Fuzzy Analogue of Conditional Probability & Transition of Fuzzy Phenomena

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Abstract: ***Fuzzy measure is an extension of probability measure and its physical meaning is similar to the membership function of a fuzzy set, the models presented in this paper are more suitable for describing the transition of fuzzy phenomena generally.***

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1. Introduction

A triple(X, $\mathscr{B}$, g) is called a fuzzy measure space, where $g(\cdot)$ is a set function & B is a Borel field of X. Fuzzy measure is a set of function by losing some properties of the probability measure $\sigma_y(\cdot \mid x)$ is a fuzzy measure of Y for any fixed $x \in X$. Then it is called a conditional fuzzy measure of Y with respect to X.

The paper is addressed to the application of fuzzy model of conditional fuzzy measure to transition of fuzzy phenomena.

The concept of fuzzy set introduced by L. A. Zadeh is a powerful tool by which we can treat many complicated problems of human behavior.

A fuzzy set A in X is characterized by a membership function A (set notation), which takes the value in the interval [0,1] i.e.

$$A : X \to [0,1]$$

The operations performed on fuzzy sets are listed as below:

Union: $A \cup B \Leftrightarrow \max(A(x), B(x))$

Intersection: $A \cap B \Leftrightarrow \min(A(x), B(x))$

Compliment: $\overline{A} \Leftrightarrow 1 - A(x)$

Algebraic Product: $A.B = A(x) \cdot B(x)$

Algebraic Sum: $A \dot{+} B = A(x) + B(x) - A(x) \cdot B(x)$

where the operation of $\vee$, $\wedge$, $+$ and – represents max, min, arithmetic sum and arithmetic difference only.

Like probability algebra it is possible to define a set function "Conditional fuzzy measure" and a relation between "a prior & posteriori measures". These are very useful for describing any kind of transition of fuzzy phenomena such as communication of rumors, the reprint of color photograph and the development of human abilities by education. The common characteristics of these phenomena are that the measures describing their status are vague and their transitions are influenced greatly by the subjectivity of things (ore people) which are involved in such fuzzy phenomena.

Since the fuzzy measure is an extension of probability measure & its physical meaning is similar to the membership function of a fuzzy set, the models presented in this paper are more suitable for describing the transition of fuzzy phenomena generally.

2. Fuzzy Measure and Integrals

2.1 Fuzzy Measures

Ordinary measures in the theory of Labesgue integrals have additivity. Here we consider a measure as a set function with monotonicity but not always with additivity.

Let X be an arbitrary set and ϕ be an empty set. x denotes a member of X. The first statement is: "The grade of $x \in X$ equals unity and that of $x \in \phi$ equals 0".

Ordering for Fuzzy Sets:
Let A and B be subsets of X. The second statement is: "If $A \subset B$, then the grade of $x \in B$ is larger than that of $x \in A$".

Definition: Let $\mathscr{B}$ be a Borel field of X. A set function $g(.)$ with the following properties is called a fuzzy measure,

1) $g(\phi) = 0, g(X) = 1$
2) If $A, B \notin \mathscr{B}$ and $A \subset B$, then $g(A) \leq g(B)$
3) If $F_n \in \mathscr{B}$ and F_n is monotonic sequence, then

$$\lim_{n\to\infty} g(F_n) = g\left(\lim_{n\to\infty} F_n\right)$$

In the above definition, (1) means boundedness & non negativity, (2) monotonicity, and (3) continuity. If X is a finite set the condition (3) can be omitted.

Let $h : X \to [0,1]$ and $F_\alpha = \{x \mid h(x) \geq \alpha\}$. If $F_\alpha \subset \mathscr{B}$ for $\alpha \in [0,1]$, then h is called $\mathscr{B}$ – measurable function. A triple $\{X, \mathscr{B}, g\}$ is called a fuzzy measure space.

In an ordinary probability space (X, $\mathscr{B}$, P), P has the following properties:

1) $0 \leq P(E) \geq 1$ for all $E \in \mathscr{B}$, particularly, $P(X) = 1$
2) If $E_n \in \mathscr{B}$ for $1 \leq n \leq \infty$ and $E_i \cap E_j = \phi$ for $i \neq j$, then

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$$P\left(\sum_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

The fuzzy measure is a set function obtained by losing some properties of the probability measures. We may use a probability measure with additivity to measure "Fuzzy phenomena". However, it is quite doubtful if there exists "additivity" behind "fuzziness" in the phenomena where human subjectivity plays an important role. We might as well as expect that we may able to measure "Fuzziness", if we construct a theory of the measures without additivity.

Monotonicity is a quite natural condition. It is not necessary that $g(A \cup B) = g(A) + g(B)$ where $A \cap B = \phi$ if the additivity is thrown away. It would be considered that a fuzzy measure is a means for measuring "fuzziness" while a probability measure is one for measuring randomness. In this sense, fuzzy measure can be regarded as subjective scales by which "grade of fuzziness" is measured.

2.2 Fuzzy Integrals

Now by using fuzzy measures, let us define fuzzy integrals which are very similar to Lebesgue integrals. Let h be a $\mathscr{B}$ - measurable function.

Definition: The fuzzy integral of h over A with respect to g is defined as follows:

$$\int h(x) o g(\cdot) = \sup_{\alpha \in [0,1]} [\alpha \wedge g(A \cap F_\alpha)]$$

$$F_\alpha = \{x \mid h \geq \alpha\} \tag{2.1}$$

The symbol $\int$ is an integral with small bar. The symbol o is the symbol of the composition used in the fuzzy set theory. For simplification, the fuzzy integral over X is written as $\int hog$. The fuzzy integrals can be also called fuzzy expections in comparison with the probabilistic counterpart. They have the following properties:

(1) $0 \leq \int hog(\cdot) \leq 1$ (2.2)

Let $\alpha \in [0,1]$, then

(2) $\int (a \vee h) og(\cdot) = a \vee \int hog(\cdot)$ (2.3)

(3) $\int (a \wedge h) og(\cdot) = a \wedge \int hog(\cdot)$ (2.4)

(4) $\int (h_1 \vee h_2) og(\cdot) \geq \int h_1 og(.) \vee \int h_2 og(\cdot)$ (2.5)

(5) $\int (h_1 \wedge h_2) og(\cdot) \leq \int h_1 og(.) \wedge \int h_2 og(\cdot)$ (2.6)

(6) $\int_{E \cup F} hof(\cdot) \geq \int_E hog(.) \vee \int_F hog(\cdot)$ (2.7)

(7) $\int_{E \cup F} hof(\cdot) \leq \int_E hog(.) \wedge \int_F hog(\cdot)$ (2.8)

Let $h_A(x)$ be a measurable function of a fuzzy set A. Then the fuzzy measure of A is defined as follows:

$$g(A) = \int h_A(x) og(\cdot) \tag{2.9}$$

The fuzzy integral of A is defined as follows:

$$\int_A h(x) og(\cdot) = \int [h_A(x) \wedge h(x)] og(\cdot) \tag{2.10}$$

The properties (6) and (7) also hold.

2.2 Conditional Fuzzy Measure

Definition: Let $\sigma_Y(\cdot \mid x)$ be a fuzzy measure of Y for any fixed $x \in X$. Then it is called conditional fuzzy measure of Y w.r.t. X.

A fuzzy measure $g_Y(\cdot)$ of Y is constructed by $\sigma_Y(\cdot \mid x)$ and $g_X(\cdot)$ as follows

$$g_Y(F) = \int_X \sigma_Y(F \mid X) o g_X(\cdot) \tag{2.11}$$

We have

$$\int_Y h(y) o g_y = \int_X [f_Y h(y) o \sigma_Y(\cdot \mid x) o g_x] \tag{2.12}$$

Analogously, we can consider $\sigma_X(\cdot \mid y)$. There holds the next relation between $\sigma_Y(\cdot \mid x)$ and $\sigma_X(\cdot \mid y)$.

$$\int_F \sigma_X(E \mid y) o g_Y = \int_E \sigma_y(F \mid X) o g(X) \tag{2.13}$$

The above equation corresponds to Baye's theorem. In this sense, g_X is called a prior fuzzy measure and $\sigma_X(\cdot \mid y)$ a posteriori fuzzy measure.

More precisely, $\sigma_y(\cdot \mid X)$ is arbitrary on a set E such that $g_X(E) = 0$ where $E \subset X$ and g_X is a fuzzy measure of X. When X & Y are finite sets, a posteriori fuzzy measure can be easily calculated from equation (2.13).

3. Transition of Fuzzy Phenomena

Using the concept of the conditional fuzzy measure two types of models are considered as for the transition of fuzzy phenomena. The first model corresponds to Markov process eqn. (2.11). Here the input is a fuzzy measure g_X and the output g_Y is the fuzzy integration of a conditional fuzzy measure $\sigma_y(\cdot \mid x)$ respect to g_X as shown in fig. 1. For example, supposing g_X is the measure of a gossip told to a person, g_Y is the measure of the gossip modified by his subjectivity σ_Y.

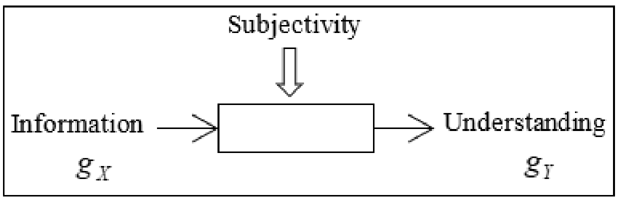

Figure 1: Model 1

The second model corresponds to Baye's theorem eqn. (2.13). The input is a priori fuzzy measure g_X as above, but the

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output is a posteriori fuzzy measure $\sigma_X(\cdot \mid Y)$ as shown in fig. 2.

One of the explanations of this model is that one's preconception g_X is improved to σ_X by information σ_Y. The example of this model is the ability of evaluation, judgment and diagnosis improved by experience, information or learning.

This is analogous to the fact that the hard headed persons do not change their opinion by outside information.

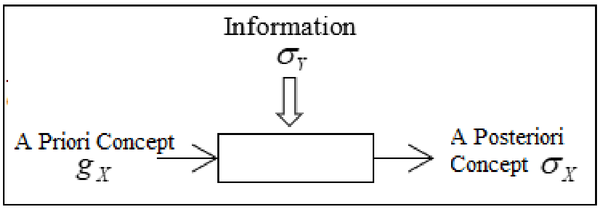

Figure 2: Model 2

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