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Numerical Study of the 2D Incompressible Navier-Stokes Equations

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Abstract: ***The numerical investigation of the two-dimensional incompressible Navier-Stokes equations using the stream function-vorticity formulation and finite-difference methods. A second-order central-difference spatial discretization combined with an explicit time-stepping scheme for vorticity transport and an iterative Poisson solver for the Stream function is implemented. The lid-driven cavity flow is used for validation and results are reported for Reynolds number $Re = 100, 400, 1000$ and 5000. Grid convergence and residual behaviour are analysed. The numerical method demonstrates good agreement with classical benchmark results for velocity profiles and vortex structure and the study discusses stability, accuracy and computational cost considerations.***

Keywords: Navier-Stokes, Stream function-vorticity, finite difference, lid-driven cavity, iterative Poisson solver, Reynolds number.

1.Introduction

The incompressible Navier-Stokes equations govern a wide range of viscous fluid flows and a central focus of computational fluid dynamics. Two-dimensional canonical problems, such as the lid-driven cavity, provide rigorous benchmarks for testing numerical schemes. This paper develops a finite-difference solve in the stream function vorticity formulation, documents the numerical approach, validates it against established benchmarks and discuss numerical behaviour as Reynolds number increases.

2.Governing Equation

For an incompressible Newtonian fluid in 2D, the velocity components are $u(x, y, t)$ and $v(x, y, t)$. The stream function ψ and vorticity ω are defined by:

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}. \qquad \dots\dots\dots (1)$$

The Vorticity is

$$\omega = \frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y} = -\nabla^2\psi \qquad \dots\dots\dots (2)$$

The vorticity transport equation (non-dimensional form) is:

$$\frac{\partial\omega}{\partial z} + u\,\frac{\partial\omega}{\partial x} + v\,\frac{\partial\omega}{\partial y} = \left(\frac{1}{Re}\right)\nabla^2\omega \qquad \dots\dots\dots (3)$$

To obtain ψ from ω we solve the Poisson equation:

$$\nabla^2\psi = -\omega \qquad \dots\dots\dots (4)$$

3.Numerical Method

3.1 Grid and Domain

We consider a square domain $0 \le x, y \le 1$, discretized on a uniform cartesian grid with $N \times N$ nodes. Grid spacing $h = \frac{1}{N-1}$. Indices i, j denoted x and y directions respectively.

3.2 Spatial Discretization

The central difference approximations for the first and second derivatives with respect to x at a grid point (i, j) are correctly represented using standard mathematical notation as:

$$\frac{\partial\omega}{\partial x}\bigg|_{i,j} \approx \frac{\omega_{i+1,j} - \omega_{i-1,j}}{2h} \text{ and } \frac{\partial^2\omega}{\partial x^2}\bigg|_{i,j} \approx \frac{\omega_{i+1,j} - 2\omega_{i,j} + \omega_{i-1,j}}{h^2}$$

These are standard second order central difference formulas used numerical methods to approximate derivatives on a uniform grid with spacing h.

- The formula for the first derivative approximates the slope using values form the points immediately before and after the central point (i, j).
- The formula for the second derivative approximates the curvature using values from the three consecutive points along the $x-$ axis.

Advection terms in (3) are discretized with central differences for second order accuracy. If higher Reynolds number cause spurious oscillations, a hybrid upwind can be used.

3.3 Time Integration

We use an explicit second order Rang - Kutta method for the verticity transport:

1. Predictor Stage:

The predictor step uses the rate of change $R(\omega^n)$ at the current time level n to estimate an intermediate value w^* at the next level, Δt later. This is an explicit Euler step.

$$w^* = w^n + \Delta t \, . \, R(w^n)$$

2. The corrector step refines the estimate by using the average of the rate of change at the beginning of the interval, $R(w^n)$ and the rate of change at the predicted end of the interval, $R(w^*)$. The is equivalent to the explicit trapezoidal rule for time integration.

$$w^{n+1} = w^n + \frac{\Delta t}{2} \, . \, (R(w^n) + R(w^*))$$

Where $R(\omega)$ includes advection and diffusion terms discretized spatially.

Time Step Δt is chosen under CFL and diffusion constraints:

$$\Delta t \le CFL * \frac{h}{\max(|u|, |v|)}$$

and $\Delta t \le 0.5 * Re * h^2$ for explicit diffusion stability (or use more precise diffusion constraint). CFL is typically $0.2 - 0.5$ for stability.

3.4 Poisson solve for stream function

At each time step, solve $\nabla^2 \psi = -\omega$ using an iterative solver. Here we Successive Over-Relaxation (SOR) with red-black ordering or Gauss-Seidel with over-relaxation.

The discrete Laplacian:

$$\psi_{i,j}^{(k+1)} = (1 - \omega)\psi_{i,j}^{k} + \frac{\omega}{4}\left(\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} + h^2 \omega_{i,j}\right)$$

SOR update Step:

$$\psi_{i,j}^{New} = (1 - \beta)\psi_{i,j}^{old} + \frac{\beta}{4}\left(\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} + h^2 \omega_{i,j}\right)$$

Here is a breakdown of the components:

- $\psi_{i,j}^{New}$: The updated value at grid point (i, j) in the current iteration.
- $\psi_{i,j}^{old}$: The value at grid point (i, j) from the previous iteration.
- β : The relaxation parameter, usually tuned between 1.5 and 1.9 experimentally to accelerate convergence.
- $\psi_{i+1,j}$, etc: Values of ψ at the neigh boring grid point.
- h^2: The square of the grid spacing.
- $\omega_{i,j}$: A source term (often related to the right-hand side of Poisson's equation, $\nabla^2 \psi = -\omega$).

The stopping criterion $\max|\text{residual}| < \text{tol}$, with a typical value like $\text{tol} = 1e - 6$.

3.5 Boundary Conditions

Lid-driven cavity:

- Top lid $(y = 1)$: The velocity components are $u = U_{lid}$ and $v = 0$. This condition drives the fluid flow inside the cavity.
- Other walls $(x = 0, y = 0)$: The no-slip condition applies, meaning the fluid velocity matches the wall velocity, which is zero for all stationary walls. Thus, $u = v = 0$ on these boundaries.

In stream function vorticity formulation:

- ψ is set constant on boundaries (can set $\psi = 0$ on solid walls).
- Wall vorticity ω_{wall} computed from boundary stream function and prescribed velocity using second order finite difference approximation:

$$\omega_{wall} = -2\frac{\psi_{interior} - \psi_{wall}}{h^2} - 2\frac{U}{h}$$

For the moving lid (where U is the tangential wall velocity), derived from discrete Laplacian at the wall using ghost points.

The example for the top boundary (at grid index $j = N$):

$$\omega_{i,N} = -2\frac{\psi_{i,N-1} - \psi_{i,N}}{h^2} - 2\frac{U_{lid}}{h}$$

3.6 Algorithm Summary

1. Initialize ψ and ω (e.g., zero).
2. Impose boundary velocities and compute initial ω on walls.
3. Loop over time:
   a. Solve Poisson equation $\nabla^2 \psi = -\omega \rightarrow \psi$ (SOR)
   b. Compute u, v from ψ via central differences.
   c. Compute advection and diffusion terms $R(\omega)$.
   d. Advance using ω using RK2 to ω^{n+1}.
   e. Enforce wall vorticity boundary conditions.
   f. Check convergence to steady-state ($\|\boldsymbol{\omega}^{n+1} - \boldsymbol{\omega}^{n}\|$) or continue until **t_max.**
4. Postprocess: contours, velocity profiles, streamlines.
5. Validation: Lid-Driven Cavity

We Validate against classical benchmark: velocity profiles along vertical centerline (u at $x = 0.5$) and horizontal centerline (v at $y = 0.5$) and vortex locations for $Re = 100, 400, 1000, 5000$.

4.Grid Convergence

Use grids: $N = 65, 129, 257$. Compare centerline velocities and maximum vorticity, compute L2 and L∞ differences between successive grids to estimate convergence rates. Expect near second order convergence for interior points.

5.Results

5.1 Numerical Setup

- Domain: $[0,1] \times [0,1]$.
- Lid velocity $U_{lid} = 1.0$.
- Time stepping: RK2 with adaptive Δt based on CFL and diffusion.
- Poisson solver: SOR with β tuned ($\beta = 1.7$ typical).
- Convergence tolerance for Poisson:$1e - 6$.
- Steady-state criterion: max change in stream function $< 1e - 8$ or max change in vorticity $< 1e - 8$ between time iterations (or run until t sufficiently large and residual small).

5.2 Representative Results

- $Re = 100$: Single primary vortex occupying most of the cavity; weak corner vortices.
- $Re = 400$: Stronger primary vortex shifted toward the center; pronounced secondary vortices in lower corners.
- $Re = 1000$: Primary vortex stronger and center shifts; clear tertices may vortices may appear near corners.
- $Re = 5000$: Flow more complex; secondary vortices larger; numerical resolution must be increased (fine grid $N \geq 257$ recommended).

5.3 Sample Quantitative Comparison

Plot $u(0.5, y)$ and $v(x, 0.5)$ against benchmark data (e.g., Ghia et al.1982). Typical agreement within a few percent for $N = 129$ at $Re \leq 1000$. For $Re = 5000$, discrepancies increase unless grid is refined.

5.4 Convergence Behaviour

- Poisson SOR iterations: iteration count grows with grid size and Re; multigrid could dramatically reduce iterations.
- Time-stepping: Explicit scheme requires small Δt at high Re; implicit or semi-implicit schemes can alleviate stiffness.

6.Error and Stability Analysis

- Spatial discretization is second-order accurate in smooth regions.
- Central differencing for advection is non-dissipative may create spurious oscillations at high Re; consider adding selective unwinding or TVD limiting.
- Explicit treatment of diffusion and advection requires small Δt for stability: $\Delta t \sim O(h^2)$ for diffusion-dominated, $O(h)$ for advection-dominated flows.
- Using RK2 improves temporal accuracy compared flows.

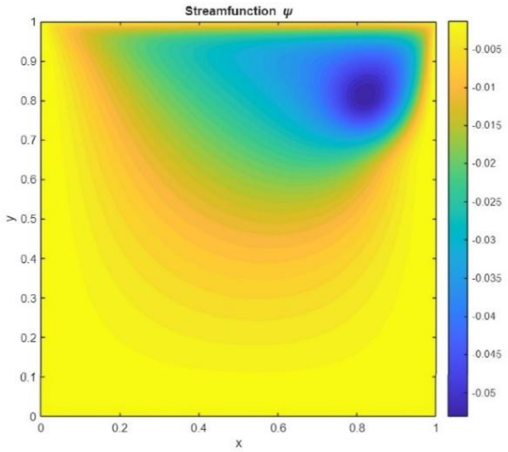
7.Numerical Experiments

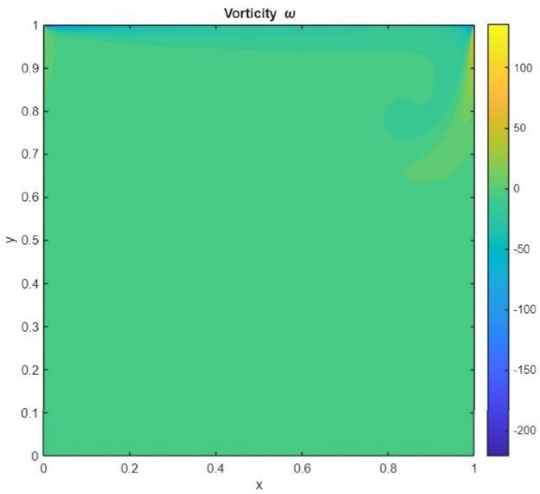
Below is a compact MATLAB-LIKE skeleton to implement the solver.

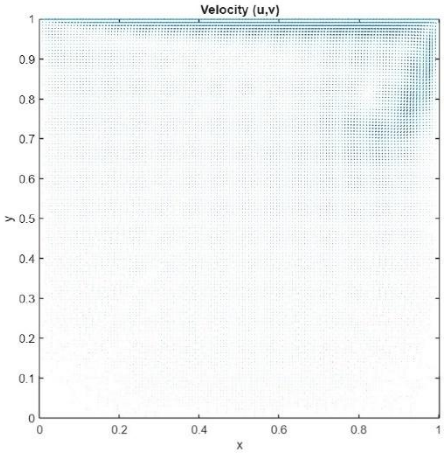
Provide sample plots (streamfunction contours, vorticity contours, velocity profiles) for Re = 100 and Re = 1000. Report quantities of interest: peak streamfunction, center of primary vortex, and comparisons with literature values.

Starting solver: Nx=129 Ny=129 dt=1.0e-04 Re=1000
step 500, max |domega| = 1.098e-01, max|omega|=2.191e+02
step 1000, max |domega| = 4.582e-02, max|omega|=2.210e+02
step 1500, max |domega| = 2.989e-02, max|omega|=2.212e+02
step 2000, max |domega| = 2.247e-02, max|omega|=2.212e+02
step 2500, max |domega| = 1.751e-02, max|omega|=2.212e+02
step 3000, max |domega| = 1.455e-02, max|omega|=2.212e+02
step 3500, max |domega| = 1.241e-02, max|omega|=2.212e+02
step 4000, max |domega| = 1.064e-02, max|omega|=2.212e+02
step 4500, max |domega| = 9.420e-03, max|omega|=2.212e+02
step 5000, max |domega| = 8.411e-03, max|omega|=2.212e+02
step 5500, max |domega| = 7.503e-03, max|omega|=2.211e+02
step 6000, max |domega| = 6.841e-03, max|omega|=2.211e+02
step 6500, max |domega| = 6.231e-03, max|omega|=2.211e+02
step 7000, max |domega| = 5.685e-03, max|omega|=2.211e+02
step 7500, max |domega| = 5.251e-03, max|omega|=2.211e+02
step 8000, max |domega| = 4.823e-03, max|omega|=2.211e+02
step 8500, max |domega| = 4.477e-03, max|omega|=2.211e+02
step 9000, max |domega| = 4.149e-03, max|omega|=2.211e+02
step 9500, max |domega| = 3.889e-03, max|omega|=2.211e+02
step 10000, max |domega| = 3.711e-03, max|omega|=2.211e+02

step 10500, max |domega| = 3.539e-03, max|omega|=2.211e+02
step 11000, max |domega| = 3.379e-03, max|omega|=2.211e+02
step 11500, max |domega| = 3.218e-03, max|omega|=2.211e+02
step 12000, max |domega| = 3.071e-03, max|omega|=2.211e+02
step 12500, max |domega| = 2.919e-03, max|omega|=2.211e+02
step 13000, max |domega| = 2.817e-03, max|omega|=2.211e+02
step 13500, max |domega| = 2.710e-03, max|omega|=2.211e+02
step 14000, max |domega| = 2.600e-03, max|omega|=2.211e+02
step 14500, max |domega| = 2.497e-03, max|omega|=2.211e+02
step 15000, max |domega| = 2.390e-03, max|omega|=2.211e+02
step 15500, max |domega| = 2.297e-03, max|omega|=2.211e+02
step 16000, max |domega| = 2.219e-03, max|omega|=2.211e+02
step 16500, max |domega| = 2.138e-03, max|omega|=2.211e+02
step 17000, max |domega| = 2.054e-03, max|omega|=2.211e+02
step 17500, max |domega| = 1.975e-03, max|omega|=2.211e+02
step 18000, max |domega| = 1.911e-03, max|omega|=2.211e+02
step 18500, max |domega| = 1.844e-03, max|omega|=2.211e+02
step 19000, max |domega| = 1.775e-03, max|omega|=2.211e+02
step 19500, max |domega| = 1.714e-03, max|omega|=2.211e+02
step 20000, max |domega| = 1.661e-03, max|omega|=2.211e+02







8.Discussion

The stream function-vorticity finite-difference approach is robust for 2D incompressible flows and avoids pressure-velocity coupling complexities. However, wall vorticity boundary treatment is delicate and strongly influences accuracy. Explicit time-stepping is simple but may be inefficient at high Re. The biggest computational bottleneck is the Poisson solver; multigrid or fast Poisson solvers are recommended for practical simulations.

9.Conclusion

A finite-difference stream function-vorticity solver was developed and validated for the 2D lid-driven cavity. Results show good agreement with classical benchmarks for moderate Reynolds number. To extend this work: implement multigrid Poisson solver, test higher-order advection schemes and apply to other canonical flows (flow past a cylinder, backward-facing step).

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Appendix

```
function lid_driven_cavity_fixed
% Lid-driven cavity (vorticity-stream)
clear; clc; close all;
%% ----- Grid & parameters -----
Nx = 129; Ny = 129;
L = 1.0;
h = L/(Nx-1);
x = linspace(0,L,Nx);
y = linspace(0,L,Ny);
[X,Y] = meshgrid(x,y);
Re = 1000;
U  = 1.0;
psi   = zeros(Nx,Ny);
omega = zeros(Nx,Ny);
% Dirichlet psi = 0 at walls (already zero)
beta = 1.7;             % unused for direct solver, kept for reference
tol_poisson = 1e-10;    % tolerance only used for diagnostic
dt = 1e-4;              % conservative timestep to avoid blowup
maxSteps = 20000;
tol_steady = 1e-8;
% initialize omega from psi BC
omega = apply_vorticity_BC(omega, psi, h, U);
% Prebuild Poisson matrix for interior (so we reuse factorization implicitly)
[Poisson.A, Poisson.nx, Poisson.ny] = build_poisson_matrix(Nx, Ny);
fprintf('Starting solver: Nx=%d Ny=%d dt=%.1e Re=%d\n', Nx, Ny, dt, Re);
%% ----- Time stepping (RK2) -----
for step = 1:maxSteps

  % 1) Solve psi from current omega (direct sparse solve)
  psi_interior = solve_poisson_direct(omega, Poisson, h);
  psi(2:Nx-1, 2:Ny-1) = psi_interior;   % place interior solution (boundaries remain 0)
  if any(~isfinite(psi(:)))
     error('Non-finite psi after Poisson solve at step %d', step);
  end

  % 2) velocity from psi
  [u,v] = velocity_from_stream(psi, h);
  [u,v] = enforce_velocity_BC(u, v, U);

  % 3) compute R at current state
  R = compute_R(omega, u, v, h, Re);

  % 4) predictor (RK2 stage 1)
  omega_pred = omega + dt * R;
  omega_pred = apply_vorticity_BC(omega_pred, psi, h, U);

  % Optional check
  if any(~isfinite(omega_pred(:)))
```

```
        fprintf('Non-finite omega_pred at step %d -> reducing dt or exiting\n', step);
        break;
    end

    % 5) solve psi_pred from omega_pred
    psi_pred_interior = solve_poisson_direct(omega_pred, Poisson, h);
    psi_pred = psi;  % copy
    psi_pred(2:Nx-1, 2:Ny-1) = psi_pred_interior;

    % 6) velocities from psi_pred
    [u2, v2] = velocity_from_stream(psi_pred, h);
    [u2, v2] = enforce_velocity_BC(u2, v2, U);

    % 7) compute Rstar at predicted state
    Rstar = compute_R(omega_pred, u2, v2, h, Re);

    % 8) RK2 corrector
    omega_new = omega + 0.5*dt*(R + Rstar);
    omega_new = apply_vorticity_BC(omega_new, psi_pred, h, U);

    % 9) diagnostics & safety checks
    if any(~isfinite(omega_new(:)))
        fprintf('Non-finite omega_new at step %d. Stopping and printing diagnostics.\n', step);
        disp('Max omega_pred, omega before stop:'), disp([max(abs(omega_pred(:))) max(abs(omega(:)))])
        break;
    end

    diffw = max(abs(omega_new(:) - omega(:)));
    omega = omega_new;

    if mod(step,500) == 0
        fprintf('step %d, max |domega| = %.3e, max|omega|=%.3e\n', step, diffw, max(abs(omega(:))));
    end

    if diffw < tol_steady
        fprintf('Converged at step %d; max change = %.3e\n', step, diffw);
        break;
    end
end

%% ----- Final fields & plotting -----
% Ensure final psi solved
psi_interior = solve_poisson_direct(omega, Poisson, h);
psi(2:Nx-1, 2:Ny-1) = psi_interior;
[u,v] = velocity_from_stream(psi, h);
[u,v] = enforce_velocity_BC(u,v,U);

figure; contourf(X,Y,psi',40,'LineStyle','none'); axis equal tight; colorbar;
title('Streamfunction \psi'); xlabel('x'); ylabel('y');

figure; contourf(X,Y,omega',40,'LineStyle','none'); axis equal tight; colorbar;
title('Vorticity \omega'); xlabel('x'); ylabel('y');

figure; quiver(X,Y,u',v'); axis equal tight;
title('Velocity (u,v)'); xlabel('x'); ylabel('y');

end

%% ---------------- helper functions ----------------

function [A, nx, ny] = build_poisson_matrix(Nx, Ny)
% returns sparse matrix A (size nx*ny) for the 5-point Laplacian
% interior grid size:
```

```
nx = Nx - 2;
ny = Ny - 2;
N  = nx * ny;

ex = ones(nx,1);
Tx = spdiags([ex -4*ex ex], -1:1, nx, nx);
Iy = speye(ny);
T = kron(Iy, Tx);

ey = ones(ny,1);
Ty = spdiags([ey ey], [-1 1], ny, ny);
A = T + kron(Ty, speye(nx));   % A has -4 on diag, 1 on neighbors
% A corresponds directly to: (psi_E + psi_W + psi_N + psi_S - 4 psi_C)
end

function psi_interior = solve_poisson_direct(omega, Poisson, h)
% Solve interior psi from omega: A * psi_vec = -h^2 * omega_interior(:)
nx = Poisson.nx; ny = Poisson.ny;
omega_interior = omega(2:end-1, 2:end-1);
b = - (h^2) * omega_interior(:);   % right-hand side
A = Poisson.A;
% Solve with MATLAB backslash (sparse direct)
psi_vec = A \ b;
% reshape
psi_interior = reshape(psi_vec, nx, ny);
% Safety guard
if any(~isfinite(psi_interior(:)))
   error('Poisson direct solve returned non-finite psi_interior.');
end
end

function [u, v] = velocity_from_stream(psi, h)
[Nx, Ny] = size(psi);
u = zeros(Nx, Ny);
v = zeros(Nx, Ny);
for i = 2:Nx-1
   for j = 2:Ny-1
      u(i,j) = (psi(i,j+1) - psi(i,j-1)) / (2*h);
      v(i,j) = -(psi(i+1,j) - psi(i-1,j)) / (2*h);
   end
end
end

function [u,v] = enforce_velocity_BC(u, v, U)
[Nx,Ny] = size(u);
u(:,Ny) = U; v(:,Ny) = 0;   % moving lid on top (j=Ny)
u(:,1)  = 0; v(:,1)  = 0;
u(1,:)  = 0; v(1,:)  = 0;
u(Nx,:) = 0; v(Nx,:) = 0;
end

function omega = apply_vorticity_BC(omega, psi, h, U)
[Nx,Ny] = size(psi);
% top (moving lid)
for i=2:Nx-1
   omega(i,Ny) = -2*psi(i,Ny-1)/h^2 - 2*U/h;
end
% bottom
for i=2:Nx-1
   omega(i,1) = -2*psi(i,2)/h^2;
end
% left
for j=2:Ny-1
```


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```
    omega(1,j) = -2*psi(2,j)/h^2;
end
% right
for j=2:Ny-1
    omega(Nx,j) = -2*psi(Nx-1,j)/h^2;
end
% corners (simple average)
omega(1,1)   = 0.5*(omega(2,1)   + omega(1,2));
omega(1,Ny)  = 0.5*(omega(2,Ny)  + omega(1,Ny-1));
omega(Nx,1)  = 0.5*(omega(Nx-1,1) + omega(Nx,2));
omega(Nx,Ny) = 0.5*(omega(Nx-1,Ny)+ omega(Nx,Ny-1));
end

function R = compute_R(omega, u, v, h, Re)
[Nx,Ny] = size(omega);
R = zeros(Nx,Ny);
for i = 2:Nx-1
    for j = 2:Ny-1
        dwdx = (omega(i+1,j) - omega(i-1,j)) / (2*h);
        dwdy = (omega(i,j+1) - omega(i,j-1)) / (2*h);
        lap  = (omega(i+1,j) + omega(i-1,j) + omega(i,j+1) + omega(i,j-1) - 4*omega(i,j)) / (h^2);
        R(i,j) = -(u(i,j)*dwdx + v(i,j)*dwdy) + (1/Re) * lap;
    end
end
end
```