Comparative Discussion Between Effect of Smoothing's

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Abstract: ***Real-world data (signals) from fields like science, economics, and biology almost always include noise, which can make accurate interpretation difficult. Therefore, denoising using techniques like filtering or smoothing is a critical initial step in any analysis. This research specifically investigates how the process of smoothing (denoising) affects a signal's memory structure. Memory Structure refers to how past data points influence future ones. Homoscedasticity: The memory structure is uniform (constant memory length, n). Heteroscedasticity: The memory structure is non-uniform (memory length varies over time). When Simple Exponential Smoothing (SES) is applied to a signal that initially has a uniform (linear homoscedastic) memory of length n, the resulting smoothed signal preserves the homoscedastic structure. Effect on Memory Length: The smoothing process increases the memory length from n to n+1 except At the (n+1)-th step, the memory length remains n. Double Exponential Smoothing (DES) is particularly useful for analyzing signals that exhibit a trend (a gradual, long-term change). DES helps remove this trend (detrending) to allow for clearer analysis of the remaining components. The study reformulates the DES method using matrix equations to gain a deeper understanding of its mathematical properties. The effectiveness of DES is further tested using signals generated by autoregressive models of order 1 and 2 (AR (1) and AR (2)), which are standard mathematical models used to represent signals with linear homoscedastic memory. In essence, the study concludes that while Simple Exponential Smoothing preserves a signal's uniform memory, it subtly increases the memory length. It also sets up a matrix-based framework to analyze how Double Exponential Smoothing affects the memory structure of trended signals***

Keywords: Signal Denoising, Memory Structure, Exponential Smoothing, Homoscedastic Signals, Autoregressive models

1. Introduction

The purpose of filtering or smoothing a signal is essentially the same—to substantially reduce noise. Noise refers to any unwanted component mixed with the true signal, and its presence may lead to incorrect interpretations. Depending on its characteristics, noise can be categorized as (i) white noise or (ii) dark noise.

White noise is defined as noise whose mean is close to zero, has finite variance, and whose values are uncorrelated. If, in addition, this noise follows a Gaussian distribution, it is known as Gaussian white noise.

Dark noise, on the other hand, refers to any noise that violates at least one of these conditions.

For a general filtering method:
$y_1 = x_1$
$y_i = f(x_i, x_{i-1}, x_{i-2}, \ldots, x_{i-j})$; $j=1, 2, \ldots, i-1$ and $i=2, 3, \ldots, N$.

For smoothing:
$y_1 = x_1$
$y_i = g(x_i, y_{i-1})$; $i=2, 3, \ldots, N$.
$y_i = \alpha_0 x_i + \alpha_1 y_{i-1}$; $i=2, 3, \ldots, N$.

Here α_0 and α_1 may or may not depend on i.

The above formula is analogous to
$y_1 = x_1$
$y_i = \alpha_0 x_i + \alpha_0\alpha_1 x_{i-1} + \alpha_0\alpha_1^2 x_{i-2} + \ldots + \alpha_0\alpha_1^{i-2} x_2 + \alpha_1^{i-1} x_1$; $i=2, 3, \ldots, N$.

When reducing noise, one should avoid distorting the intrinsic pattern and characteristics of the discrete signal. However, this limitation is not well respected in 3-point and 5-point moving average techniques. In the 3-point moving average, the value at position iii is replaced by the average of its neighbors, and a similar idea applies to the 5-point average. Consequently, important information at the actual position iii is diluted, causing noise to spread across adjacent points.

Empirical Mode Decomposition (either low-pass or high-pass) relies on identifying all local extrema. These extrema are used to construct upper and lower envelopes using cubic splines. However, extrema are usually determined visually from the plotted data, and identifying them precisely is often difficult, thereby introducing error at the very beginning.

Monte Carlo noise reduction assumes Gaussian noise and uses random number generation, which may be unrealistic because physical systems seldom obey such idealized assumptions. Wavelet-based methods rely on signal compression; this tends to redistribute and smear sharp noise peaks across other locations and often produces exaggerated values near the boundaries, leading to misleading conclusions.

The Kalman filter on a discrete signal $\{x_i\}$; $i=1, 2, \ldots, N$ is analogous to updating a running average:

$$y_i = \{(i-1)/i\}\, x_{i-1} + (1/i)\, x_i;\ i=1, 2, \ldots, N.$$

and therefore, usually performs better than moving averages because it only uses the immediate previous data instead of wider neighbourhood averaging. However, it still does not preserve the positional significance of the original samples. Simple Exponential Smoothing (SES) is ineffective when the underlying signal exhibits a trend. In such cases, one applies Double Exponential Smoothing (Holt–Winters method). In the present work, we examine the difference between SES and Double Exponential Smoothing (DES) with respect to their influence on the memory behaviour of linear signals that possess homogeneous memory.

2. Theory

2.1 Simple Exponential Smoothing (SES):

Proposed by Brown (1956) and extended by Holt (1957) and Winter (1961), SES smooths data using the following formula:

1) **Initial condition**
   $y_1 = x_1$
2) **Recursive formula:**
   $y_i = \alpha x_i + (1-\alpha)y_{i-1}$ for i = 2, 3, ..., N and $0 < \alpha < 1$

where: $\{x_i\}$ is the original signal,
$\{y_i\}$ is the smoothed signal,
α is the smoothing factor.

Special cases:
$\alpha = 1 \rightarrow$ No smoothing; $y_i = x_i$
$\alpha = 0 \rightarrow$ All y_i equal x_1 (complete smoothing)

Because α cannot be 0 or 1, it's constrained to the interval (0, 1).

Requirements for Effective Smoothing:

- It should limit the carryover of noise across time steps.
- The smoothed value at any point should be mainly derived from the original signal at the same point.

Autoregressive Signal Model:
For computational purposes, the study uses signals generated via an **Autoregressive (AR) model of order n**, which exhibit linear homoscedastic memory with memory length *n*. The goal is to examine how SES and DES effect such memory characteristics.

2.2 Effect of Simple Exponential Smoothing on Memory of linear homoscedastic Signal:

The prescribed formulation for a discrete signal $\{x_i\}$; $i=1, 2, \ldots, N$ is $y_1=x_1$ and $y_i=\alpha x_i+(1-\alpha)y_{i-1}$ (1) $i=2, 3, \ldots, N$ where y_i is the smoothed data at the i-th position and α ($0<\alpha<1$) is a parameter. This is analogous to $y_1=x_1$ and $y_i=\alpha x_i+\alpha(1-\alpha)x_{i-1}+\alpha(1-\alpha)^2x_{i-2}+\ldots+\alpha(1-\alpha)^{i-2}x_2+(1-\alpha)^{i-1}x_1$ for $i=2, 3, \ldots, N$ where the sum of the corresponding weights α, $\alpha(1-\alpha)$, $\alpha(1-\alpha)^2$, …., $\alpha(1-\alpha)^{i-2}$ and $(1-\alpha)^{i-1}$ is equal to unity. Thus in effect, each smoothed value is a convex linear combination of all the previous observations as well as the current observation. To maintain the positional importance in this case we must have $\alpha>0.5$. But as one goes with increasing the value of α at the right side of 0.5 it can be observed that the contributions of distant observations will be getting fader. So rationally we may choose α at the right-hand neighborhood of 0.5.

We consider a homoscedastic signal $\{x_i\}_{i=1}^{N}$ governed by an Autoregressive method of order n or AR (n) i.e., with linear homoscedastic memory with memory length n. Then we have (Box, Jenkins and Reinsel, (1976)

$$x_k=w_1x_{k-1}+w_2x_{k-2}+w_3x_{k-3}+\ldots\ldots+w_nx_{k-n} \quad \text{for all } k\geq n+1 \quad (2)$$

The entry at the $(n+m)$-th stage of the signal after smoothing [using (1)] is

$$y_{n+m}=\alpha x_{n+m}+(1-\alpha)y_{n+m-1} \quad (3)$$

Using (2) in (3) we get

$$y_{n+m}=[w_1+(1-\alpha)]y_{n+m-1}+[w_2-w_1(1-\alpha)]y_{n+m-2}+[w_3-w_2(1-\alpha)]y_{n+m-3}+\ldots\ldots$$
$$+[w_n-w_{n-1}(1-\alpha)]y_{m-}-w_n(1-\alpha)y_{m-1} \quad (4)$$

By taking a signal with linear homoscedastic memory with memory length n we have established that the signal after smoothing keeps its homoscedastic nature intact with increased memory length $(n+1)$ except at $(n+1)$-th step where the memory length preserves the original length n. This observation holds true for all the choices of α within the prescribed range.

2.3 Effect of Simple Exponential Smoothing on a Polynomial time series model of order 1

Let us consider a data following memory of a polynomial time series model (5) of order 1. We perform simple exponential smoothing on it ignoring the noise term.

$$x_k = Ax_{k-1}{}^m + \varepsilon_k \quad k=2,3,\ldots\ldots n \quad (5)$$

$$y_1 = x_1$$
$$y_2 = \alpha x_2 + (1-\alpha)y_1 = \alpha A x_1{}^m + (1-\alpha)y_1 = \alpha A y_1{}^m + (1-\alpha)y_1$$
$$y_3 = \alpha x_3 + (1-\alpha)y_2 = \alpha A x_2{}^m + (1-\alpha)y_2$$
$$= \alpha A\left(\frac{y_2-(1-\alpha)y_1}{\alpha}\right)^m + (1-\alpha)y_2 = \alpha^{1-m}A\left(y_2-(1-\alpha)y_1\right)^m + (1-\alpha)y_2 = \phi(y_2, y_1)(\text{say})$$

Then

$$y_4 = \alpha x_4 + (1-\alpha)y_3 = \alpha A x_3{}^m + (1-\alpha)y_3$$
$$= \alpha A\left(\frac{y_3-(1-\alpha)y_2}{\alpha}\right)^m + (1-\alpha)y_3 = \alpha^{1-m}A\left(y_3-(1-\alpha)y_2\right)^m + (1-\alpha)y_3 = \phi(y_3, y_2)$$

…

In general,

$$y_k = \alpha x_k + (1-\alpha) y_{k-1} = \alpha A x_{k-1}{}^m + (1-\alpha) y_{k-1}$$

$$= \alpha A \left(\frac{y_{k-1} - (1-\alpha) y_{k-2}}{\alpha} \right)^m + (1-\alpha) y_{k-1} = \alpha^{1-m} A \left(y_{k-1} - (1-\alpha) y_{k-2} \right)^m + (1-\alpha) y_{k-1} = \phi(y_{k-1}, y_{k-2}),\ k=3,4,\ldots,N$$

Hence, a polynomial time series model of memory order 1, after applying simple exponential smoothing, is transformed to a complex polynomial time series model of memory order 2 which indicates that order of the polynomial memory increases after smoothing.

2.4 Effect of Simple Exponential Smoothing on a Polynomial time series model of order 2

Let us take the data following memory of a polynomial time series model of order 2 given by (6). We apply simple exponential smoothing on it ignoring the noise term.

$$x_k = A x_{k-1}{}^m + B x_{k-2}{}^n + \varepsilon_k,\ k=1,2,\ldots,N \qquad (6)$$

$$y_1 = x_1$$

$$y_2 = \alpha x_2 + (1-\alpha) y_1$$

$$y_3 = \alpha x_3 + (1-\alpha) y_2 = \alpha \left[A x_2{}^m + B x_1{}^n \right] + (1-\alpha) y_2 = \alpha \left[A \left\{ \frac{y_2 - (1-\alpha) y_1}{\alpha} \right\}^m + B y_1{}^n \right] + (1-\alpha) y_2$$

$$y_4 = \alpha x_4 + (1-\alpha) y_3 = \alpha \left[A x_3{}^m + B x_2{}^n \right] + (1-\alpha) y_3 = \alpha \left[A \left\{ \frac{y_3 - (1-\alpha) y_2}{\alpha} \right\}^m + B \left\{ \frac{y_2 - (1-\alpha) y_1}{\alpha} \right\}^n \right] + (1-\alpha) y_3$$

$$= \alpha^{1-m} A \left\{ y_3 - (1-\alpha) y_2 \right\}^m + \alpha^{1-n} B \left\{ y_2 - (1-\alpha) y_1 \right\}^n + (1-\alpha) y_3 = \psi(y_3, y_2, y_1) \text{ (say)}$$

Then

$$y_5 = \alpha x_5 + (1-\alpha) y_4 = \alpha \left[A x_4{}^m + B x_3{}^n \right] + (1-\alpha) y_4 = \alpha \left[A \left\{ \frac{y_4 - (1-\alpha) y_3}{\alpha} \right\}^m + B \left\{ \frac{y_3 - (1-\alpha) y_2}{\alpha} \right\}^n \right] + (1-\alpha) y_4$$

$$= \alpha^{1-m} A \left\{ y_4 - (1-\alpha) y_3 \right\}^m + \alpha^{1-n} B \left\{ y_3 - (1-\alpha) y_2 \right\}^n + (1-\alpha) y_4 = \psi(y_4, y_3, y_2)$$

In general,

$$y_k = \alpha x_k + (1-\alpha) y_{k-1} = \alpha \left[A x_{k-1}{}^m + B x_{k-2}{}^n \right] + (1-\alpha) y_{k-1} = \alpha \left[A \left\{ \frac{y_{k-1} - (1-\alpha) y_{k-2}}{\alpha} \right\}^m + B \left\{ \frac{y_{k-2} - (1-\alpha) y_{k-3}}{\alpha} \right\}^n \right] + (1-\alpha) y_{k-1}$$

$$= \alpha^{1-m} A \left\{ y_{k-1} - (1-\alpha) y_{k-2} \right\}^m + \alpha^{1-n} B \left\{ y_{k-2} - (1-\alpha) y_{k-3} \right\}^n + (1-\alpha) y_{k-1} = \psi(y_{k-1}, y_{k-2}, y_{k-3}),\ k=4,5,\ldots,N$$

Therefore, a polynomial time series model of memory order 2, after applying simple exponential smoothing, is converted to a complex polynomial time series model of memory order 3. So, order of the polynomial memory increases after smoothing if the original order is 2.

2.5 Effect of Double exponentials smoothing in the memory of a linear Homoscedastic Signal

Double exponential smoothing is an effective tool of denoising used in signals. In the present work an effort has been put up to establish the recurrence matrix formulation of the associated weights in double exponential smoothing. In addition to this an analytic study has been incorporated to identify whether there is any change in the memory of a discrete signal with linear homoscedastic memory after employing double exponential smoothing in it. It has been observed that the memory of a discrete signal governed by autoregressive method of order 1 becomes heterogeneous while order 2 autoregressive method probably becomes random on application of double exponential smoothing

2.6 Double exponential smoothing:

The method of double exponential smoothing is governed by the following system of equations shown below

$$\left.\begin{aligned} x_1^{(p)} &= x_1\ ;\ b_1 = x_2 - x_1 \\ \text{and} \quad x_i^{(p)} &= \alpha x_i + (1-\alpha)(x_{i-1}^{(p)} + b_{i-1}) \\ b_i &= \beta\,(x_i^{(p)} - x_{i-1}^{(p)}) + (1-\beta) b_{i-1} \\ &\qquad (i = 2,3,4,\ldots\ldots,N) \end{aligned}\right\} \qquad (7)$$

where $\{x_i\}_{i=1}^{N}$ is the observed discrete signal, $\left\{x_i^{(p)}\right\}_{i=1}^{N}$ is the smoothed signal and $\{b_i\}_{i=1}^{N}$ is the trend traced in the signal, α and β are the 'signal smoothing parameter' and 'trend smoothing parameter' respectively. We have $0 < \alpha < 1$ and $0 < \beta < 1$.

2.7 Application of Double exponential smoothing in Homoscedastic Signal:

Next we investigate the effect in the memory in a linear homoscedastic memory driven signal after the application of

Simple Exponential Smoothing in it. Another interesting issue in this regard is to examine whether the estimated range of α as above remains consistent and sufficient to tally with the observed status of the memory in the smoothed signal.

3. Imapct of Double Exponential Smoothing on Discrete Signal

The present work focuses on the effect of *Double Exponential Smoothing* on the memory of a linear homoscedastic discrete signal. To serve the present purpose we have taken into account discrete signals governed in particular by autoregressive methods of order 1 and 2.

3.1 Case study on a discrete signal governed by autoregressive method of order *1*:

In this section we investigate the effect of *Double Exponential Smoothing* on the memory of a signal governed by autoregressive method of order *1*.

Let the observed discrete signal be $\{x_i\}_{i=1}^{N}$ and smoothed signal be$\{x_i^p\}_{i=1}^{N}$

As $\{x_i\}_{i=1}^{N}$ is governed by autoregressive of order *1* we have

$x_i=w_1x_{i-1}$ for $i=2,3,\ldots.N$ (8)

where w_1 is the process parameter (ignoring the noise).

Now by applying *Double Exponential Smoothing* as in (7) in the present signal we have by using (8)

$$\left.\begin{array}{c} b_1=(w_1-1)x_1 \\ x_2^{(p)}=w_1x_1 \\ b_2=(w_1-1)x_1 \\ x_3^{(p)}=\alpha x_3+(1-\alpha)(x_2^{(p)}+b_2) \end{array}\right\} \begin{array}{c}(9)\\ \\ (10)\end{array}$$

Using (8) and (9) in (10) we get

$$x_3^{(p)}=\{\alpha w_1+2(1-\alpha)\}x_2-(1-\alpha)x_1 \quad (11)$$

From (10) it is clear that $x_3^{(p)}$ shows a memory of length *2*.

Now from (7) we can have

$$b_3=\beta\,(x_3^{(p)}-x_2^{(p)})+(1-\beta)b_2 \quad (12)$$

Using (9), (10), (11) in (12) we get

$$b_3=\beta(\alpha w_1-2\alpha+2)x_2+\{\beta(\alpha-2w_1)+(w_1-1)\}x_1$$
(13)

Again using (11) and (13) we can have from (3.1) the following

$$x_4^{(p)}=\alpha w_1x_3+(1-\alpha)(\alpha w_1+2-2\alpha+\alpha\beta w_1-2\alpha\beta+2\beta)x_2+(1-\alpha)(\alpha-2+\alpha\beta-2\beta w_1+w_1)x_1$$
(14)

So $x_4^{(p)}$ shows a memory of length *3*.

Again using (11), (13) and (14) we can get from (7) the following

$$b_4=\alpha\beta w_1x_3+(\alpha\beta w_1-2\alpha\beta+2\beta-\alpha^2w_1-2\alpha+2\alpha^2-\alpha^2\beta w_1+2\alpha^2\beta-2\alpha\beta)x_2+(1-\alpha)(\alpha-1+\alpha\beta-2\beta w_1+w_1)x_1$$
(15)

Next using (14) and (15) we get

$$x_5^{(p)}=\alpha w_1x_4+(1-\alpha)(\alpha w_1x_3)+(1-\alpha)(\alpha w_1+2-4\alpha+2\alpha\beta w_1-6\alpha\beta+4\beta+4\alpha^2-2\alpha^2\beta w_1+4\alpha^2\beta)x_2+(1-\alpha)^2(2\alpha-3+2\alpha\beta-4\beta w_1+2w_1)x_1$$
(16)

So $x_5^{(p)}$ shows a memory of length *4*.

3.2 Case study on a discrete signal governed by autoregressive method of order *2*:

In this section we investigate the effect of *Double Exponential Smoothing* on the memory of a signal governed by autoregressive method of order *2*.

As $\{x_i\}_{i=1}^{N}$ is governed by AR (2) we can have

$$x_i=w_1x_{i-1}+w_2x_{i-2} \quad \text{for } i=3,4,\ldots.N \quad (17)$$

where w_1 and w_2 are process parameters (ignoring the noise).

If possible, let us assume that $\{x_i^{(p)}\}$ is also governed by AR (2).

So we can write

$$x_i^{(p)}=w_1'x_{i-1}^{(p)}+w_2'x_{i-2}^{(p)} \quad \text{for } i=3,4,\ldots..N \quad (18)$$

where w_1' and w_2' are the corresponding process parameters.

Using (7) we have from (18) for *i=4*

$$\alpha x_4=\alpha\{w_1'-(1-\alpha)(1+\beta)\}x_3+\{(2w_1'-3+2\alpha+2\alpha\beta)(1-\alpha)+w_2'\}x_2-(1-\alpha)\{w_1'+\alpha\beta+\alpha-2\}x_1 \quad (19)$$

As x_4 is governed by AR (2) we must have

$w_1'+\alpha\beta+\alpha-2=0$

$$\text{i.e. } w_1'=2-\alpha\beta-\alpha \quad (20)$$

Using (7) we have from (18) for *i=5*

$$\alpha x_5=\alpha\{w_1'-(1-\alpha)(1+\beta)\}x_4+\alpha[(1-\alpha)\{(1+\beta)w_1'+\alpha+\alpha\beta^2-1-2\beta+2\alpha\beta\}+w_2']x_3+(1-\alpha)\{(3-2\alpha-2\alpha\beta)w_1'+2w_2'+5\alpha-4+7\alpha\beta-2\alpha^2-4\alpha^2\beta-2\alpha^2\beta^2\}x_2+(1-\alpha)\{w_1'(\alpha\beta+\alpha-2)-w_2'-4\alpha\beta-3\alpha+3+2\alpha^2\beta+\alpha^2+\alpha^2\beta^2+\alpha\beta^2\}x_1$$
(21)

As by our assumption x_5 also obeys AR (2) coefficients of x_1 and x_2 both will be separately zero.

Hence we can have

$$w_1'(\alpha\beta+\alpha-2)-w_2'-4\alpha\beta-3\alpha+3+2\alpha^2\beta+\alpha^2+\alpha^2\beta^2+\alpha\beta^2\}=0$$

$$(3-2\alpha-2\alpha\beta)w_1'+2w_2'+5\alpha-4+7\alpha\beta-2\alpha^2-4\alpha^2\beta-2\alpha^2\beta^2=0 \quad (22)$$

Solving equation (22) we get

$$w_2'=\alpha-1+\alpha\beta^2 \quad (23)$$

Comparing (20), (22) and (23) we get

$$\alpha\beta^2=0 \quad (24)$$

So at least one of $\alpha$and β is 0 .which is not possible as $\alpha>0$, $\beta>0$.

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So we arrive at a contradiction. Hence our previous assumption that $\{x_i^p\}$ is governed by AR (2) is wrong.

If possible, let us assume that $\{x_i^p\}$ is governed by AR (1) so $x_i^p = w_1' x_{i-1}^p$ for i=2,3,4….N (25)

Using (1) we have from (25) for *i=4*

$\alpha x_4 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_3 + (1-\alpha)\{2w_1' - (3-2\alpha-2\alpha\beta)\}x_2 - (1-\alpha)\{w_1' + (\alpha\beta+\alpha-2)\}x_1$ (26)

As x_4 obeys AR(2) we must have

$$w_1' + (\alpha\beta+\alpha-2) = 0$$

$$w_1' = (2-\alpha\beta-\alpha) \quad (27)$$

Using (7) we have from (25) for *i=5*

$\alpha x_5 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_4 + \alpha(1-\alpha)\{(1+\beta)w_1' - (1-\alpha+2\beta-2\alpha\beta-\alpha\beta^2)\}x_3 + (1-\alpha)\{w_1'(3-2\alpha\beta-2\alpha) - (4-5\alpha-7\alpha\beta+2\alpha^2+4\alpha^2\beta+2\alpha^2\beta^2\}x_2 + (1-\alpha)\{w_1'(\alpha\beta+\alpha-2) - (4\alpha\beta+3\alpha-3-\alpha^2-2\alpha^2\beta-\alpha^2\beta^2-\alpha\beta^2)\}x_1$ (28)

As by our assumption x_5 obeys AR(2) coefficient of x_2 and x_1 must be zero so we get

$w_1'(3-2\alpha\beta-2\alpha) = 4-5\alpha-7\alpha\beta+2\alpha^2+4\alpha^2\beta+2\alpha^2\beta^2$ (29)

Solving equation (27) and (29) we get $\alpha=1$

Which is a contradiction as $0<\alpha<1$ .hence our assumption that $\{x_i^p\}$ is governed by AR(1) is wrong .

Now let $\{x_i^p\}$ is governed by AR(3)

So $x_i^p = w_1' x_{i-1}^p + w_2' x_{i-2}^p + w_3' x_{i-3}^p$ for i=4,5,…..N (30)

Using (7) we have from (30) for *i=4*

$\alpha x_4 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_3 + 2\{(1-\alpha)w_1' + w_2 - (1-\alpha)(3-2\alpha-2\alpha\beta)\}x_2 - \{w_3' - (1-\alpha)w_1' - (1-\alpha)(\alpha\beta+\alpha-2)\}x_1$

As $\{x_i\}$ is governed by AR(2) we must have

$w_3' - (1-\alpha)w_1' = (1-\alpha)(\alpha\beta+\alpha-2)$ (31)

Again, we have $x_5^p = w_1' x_4^p + w_2' x_3^p + w_3' x_2^p$.

Using (7) putting the expressions of $x_5^p, x_4^p, x_3^p, x_2^p$ we get

$\alpha x_5 = \alpha\{w_1' - (1-\alpha)(1+\beta)\}x_4 + \alpha\{(1-\alpha)(1+\beta)w_1' + w_2' - (1-\alpha)(1-\alpha+2\beta-2\alpha\beta-\alpha\beta^2)\}x_3 + \{(1-\alpha)(3-2\alpha\beta-2\alpha)w_1' + 2(1-\alpha)w_2' + w_3' - (1-\alpha)(4-5\alpha-7\alpha\beta+2\alpha^2+4\alpha^2\beta+2\alpha^2\beta^2\}x_2 + (1-\alpha)\{w_1'(\alpha\beta+\alpha-2) - w_2' - (4\alpha\beta+3\alpha-3-\alpha^2-2\alpha^2\beta-\alpha^2\beta^2-\alpha\beta^2)\}x_1$ (32)

As by our assumption $x_5$isgoverned by AR(2) coefficient of x_1 and x_2 is zero .so we get

$w_1'(\alpha\beta+\alpha-2) - w_2' = 4\alpha\beta+3\alpha-3-\alpha^2-2\alpha^2\beta-\alpha^2\beta^2-\alpha\beta^2$ (33)

And $(1-\alpha)(3-2\alpha\beta-2\alpha)w_1' + 2(1-\alpha)w_2' + w_3' = (1-\alpha)(4-5\alpha-7\alpha\beta+2\alpha^2+4\alpha^2\beta+2\alpha^2\beta^2)$ (34)

By applying Cramer's rule, it is examined that the above system of three equations (31, 33 and (34) is inconsistent .so we arrive a contradiction. Hence our assumption that $\{x_i^p\}$ is governed by AR(3) is wrong.

3.3 Example: A practical demonstration of the scheme proposed in section 2.2.

To demonstrate the proposed scheme as in Section 3.2.1 here the monthly average data of solar radio flux is taken at a wavelength of 10.7 cm ranging from 14 February 1947 to 31 March 2015 observed at National Research Council and Natural Resources, Canada with support from the Canadian Space Agency, previously known as Dominian Radio Astrophysical Observatory (DRAO) at local noon in a bandwidth of 100 MHz. The units are in solar flux units (1 s.f.u. = 10−22 m−2 Hz−1). (Source: Solar radio flux-Archive of measurements http://www.spaceweather.gc.ca/solarflux/sx-5-en.php). Figure 1 and 2 shows the profiles of the original monthly average data and corresponding smoothed data respectively. Figure 3 depicts the story of the observed trends and Figure 4 exhibits the estimated noise.

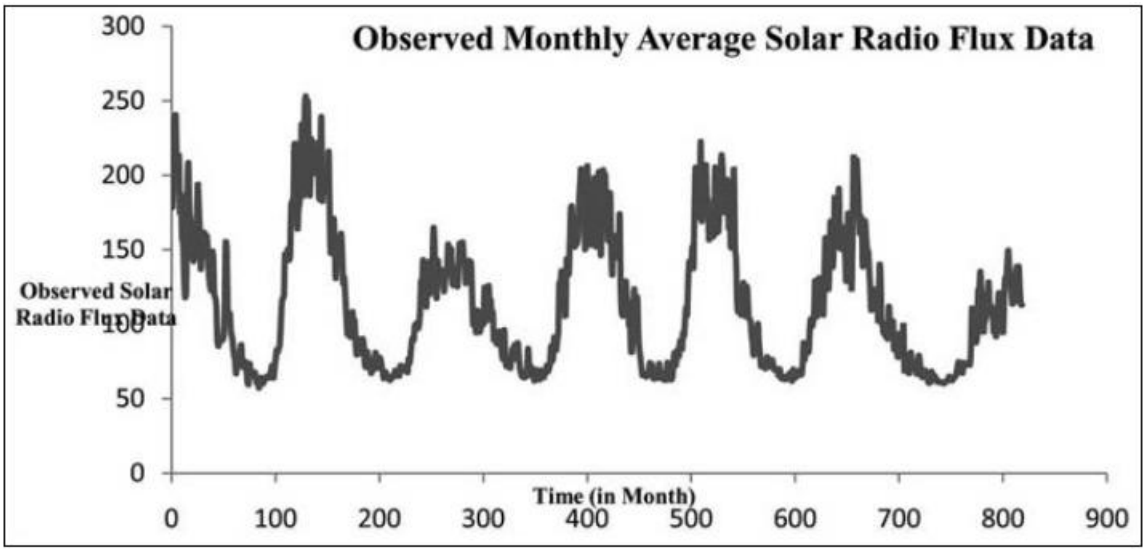

Figure 1

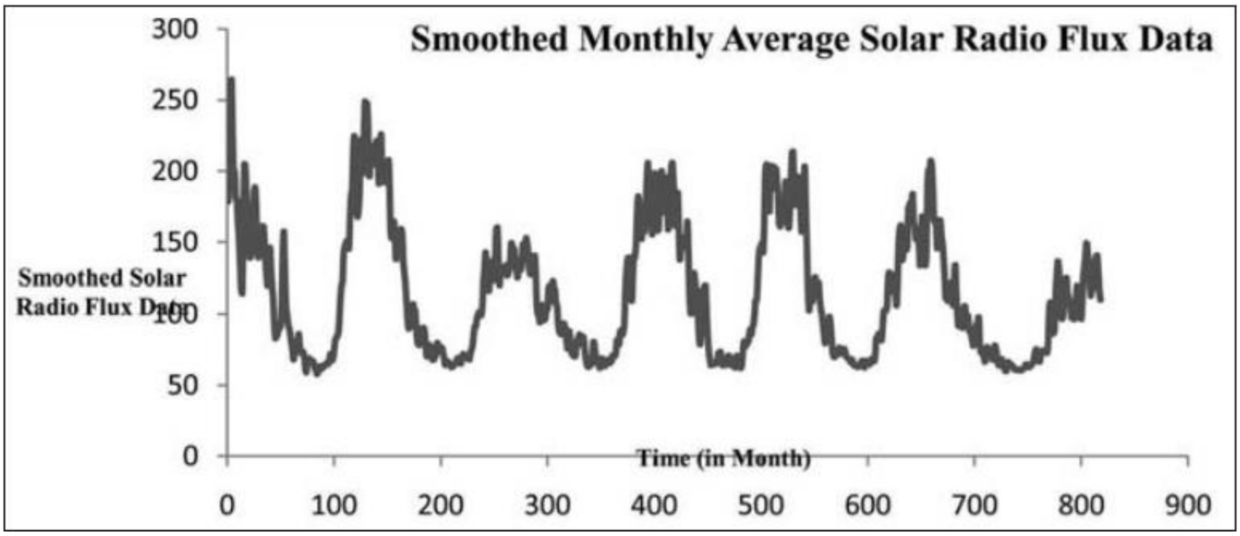

Figure 2

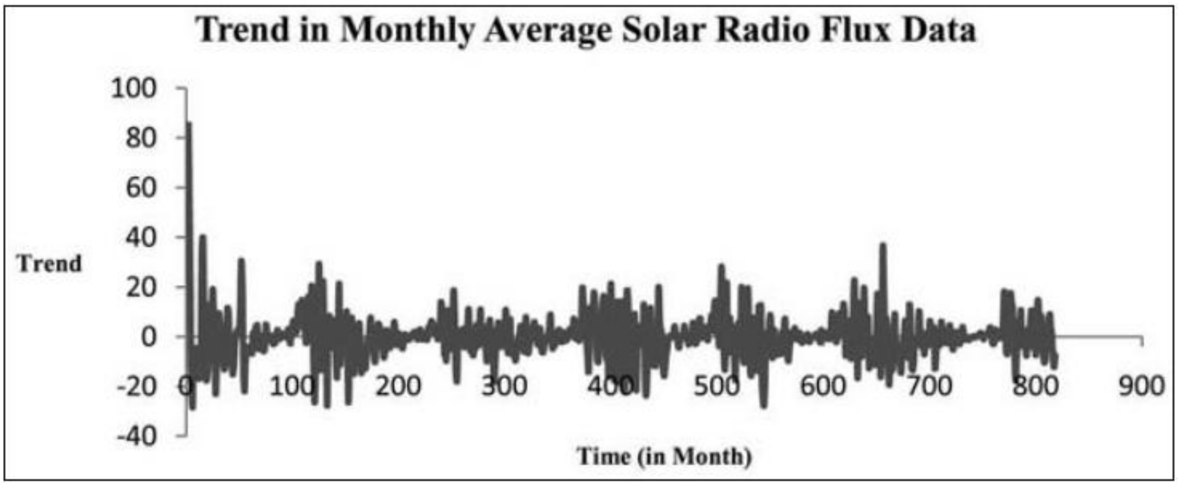

Figure 3

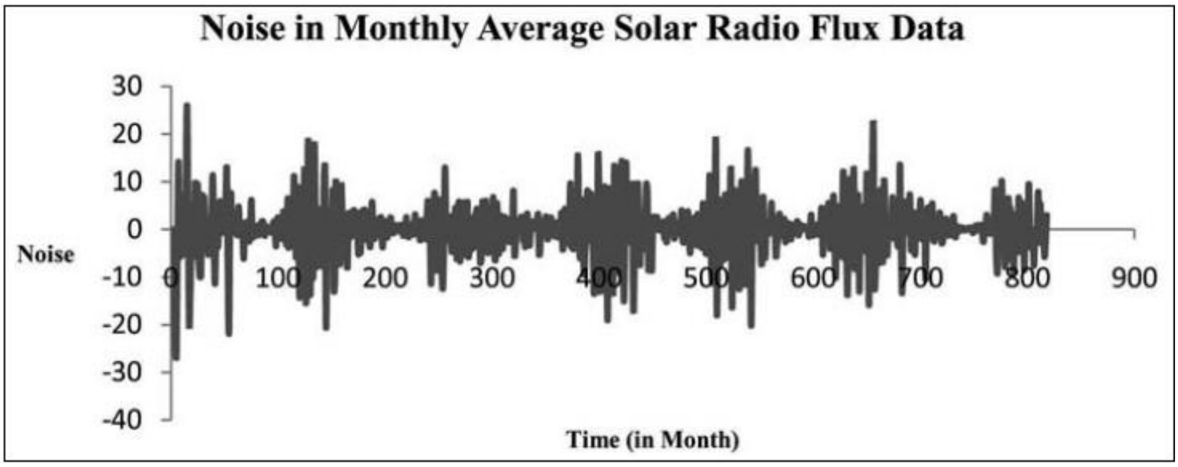

Figure 4

4. Result

For signals with linear n-length homoscedastic memory, we have shown that simple exponential smoothing preserves its homoscedastic behaviour while extending the memory length (n + 1) The only exception occurs in the step (n + 1), where the memory length remains at the original value n. In contrast, when a discrete signal is originally followed by the AR(2) structure, the application of double exponential smoothing tends to destroy its memory properties; the resulting series cannot show recognizable memory of the order 1, 2 or 3. Since the initial memory length is 2, it is not unreasonable to expect the smooth series to have a longer memory, such as the fourth or higher order. However, a polynomial time series model with memory order 2 is transformed into a more complex polynomial model with memory order 3 by applying Simple Exponential Smoothing Thus, if the order of the original polynomial is 2, smoothing increases the order of the memory. However, under double-exponential smoothing, the memory structure is randomly lost and lost its original order.

5. Conclusion

The result is interesting from the perspective of the previous conclusion (Saha, Ghosh, and Choudhuri, 2016) that a discrete signal, which obeys AR (2), is converted to an AR (3) signal by simple exponential smoothing. In addition, in a two-order polynomial time series model, it is converted to a three-order complex polynomial time series model after applying simple exponential smoothing. This phenomenon is not observed in the current application of double exponential smoothing. Furthermore, this study shows that when applying double exponential smoothing to a signal controlled by AR (1), we can eventually find a sequential heterogeneous memory, and that AR(2) can also be converted to random behavior. Further research could be carried out to identify possible memory changes in discrete signals with linear homogeneous memory lengths (3 or greater) after applying double exponential smoothing. Future studies could also be conducted to understand the influence of double-expanding smoothness in discrete signals with nonlinear homosedastic memory. Monthly average data on solar radiation flow from February 14, 1947 to March 31, 2015 were recorded by the National Research Council and the National Resources Commission of Canada, and profiles of corresponding smooth signals, trends and noises were presented.

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