Dynamics of Differential Equations through the Applications

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Abstract: ***Mathematics is the language of science and differential equation is one of the most important parts of this as far as science and engineering are concerned. The laws of physics are generally written down as differential equations. This chapter presents a systematic and comprehensive introduction to ordinary differential equations for engineering. In this paper mathematical concepts and various techniques are presented in a clear, logical, and concise manner. Various visual features are used to highlight focus areas. Complete illustrative diagrams are used to facilitate mathematical modeling of application problems. Thus, the emphasis is given on the relevance of differential equations through their applications in various engineering disciplines. Detailed step-by-step analysis is presented to model the engineering problems using differential equations. Such a detailed, step-by-step approach, especially when applied to practical engineering problems, helps the readers to develop problem-solving skills.***

Keywords: Differential equation; Projectile motion; tuned mass dampers; bending of beams; Newton's law of Cooling

1. Introduction on Differential Equations

In general, modeling of the variation of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain, current, voltage, or concentration of a pollutant, with the change of time or location, or both would result in differential equations. Similarly, studying the variation of some physical quantities on other physical quantities would also lead to differential equations. In fact, many engineering subjects, such as mechanical vibration or structural dynamics, heat transfer or theory of electric circuits, are founded on the theory of differential equations. This chapter will enable you to develop a more profound understanding of engineering concepts and enhance your skills in solving engineering problems. In other words, you will be able to construct relatively simple models of change and deduce their consequences. By studying these, you will learn how to monitor and even control a given system to do what you want it to do.

2. Applications of Differential Equations in Engineering

2.1 Applications to Projectile Motion of a body

The first important area which comes to mind is ball games and sports.

- Footballs are heavy enough to follow a nearly parabolic trajectory, without spin, with the effect of spin often being spectacular.
- A cricket ball is small and dense enough to follow a nearly parabolic path and it is up to the batsman to judge this, in playing his shot. But, the extraordinary thing is that a cricket ball can swing in the air, when bowled in a certain way, making the batsman's job much more difficult. The parabolic trajectory of the ball is also very important when a fielder tries to catch a very high, long ball, on the boundary. Getting a feel for parabolic flight is essential.
- With tennis and table tennis, the constant use of spin, even with lob shots, makes parabolic trajectories less important. This is also true in games such as golf, where spin predominates, and in rugby, where the shape of the ball affects its motion.
- Scoring from distance in basketball is an example of the pure judgment of parabolic flight, and can only be mastered by constant practice.
- Another general and important area which is to do with projectile motion is projectile weapons, of which there are and have been many. For instance, arrows and thrown spears are projectile weapons, where the angle of projection which the general graph of projectile motion is as below:

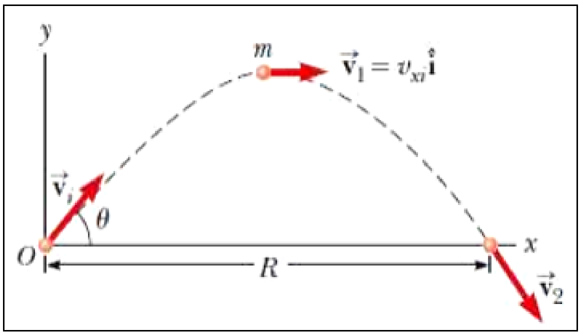


Figure: Projectile motion

The equations of motion in the X and Y direction respectively are given by

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} = 0$$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} = -mg$$

In which the initial conditions are at time

$t = 0 : x(0) = 0,\ y(0) = 0,\ x(0) = v_0 \cos\theta_0, y(0) = v_0 \sin\theta_0.$

The equations of motion are two equations involving the first and second order derivatives $\frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$.

These equations are called as a system of two second-order ordinary differential equations.

2.2 Application to tuned mass dampers (TMD):

Tuned mass dampers (TMDs) can be successfully employed to control steady-state vibration problems of other composite floor systems [1,2]. However, the problems such as insufficient control force capacity and excessive power demands encountered by current technology in the context of structural control against earthquakes are unavoidable and need to be overcome. The optimum parameters of TMD results in considerable reduction in the response of structures to seismic loading are presented. The criteria used to obtain the optimum parameters is to select, for a given mass ratio, the frequency (tuning) and damping ratios that would result in equal and large modal damping in the first two modes of vibration. The parameters are used to compute the response of several single and multi-degree of freedom structures with TMDs to different earthquake excitations. The results indicate that the use of the proposed parameters reduces the displacement and acceleration responses significantly. The method can also be used in vibration control of tall buildings using the so-called mega-substructure configuration, where substructures serve as vibration absorbers for the main.

Consider the vibration of a single story shear building under the excitation of earthquake. The shear building consists of a rigid girder of mass m supported by columns of combined stiffness k . The vibration of the girder can be described by the horizontal displacement $x(t)$. The earthquake is modeled by the displacement of the ground $x_0(t)$ as shown. When the girder vibrates, there is a damping force due to the internal friction between various components of the building, given by $c\frac{dx}{dt} - \frac{dx_0}{dt}$,

where c is the damping = − coefficient.

The relative displacement $y(t)$, $x(t)$ and $x_0(t)$ between the girder and the ground is governed by the equation

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} = ky(t) = -m\frac{d^2x_0}{dt^2}$$

This is second-order linear ordinary differential equation.

2.3 Application to bending of beams:

The solution of bending and buckling problems is integral to the study of civil, mechanical and aerospace engineering. The academic introduction to the bending of beams with constant cross section is usually given to students of these disciplines early in the engineering curriculum in a course in mechanics of materials.

For mechanical engineering analyses, frequently used laws of physics include the following:

- The Newton's laws for statics, dynamics and kinematics of solids.
- The Fourier's law for heat conduction in solids.
- The Newton cooling law for convective heat transfer in fluids.
- The Bernoulli's principle for fluids in motion.
- Fick's law for diffusion of substances with different densities
- Hooke's law for deformable solids

In real-world, there are many physical quantities that can be represented by functions involving only one of the four independent variables e.g., (x, y, z, t), in which variables (x, y, z) used for space and variable t for time.

3. Applications of Differential Equations in Physics

3.1 Newton's law of Cooling / Warming:

Newton's empirical law of cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and that of the temperature of the surrounding medium, the so-called ambient temperature. Let *T(t)* be the temperature of a body and let *T(t)* denote the constant temperature of the surrounding medium. Then the rate at which the body cools is denoted by $\frac{dT}{t}$ is proportional to $T(t) - T_A$. This means that

$$\frac{dT}{t} = k\left(T(t) - T_A\right)$$

where k is a constant of proportionality. The value of the constant k is determined by the physical characteristics of the object. If the object is large and well-insulated then it loses or gains heat slowly and the constant k is small. If the object is small and poorly insulated then it loses or gains heat more quickly and the constant k is large. We assume the body is cooling, then the temperature of the body is decreasing and losing heat energy to the surrounding. Then we have $T > T_A$. Thus $\frac{dT}{t} < 0$ and hence the constant k must be negative. If the body is heating, then the temperature of the body is increasing and gain heat energy from the surrounding and $T < T_A$. Thus $\frac{dT}{t} > 0$ and the constant k is positive.

Under Newton's law of cooling, we can predict how long it takes for a hot object to cool down at a certain temperature. Moreover, we can tell us how fast the hot water in pipes cools off and it tells us how fast a water heater cools down if you turn off the breaker and also it helps to indicate the time of death given the probable body temperature at the time of death and current body temperature. Newton's law of cooling leads to the classic equation of exponential decay over time which can be applied to many phenomena in science and engineering including the decay in radioactivity.

$$\frac{dT}{t} = k(T - T_A)$$

$$\frac{dT}{(T - T_A)} = kdt$$

$\ln|T - T_A| = kt + c_1$ where c_1 is constant

$$|T - T_A| = e^{kt+c}$$

Hence $T(t) = T_A + c_2 e^{kt}$ where c_2 is constant

$$T(t) = T_A + c_2 e^{kt}$$

This equation represents Newton's law of cooling.

3.2 Population Growth:

In order to illustrate the use of differential equations with regard to population problems, we consider the easiest mathematical model offered to govern the population dynamics of a certain species. One of the earliest attempts to model human population growth by means of mathematics was by the English economist Thomas Malthus in 1798[3]. Essentially, the idea of the Malthusian [4] is the assumption that the rate at which a population of a country grows at a certain time is proportional to the total population of the country at that time. In mathematical terms, if $P(t)$ denotes the total population at time t, then this assumption can be expressed as

$$\frac{dP}{T} = kP(t)$$

Where k is called the growth constant or the decay constant, as appropriate. Solution of the equation will provide population at any future time t. This simple model which does not take many factors into account (immigration and emigration, for example) that can influence human populations to either grow or decline, nevertheless turned out to be fairly accurate in predicting the population. The differential equation $\frac{dP}{T} = kP(t)$, where $P(t)$ denotes population at time t and k is a constant of proportionality that serves as a model for population growth and decay of insects, animals and human population at certain places and duration. It is fairly easy to see that if $k > 0$, we have grown, and if $k > 0$, we have decay. This is a linear differential equation that solves into $P(t) = P_0 e^{kt}$ where the initial population, i.e. $P(0) = P_0$, and k is called the growth or the decay constant. Therefore, we conclude that:

- if $k > 0$, then the population grows and continues to expand to infinity, that is, lim t→∞
- if $k < 0$, then the population will shrink and tend to 0. In other words, we are facing extinction.

3.3 Bernoulli's Equation

Bernoulli's principle states that, an increase in the speed of a fluid occurs simultaneously with a decrease in static pressure or a decrease in the fluid's potential energy. Bernoulli's principle can be applied to various types of fluid flow, resulting in various forms of Bernoulli's equation. Bernoulli's principle can be derived from the principle of conservation of energy. This states that, in a steady flow, the sum of all forms of energy in a fluid along a streamline is the same at all points on that streamline. This requires that the sum of kinetic energy, potential energy and internal energy remains constant.

Consider the differential equation given by

$$y' + P(x)y = Q(x)y^n$$

This equation is linear if $n = 0$ and has separable variables if $n = 1$. Thus, we assume that $n \neq 0$ and $n \neq 1$. Multiplying both sides of above equation by y^{-n} and then $1-n$ gives

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

$$\frac{d}{dx}\left[y^{1-n}\right] + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

Let $z = y^{1-n}$,

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

Finally, the general solution of the Bernoulli's equation is

$$y^{1-n}e^{\int(1-n)P(x)dx} = \int(1-n)Q(x)e^{\int(1-n)P(x)dx} + c$$

3.4 Newton's Second Law of Motion:

Newton's Second Law of Motion states that If an object of mass m is moving with acceleration a and being acted on with force F then Newton's Second Law tells us that,

$$F = ma$$

To see that this is in fact a differential equation we need to rewrite it a little. First, remember that we can rewrite the acceleration a, in one of two ways.

$$a = \frac{dv}{dt} = \frac{d^2u}{dt^2}$$

Where v is the velocity of the object and u is the position function of the object at any time t. We should also remember at this point that the force F may also be a function of time, velocity, and/or position. So, with all these things in mind Newton's Second Law can now be written as a differential equation in terms of either the velocity, v or the position u of the object as follows.

$$m\frac{dv}{dt} = F(t, v)$$

$$m\frac{d^2u}{dt^2} = F\left(t, v, \frac{du}{dt}\right)$$

3.5 Exponential Growth of Bacteria

In the biomedical field, bacteria culture growth takes place exponentially. For exponential growth, we use the formula;

$$L(t) = L_0 e^{kt}$$

Let L_0 is positive and k is constant, then

$$\frac{dL}{dt} = k$$

$L(t)$ increases as time increases. L_0 is the value when $t = 0$

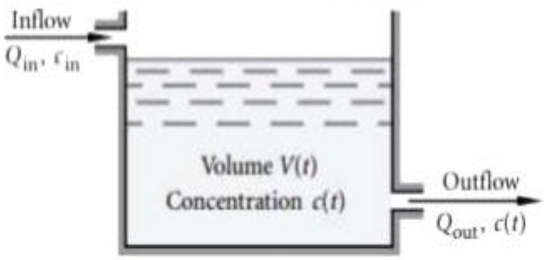
L is the exponential growth model.

4. Applications of Differential Equations in Chemistry

4.1 Application to Mixing problems:

These problems arise in many settings, such as when combining solutions in a chemistry lab.

Adding ingredients to a recipe, e.g. a lemonade mixture problem may ask how tartness changes when pure water is added or when different batches of lemonade are combined.



Inflow: Solution of concentration c_{in} grams/liter flows in at a rate of r_1 liters/minute

In tank: $A(t)$- amount of chemical in the tank at time t, $V(t)$-volume of solution in the tank at time t, $c(t) = A(t)/V(t)$ concentration of chemical in the tank at time t out flow: Solution of concentration $c(t)$ grams/liter flows out at a rate of r_2 liters/minute at time $t = 0$, the volume of the liquid is V_0 with a pollutant concentration of c_0. The equation governing the pollutant concentration $c(t)$ is given by

$$\left[v_0 + (Q_{in} + Q_{out})t\right]\frac{dc(t)}{dt} + Q_{in}\,c(t) = Q_{in}c_{in}$$

5. Conclusion

Differential equations are of great importance as almost in every area of engineering, almost all real life situations can be expressed using differential equations. Here we discussed very few of the applications. Some are listed below:

- In the field of medical science to study the growth or spread of certain diseases in the human body.
- In the prediction of the movement of electricity.
- In the description of various exponential growths and decays.
- In the calculation of optimum investment strategies to assist the economists.
- In describing the equation of motion of waves or a pendulum.
- There are various other applications of differential equations in the field of engineering (determining the equation of a falling object, Newton's Law of Cooling, the RL circuit equations, etc), physics, chemistry, geology, economics, etc.

These differential equations can be solved by any method either by using analytic methods or by using numerical methods.

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Paper ID: SR251226153313 DOI: https://dx.doi.org/10.21275/SR251226153313