The Studies of Usefulness of Compact Finite Derivatives and Navier-Stokes Equations to Identify the Significance of Interpolation Strategies

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Abstract: ***Having the limitations like Grid requirements and complexity and enabling the advantages like high accuracy, computational efficiency as well as numerical stability Compact Finite Differential Methods has applications like Navier-Stokes equations, Burger's equation, Boundary layer flows and Advection-dispersion equations in the field of fluid dynamics, a field where Compact Finite Methods (CFMs) seems to better fit for finding solutions. In this paper we are going to study about usefulness of compact finite derivatives in certain conditions like Navier-Stokes equations. Implementation of Navier-Stokes equation in the field of fluid dynamics results to the accuracy attribute of CFMs.***

Keywords: Fluid Dynamics, Navier-Stokes equation, Interpolation strategies, Reynolds number

1.Introduction

Fluid dynamics has applications in the field of science, technology and environment that includes designing cars and aircrafts, developing renewable energy sources like wind turbines, weather forecasting, and understanding blood circulation in body as well as for engineering applications such as HVAC systems, pipelines and optimizing sports equipments. Numerical methods are applied in fluid dynamics, particularly through Computation Fluid Dynamics, to solve the partial differential equations that govern fluid flow by discretizing them into a solvable form. This enables to develop applications to design vehicles, aircrafts, and hydraulic structures like dams and water supply networks by executing simulations and analysis of fluid behaviour in a number of fields like civil, environment, automotive, and aerospace, where engineering is likely to be implemented. In comparison with the other methods like Finite Difference (FD) and Finite Volume (FV) that can be applied in the field of fluid dynamics to have applications Compact Finite Difference (CFD) method seems beneficial enabling to achieve high-order accuracy and to execute wider stencil to compute derivatives.

Compact finite methods are a class of high-accuracy numerical methods used in computational Fluid Dynamics to solve fluid flow equations like the Navier-Stokes equation.

In our earlier paper [1] we have already discussed about the basic principles, accuracy and stability, and boundary conditions of Compact Finite Difference (CFD). By using compact stencils, CFD requires fewer grid points and less memory, leading to faster computation compared to methods with a wider stencil. To achieve fourth-order accuracy scheme to minimize errors are constructed carefully, that plays important role to simulate high-Reynolds-number flows. In mathematics, the Navier-Stokes equations are examined to solve complex problems in fluid mechanics, by focusing on their mathematical properties, particularly the existence and smoothness of solutions for three dimensional turbulent flows. Recent researches continue to refine and extend CFD methods for the Navier-Stokes equations to examine and conclude high-order accuracy, non-uniform and curvilinear grids, multiphysics applications, adaptive mesh refinement, high performance computing (HPC) and Artificial compressibility method. Ashwani Punia et. al. [2] in their study using a time-marching methodology and modified artificial compressibility method by employing pressure calculated via a pressure- correction technique attempted the benchmark problem of 3D lid-driven cavity, analyzed the varied rheological behaviour of shear-thinning ($n = 0.5$ and $n = 1.5$), and Newtonian ($n = 1.0$) fluids across different Reynolds numbers ($Re = 1,50,100,200$) concluded by investigating Non-Newtonian and Newtonian results in terms of velocity variation, streamlines, viscosity contours, pressure distribution and validated the computed results with the existing benchmark results that existing results were in agreement with their study.

2.Description

It is common fact that processing speed has increased significantly faster than memory speed in recent decades [3]. In the time needed to retrieve some data from main memory, modern computers can do tens or even hundreds of floating-point operations. Therefore, it is usually advisable to decrease memory use, even if doing so requires more processing, in order to increase speed and efficiency.

Due to their low precision, classic first- and second-order finite difference methods require narrow grids, which mean they use a lot of memory when dealing with numerical simulations. Higher-order techniques can preserve accuracy while drastically lowering memory needs. Reducing memory access frequently results in a speed boost that is far larger than the expense of extra processing. One of the primary drivers behind the creation of high-order numerical techniques is this.

Compact finite differences are one method to get high-order discretizations. Compact finite difference schemes have been the subject of numerous investigations. A series of such high-order compact formulations was proposed by Lele in [4]. Some benefits of fourth-order compact methods over conventional techniques were covered by Adam [5] and Hirsh [6]. The sixth-order compact scheme has good agreement with linear stability theory, as shown by Souza et al. in [7].

Compact finite differences are typically used to compute function derivatives implicitly, meaning that the derivative values at a given node are calculated using both the function values and the derivative values at nearby nodes. Compact finite difference methods are still being researched since they are difficult to apply to the Navier-Stokes equations.

For the numerical solution of the Navier-Stokes equations, there are two common families of approaches: the methods based on primitive variables and the methods based on the vorticity-stream function formulation. Compact finite difference schemes in conjunction with the vorticity-stream function formulation [8] [9] [10] have been the subject of several recent investigations. They also have the benefit of not having pressure-decoupling issues. However, vorticity-stream function approaches are frequently regarded as ineffective for handling 3D situations and complex boundary conditions [11].

The use of projection methods, which are based on primitive variables, is a common substitute [12], [13]. When compared to vorticity-stream function methods, the computing cost for 2D problems is almost the same, and there is a significant benefit for 3D problems. Furthermore, a staggered grid can be used to address the pressure-decoupling problem.

The works of Bell et al. [14] and Kim and Moin [15] are two of the most well-known primitive variables projection techniques for Navier-Stokes problems. The accuracy of both approaches is second-order. The primary distinction between the two is that Kim and Moin do not employ an approximation for the pressure in the momentum equation to achieve second-order, whereas Bell et al. do. Rather, Kim and Moin employ more complex boundary constraints for the velocity field in between. In contrast to pressure-free projection methods, which do not employ an approximation for the pressure in the momentum equation, incremental pressure projection methods are a class of projection methods that do [16]. Other well-liked techniques include Gresho's [17], [18], which tackles the problem of boundary conditions for the intermediate velocity field in a finite element framework, and Strikwerda's [19], which applies to non-uniform finite-difference grids.

3.Literature Review

Various temporal discretizations in conjunction with fourth-order in-space compact finite difference approaches have been documented in the literature for incompressible flows. The momentum equation is treated using a semi-implicit Crank–Nicholson/Adams–Bashforth technique in [20]. For the pressure field, only first-order temporal precision is seen, when second-order accuracy would be anticipated. The formal precision of the compact stencils employed in [21] varies. Rather, second- to sixth-order accuracy varies. This is reflected in the outcomes, and the accuracy of the procedure varies according on the problem. Similar outcomes are obtained when variable accuracy stencils are employed in [22]. In [23], traditional 5- or 7-point stencil discretization is employed in an iterative manner rather than immediately applying compact techniques to the Poisson equation.

4.Results

Compared to earlier works, this one is distinct because;

- It uses primitive variables on a staggered grid.
- The only basis for spatial discretizations is fourth-order compact finite differences. Poisson's equation and domain boundaries are included in this.
- Instead of being applied roughly (up to the discretization accuracy), the discrete incompressibility requirement is applied precisely (up to machine precision).
- A second-order adams–bashforth approach is used to explicitly integrate the momentum equation in time.
- Both velocity and pressure are shown to have second-order temporal accuracy and fourth-order spatial accuracy.

The structure of this paper is as follows: A concise summary of well-known information regarding the Navier-Stokes equations, projection techniques, and compact finite differences is given in Sections 5–8, that forms the base of the method that we suppose to be beneficial to develop exact projection method for Navier-Stokes equation on a staggered grid.

5.Governing Equations

A Newtonian fluid can be regarded as incompressible at low Mach numbers (< 0.3) [24]. The Navier-Stokes equations in this instance can be reduced to two spatial dimensions,

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u}.\nabla\mathbf{u} - \nabla p + v\nabla^2\mathbf{u} + \mathbf{S} \qquad (1a)$$

$$\nabla.\mathbf{u} = 0 \qquad (1b)$$

where the applied force vector per unit mass is $S = (S_u, S_v)$, the fluid velocity vector is $\mathbf{u} = (u, v)$, the pressure over density ratio is p, and the kinematic viscosity is v. In this paper, density and viscosity are taken to be constant. The above presented equations (1a) and (1b) are referred to as the momentum equation and continuity equation and the derivative of **u** is over time t.

Applying Cartesian coordinates, the above equations take the form:

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = -\left(\frac{\partial(\mathrm{u}^2)}{\partial x} + \frac{\partial(uv)}{\partial y}\right) - \frac{\partial p}{\partial x} + v\left(\frac{\partial u}{\partial x^2} + \frac{\partial^2 u}{\partial \mathrm{y}^2}\right) + \mathrm{S}_{\mathbf{u}} \quad (2a)$$

$$\frac{\partial v}{\partial \mathrm{t}} = -\left(\frac{\partial(v^2)}{\partial y} + \frac{\partial(uv)}{\partial x}\right) - \frac{\partial p}{\partial y} + v\left(\frac{\partial v}{\partial x^2} + \frac{\partial^2 v}{\partial \mathrm{y}^2}\right) + \mathrm{S}_{v} \quad (2b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2c)$$

For the velocity field, initial and boundary conditions are specified. According to the continuity equation, the velocity boundary conditions must be such that there is no net flow into or out of the domain. Only Dirichlet boundary conditions for the velocity field will be taken into consideration in our presented work.

Although some numerical techniques might call for them, there are no physical boundary or beginning conditions for the pressure field. Nevertheless, only an additive constant can be used to calculate the pressure. Nevertheless, certain techniques for linear systems will solve them without the need for extra care.

6.Projection Methods

The absence of an explanation for the pressure's time evolution in the incompressible Navier-Stokes equations is a significant obstacle to the numerical simulation of incompressible flows. The "pressure equation," or continuity equation (1b), actually has nothing to do with pressure. The pressure functions as the Lagrange multiplier for the continuity equation, a restriction that requires the resulting velocity field to be divergence-free. The pressure and velocity fields become coupled as a result. By dividing the solution of the coupled system into distinct phases for the velocity and pressure, projection methods, first independently presented by [12,13], seek to address this issue. By successfully decoupling the equations, this lowers computational costs while preserving accuracy and consistency in solving time-dependent issues.

A second-order projection method can be applied to obtain exact projection method for Navior-Stokes equation on a staggered grid [25]. An intermediate velocity field $'\mathbf{u}'$ is defined, and it can be computed using:

$$\frac{\boldsymbol{u}^{*}-\boldsymbol{u}^{n}}{\Delta t} = -\boldsymbol{u}^{\mathrm{n}+\frac{1}{2}}.\nabla\mathbf{u}^{\mathrm{n}+\frac{1}{2}} + v\nabla^2\boldsymbol{u}^{\mathrm{n}+\frac{1}{2}} + \mathrm{S}^{\mathrm{n}+\frac{1}{2}} \quad (3)$$

In the above presented equation at discrete time t^{n}, $\mathbf{u}^{\mathrm{n}}$ is the velocity, along with other quantities with superscripts. The time step is here Δt. At next discrete time the velocity is computed using

$$\frac{\mathbf{u}^{\mathrm{n}+1}-\mathbf{u}^{*}}{\Delta t} = -\nabla p^{\mathrm{n}+1/2} \quad (4)$$

With a second-order time discretization of (1a) Eqs. (3) and (4) remains consistent. The following Adams-Bashforth method has been used in our study to explicitly calculate the intermediate velocity:

$$\mathbf{F}\left(\mathbf{u}^{\mathrm{n}+1/2}\right) - \frac{3}{2}\mathbf{F}(\mathbf{u}^{\mathrm{n}}) - \frac{1}{2}\mathbf{F}(\mathbf{u}^{\mathrm{n}-1}) + \mathrm{o}(\Delta \mathrm{t}^2) \quad (5)$$

Where

$$\mathbf{F}(\mathbf{u}) = -\mathbf{u}.\nabla\mathbf{u} + v\nabla^2\mathbf{u} + \mathbf{S} \quad (6)$$

By taking the divergence of (4) and replacing the fact $\nabla.\mathbf{u}^{\mathrm{n}+1} = 0$ implicit calculation of pressure is executed to yield

$$\nabla.\nabla p^{\mathrm{n}+1/2} = \frac{1}{\Delta t}\nabla.\mathbf{u}^{*} \quad (7)$$

If required, at time $t^{\mathrm{n}+1}$, the pressure can be calculated from

$$p^{\mathrm{n}+1} = \frac{3}{2}p^{\mathrm{n}+1/2} - \frac{1}{2}p^{\mathrm{n}-1/2} + \mathrm{o}(\Delta t^2). \quad (8)$$

Using a first-order explicit Euler method for the first time step is equal to initializing the Adams–Bashforth method with the assumption that $\mathbf{u}^{-1} = \mathbf{u}^{\mathrm{o}}$, and similarly for other quantities. The numerical testing will show that this had no appreciable detrimental effects on the temporal precision. Similar to [15], this projection method is non-incremental or pressure-free because the velocity is independent of past pressure values [16]. However, in contrast to [15], the current approach achieves second-order temporal accuracy without the need for complex boundary conditions for the intermediate velocity field $\mathbf{u}^{*}$. For $\mathbf{u}^{*}$, the same boundary requirements that apply to $\mathbf{u}^{\mathbf{n}+1}$ also apply.

7.Spatial Discretization

The issue of velocity–pressure decoupling, which results in misleading pressure oscillations [26], plagues numerical simulations using collocated grids and primitive variables. The current study uses a staggered grid [27], as depicted in Fig. 1, to get around this issue.

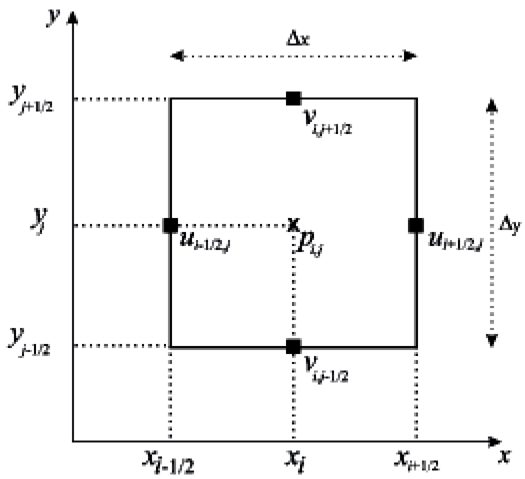

Figure 1: In a staggered grid, the velocities are samples on the cell's edges, while the pressure is in the center

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Particular consideration must be given when using compact finite differences in conjunction with a staggered grid. In order to avoid the unknowns being calculated in the same locations as the known quantities, several compact finite differences stencils will first be staggered. Second, there will be some irregular boundary stencils. And third, for convective terms interpolation plots are required.

8.Compact Finite Derivatives

Because the increase in order of approximation is not exclusively attributable to the addition of more points to the computational stencil, the so-called compact finite difference approaches are especially tempting. Rather, the equation that approximates the derivative implicitly includes the number being calculated, or the derivative. The benefits of this class of approaches have been demonstrated in a number of published articles [4,7,28].

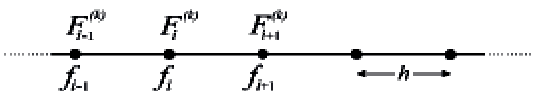
For instance, if the derivative approximations themselves are incorporated into the equation, it is feasible to obtain up to fourth-order precision when approximating the first or second derivatives using no more than three points, which becomes

$$L_{i-1}F_i^k + L_iF_i^k + L_{i+1}F_{i+1}^k - R_{i-1}f_{i-1} + R_if_i + R_{i+1}f_{i+1} \quad (9)$$

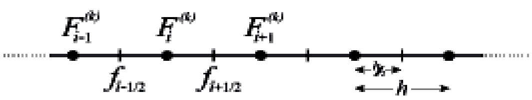
In above equation $F^k, k = \{1,2\}$ that is for kth derivative is the approximation and $f$is the function, as depicted in Fig. 2a. By matching the Taylor series of a number of orders left and right-side coefficients L_i and R_i can be derived. A number of studies have been conducted earlier for several precision, central schemes and stability [4,29,30].

Since ghost cells are not being used in this work, Eq. (9) cannot be used close to the domain boundary for nonperiodic boundary conditions. Non-central schemes are therefore necessary, as depicted in Fig. 3. Lower-order systems are occasionally employed to increase stability [4,31,32,21]. The behavior of fourth-order border schemes in conjunction with equal-order central schemes was examined by the authors in [29].

Another issue with staggered meshes is that, as seen in Figs. 2b and 3c, the unknowns are not sampled in the same locations as the function. The creation of particular staggered stencils is necessary for this. Compact approaches can also be useful for interpolation strategies, which are necessary for certain situations, such as the incompressible Navier-Stokes equations.

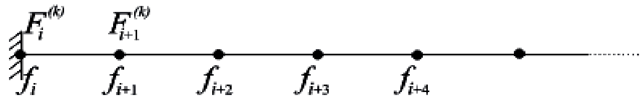


(a) Collocated grid

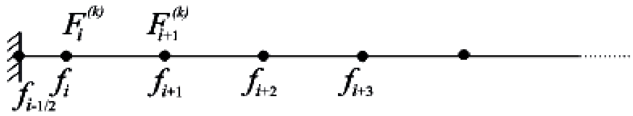


(b) Staggered Grid

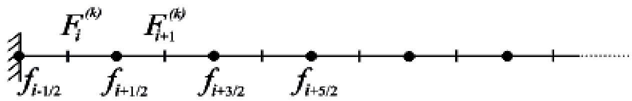
Fig.2: Interior stencils for collocated (a) and staggered (b) grids $f_i = f(x_i), F_i^k \approx \frac{\partial^k f}{\partial x^k}|_{x_i}, k = \{1,2\}$ and h is the grid spacing. In the collocated grid, the derivatives of f are evaluated at the same nodes as f itself, while in the staggered grid, the derivatives are evaluated half a grid spacing away.



(a) Uniform collocated grid.



(b) Non-uniform collocated grid.



(c) Staggered Grid

Fig. 3: Stencils for uniform collocated (a), non-uniform collocated (b) and staggered (c) grids near the boundary. $f_i - f(x_i), F_i^k \simeq \frac{\partial^k f}{\partial x^k}|_{x_i}, k = \{1,2\}$. In the non-uniform grid, the boundary is half a grid spacing away in the derivative direction.

The boundary (non-central) and interior (central) schemes taken into consideration in this work are both fourth-order precise.

9.Conclusion

Interpolation strategies are required for numerically solving the Navier-Stokes equations in computation fluid dynamics (CFD). Interpolation is used to approximate flow-variables, such as velocity, pressure and temperature, at points where they are not directly computed. Interpolation strategies using the Navier-Stokes equation within the context of compact finite methods are useful for achieving high accuracy, stability, and efficiency in computational fluid dynamics.

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