (X, d) is said to be a weakly commuting pair (w.c.p.) if $d(f_1 f_2 x, f_2 f_1 x, a) \le d(f_1 x, f_2 x, a)$ ∀ x, y, a in X.

Definition 7: (see [2]). A pair (f_1, f_2) of self-maps on (M, ρ) is said to be ac compatible pair (co.p.) if $\{\rho(f_1f_2x_n, f_2f_1x_n)\}$ converges to zero whenever $\{x_n\}$ is a sequence in M such that $\{f_1x_n\}$ and $\{f_2x_n\}$ are convergent in M and have the same limit.

Definition 8: (see [2]). A pair (f_1, f_2) of self-maps on an arbitrary set E is said to be a weakly compatible pair (w.co.p.) if $f_1f_2x = f_2f_1x$ whenever x ∈ E is such that $f_1x = f_2x$.
In analogy with Definition 7 we introduce the concept of compatibility for a pair of self-maps on a 2-metric space.

Definition 9: A pair (f_1, f_2) of self-maps on (X, d) is called a compatible pair (co.p.) if $\{d(f_1f_2x_n, f_2f_1x_n, a)\}$ converges to zero for each a in X whenever $\{x_n\}$ is a sequence in X such that $\{f_1x_n\}$ and $\{f_2x_n\}$ are convergent sequences in X having the same limit and $\{d(f_2x_n, f_1x_n, a)\}$ converges to zero for each a in X.

Remark 3: Naidu [3] established the idea of asymptotic weak commutativity for a pair of self-maps on a 2-metric space, which is a little stricter than the idea of compatibility presented here. Compatibility is implied by weak commutativity in 2-metric spaces. However, the opposite is untrue.

Theorem 4: Suppose that Ψ is monotonically increasing on R^+, $\Psi(t+) < t$ ∀ t in (0,∞),

$$d^2(fx, gy, a) \le \varphi(\max\{d^2(Sx, fx, a), d^2(Ty, gy, a)\}) + \Psi(d(fx, Ty, a)d(Sx, gy, a)) \quad (1g)$$

for all x, y, a in X and that there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ as stated in Proposition 1. Then $\{y_n\}$ is Cauchy. Suppose that it converges to an element z of X.

Corollary 2: Assume that Ψ is monotonically increasing on R^+, $\Psi(t+) < t$ ∀ t in (0,∞),

$$d^2(fx, gy, a) \le \varphi(\max\{d^2(Sx, fx, a), d^2(Sy, gy, a)\}) + \Psi(d(fx, Sy, a)d(Sx, gy, a)) \quad (1h)$$

for all x, y, a in X and that there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in X such that

$$fx_{2n} = sx_{2n+1} = y_{2n}, \ gx_{2n+1} = sx_{2n+2} = y_{2n+1} \quad (n = 0, 1, 2, \ldots \ldots \quad (1i)$$

Then $\{y_n\}$ is Cauchy. Presume that it converges to X's element z. If one of the following sets of requirements holds, then z is a unique common fixed point of f, g, and S.

- Either (f, S) or (g, S) is a w.co.p. and z ∈ S(X).
- Either f or g is continuous at z and (f, S) is a co.p., or S is continuous at z and (g, S) is a co.p.
- S is continuous at z, d is weakly continuous at Sz and either (f, S) or (g, S) is a co.p.
- S^k is continuous at z and d is weakly continuous at S^kz for some positive integer k, and S commutes with either f or g.
- For some positive integer k, S^k is continuous at z, and S commutes with each of the maps f and g.

Remark 4: In Theorem 4 if inequality (1g) is replaced with the following more stringent inequality

$$d^2(fx, gy, a) \le \varphi(\max\{d^2(Sx, fx, a), d^2(Ty, gy, a)\}) + \Psi(d(fx, Ty, a)d(Sx, gy, a)) \quad (1j)$$

Then the weak continuity d can be dropped from all those numbered statements in which it appears. A similar remark applies to Corollary 2 also.

We now state without proof the metric space versions of some of the results we obtained in 2-metric spaces. Hereafter, unless otherwise stated, f, g, S, T are self-maps on M.

Preposition 2: Assume that

$$\rho^2(fx, gy) \le \varphi(K\rho(fx, Ty)\rho(Sx, gy) + \max\{\rho^2(Sx, Ty), \rho^2(Sx, fx), \rho^2(Ty, gy)\}) + \Psi(\rho(fx, Ty)\rho(Sx, gy)) \quad (1k)$$

for all x, y in M and that there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in M satisfying equations (2) then $\{y_n\}_{n=0}^{\infty}$ is Cauchy.

Remark 5: Proposition 2 fails if the condition $\Sigma_{(n=1)}^{\infty}\sqrt{\varphi^n(t)} < +\infty$ ∀ t in R^+ is replaced by the condition $\varphi(t+) <$

$t\ \forall\ t$ in $(0,\infty)$. Exaple 3 illustrates this when $g = f$, $S = T = I$ (the identity map on M) and $\Psi(t) = t\ \forall$ t in R^+.

Example 3: Let $M = \{x_n: n = 1, 2, 3, ...\}$, where $x_n = \Sigma_{(n=1)}^{\infty} \frac{1}{k}$. Define f: M →M as $f x_n = x_{n+1}$ for all n=1, 2, 3, ... Define φ: R+ →R^+ as $\varphi(t) = \frac{t}{1+t}\ \forall$ t in R^+. Then φ is a strictly increasing continuous function on R^+ with $\varphi(t) < t\ \forall$ t in $(0,\infty)$ and

$$|fx-fy|^2 \le \varphi\ |x-y|^2 + |fx-y||x-fy| \qquad (1l)$$

for all x, y in M. Evidently for any x in M the sequence $\{f^n x\}$ diverges to + ∞ and hence is not Cauchy.

Theorem 5: Assume that Ψ is right continuous at zero and

$$\rho^2(fx, gy) \le \varphi(\max\{\rho^2(x,y), \rho^2(x,fx), \rho^2(y,gy), k\rho(fx,y)\rho(x,gy)\}) + \Psi(\rho(fx, y) + \rho(x, gy)) \qquad (1m)$$

for all x, y in M. For any x_0 in M, let $\Sigma_{n=1}^{\infty}$ be defined iteratively as in Theorem 2. Then $\{x_n\}$ is Cauchy. If $\{x_n\}$ converges to an element z of M, then z is a common fixed point of f and g. Further the fixed point sets of f and g are the same.

Theorem 6: Assume that $\varphi(t+) < t\ \forall$ t in $(0,\infty)$, Ψ is the right continuous at zero and

$$\rho^2(fx, gy) \le \varphi(k\rho(fx,y)\rho(x,gy) + \max(\{\rho^2(x,y), \rho^2(x,fx), \rho^2(y,gy)\} + \Psi(\rho(fx, y) + \rho(x, gy)) \qquad (1n)$$

for all x, y in M. For any x_0 in M, let $\Sigma_{n=1}^{\infty}$ be defined iteratively as in Theorem 2. Then $\{x_n\}$ is Cauchy. An element z of M is a common fixed point of f and g if $\{x_n\}$ converges to it. Additionally, the fixed point sets of f and g are identical.

Remark 6: In the lack of the constraint φ(t+), even if (M, ρ) is complete, f = g, K = 1, and Ψ is identically zero on R^+, the continuous z is a common fixed point of f and g fails in the theorem 6. Instances 4 and 5 demonstrate this. In Example 5, the function f has a fixed point; in Example 4, it does not.

Example 4: Let M= $\left\{\frac{1}{2^n} n = 0, 1, 2 \ldots \ldots.\right\} \cup \{0\}$. Then M is a complete metric space under the metric induced by the modulus function. Define f: M →M as fx = $\frac{1}{2}$ (x) if x≠ 0 and f 0 = 1. Define φ: R^+ →R^+ as φ (t) = 1if t >1 and φ (t) = ($\frac{1}{2}$) t if t ≤ 1. Then φ is monotonically increasing on R^+,$\Sigma_{(n=1)}^{\infty} \sqrt{\varphi^n(t)} < +\infty\ \forall$ t in R^+, φ (1+) = 1 and

$$|f x-f y|^2 \le \varphi\ (|f x-y||x-f y| + \max |x-y|^2, |x-f x|^2, | y-f y|^2) \qquad (1o)$$

for all x, y in M. We note that for any x_0 in M the sequence $\{f^n x_0\}$ converges to zero. But 0 is not a fixed point of f. In fact, f has no fixed point.

Example 5: Consider M as shown in Example 4. Define f: M →M as fx = $\frac{1}{2}$ (x) if x∉{0,1} and f 0 = f 1=1. Define φ: R+ →R^+ as φ (t) = 1if t > 1and φ (t) = ($\frac{9}{10}$) t if t ≤ 1. Then φ is monotonically increasing on R^+,$\Sigma_{(n=1)}^{\infty} \sqrt{\varphi^n(t)} < +\infty\ \forall$ t in R^+, φ (1+) = 1 and inequality (1o) is satisfied ∀ x, y in M. We note that for any x0 in M\ {0,1} the sequence $\{f^n x_0\}$ converges to zero. However, zero is not a fixed point of f.

Remark 7: A pair (f_1, f_2) of self-maps on (M, ρ) is a w*.c.p. if ρ (f_1f_2x, f_2f_1x) ≤ γρ (f_2x, f_1x) for all x in M for some nonnegative real number γ and a w.c.p. (weakly commuting pair) if ρ(f_1f_2x, f_2f_1x) ≤ ρ(f_2x,f_1x) for all x in M. (The notion of weak commutativity for a pair of self-maps on a metric space was introduced by Sessa [5].) Clearly a w.c.p. is a w*.c.p. and a w*.c.p. is a co.p. But the converse is false in either instance. Examples 6 and 7 prove this.

Example 6: Define f_1, f_2 from R to R as $f_1x = x^2$ and $f_2x = 2x-1$ ∀ x in R. Then $|f_1f_2x - f_2f_1x| = 2(x-1)^2 = 2|f_1x - f_2x|$ ∀ x in R. Hence (f_1, f_2) is a w*.c.p. but not a w.c.p.

Example 7: Define f_1, f_2 from R to R as $f_1x = x^2$ and $f_2x = -x^2$ ∀ x in R. Then $|f_1f_2x-f_2f_1x| = 2x^4$ and $|f_1x - f_2x| = 2x^2$ ∀ x in R. Clearly there is no nonnegative real number γ such that $2x^4 \le \gamma\ (2x^2)$ ∀ x in R. Hence (f_1, f_2) is a w*.c.p. Clearly it is a co.p.

Theorem 7: Assume that Ψ increases monotonically on R^+, Ψ(t+) < t ∀ t in (0, ∞), $\rho^2(fx, gy) \le \varphi\ (\max\{\rho^2(Sx, fx), \rho^2(Ty, gy)\}) + \Psi(\rho(fx, Ty), \rho(Sx, gy))$ (1p)
∀ x, y in M and that there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in M satisfying the equations (2)
$\{y_n\}_{n=0}^{\infty}$ is Cauchy because of this. Suppose it converges to a z-element in M. Consequently, the following claims are accurate.

- The fixed point z is the only one that both the maps f and S and g and T pair can share.
  When Sz = z, fz = z. Tz = z implies that gz = z.
- Z is a common fixed point of f and S if and only if it is a common fixed point of g and T, provided that Sz = Tz.
- If z ∈ S(X) and (f, S) is a w.co.p., then fz = Sz = z.
- If S is continuous at z and (f, S) is a co.p., then fz = Sz = z.
- If (f, S) is a w.co.p. and, for some positive integer k, $fS^k = S^kf$ and Sk is continuous at z, then fz = Sz = z.
  fz = z if f is continuous at z and (f, S) is a co.p.
- Statements (3), (4), (5), and (6) with f and S replaced by g and T, respectively

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Author Profile

N. Sreenivasa Reddy, working as Lecturer in Mathematics at Government Degree College, Uravakonda, A.P, India. Email ID: *nsr.bpl9[at]gmail.com*