Möbius Twist, Möbius Band, LMC, Interlocking and Knot Formation: Operator Formalism

Rabindranath Chattopadhyay

MTA, IAPT, IPS, ASI, ISEC, IACS, ICSP
Email: *rchattopadhyay03[at]yahoo.co.in*

Abstract: ***Möbius Band or Möbius surface is a century-old topic of research which, like many other similar entities has the potentiality of being experimented, investigated and explored endlessly beyond ages. Here in this article the author, after reviewing briefly some of the major researches and studies as the background tries to explore a relatively newer type of representations of related and consequent properties involved in operations with Möbius surface.***

Keywords: Möbius Band, Twist Angle, Knot, Operator, Group

1. Introduction

A thin A4-size paper-sheet is taken and a number of parallel-edged strips with sufficient length and relatively much smaller width are cut of that sheet. Each of these strips form a two-sided surface with up and down and open at both end. A few of those strips are rounded and are held touched end to end untwisted i,e. up-side of the strip at one end touches the down-side of it at other end [Fig.1(a)]. The ends are then glued one on another as held. These form simple closed-loop paper- band. This paper-band is a two-sided surface as this has an inside-surface and a outside-surface. When a pencil-tip is to run through on its surface a closed line is produced on the outside-surface leaving inside-surface untouched/blank [Fig.1(b)]. One of those strips are then twisted by an angle equals 180° (a single twist defined) before being pasted keeping the sides same at ends and as a consequence a twisted closed-loop surface is produced [Fig.1(c)]. Now if the tip of a pencil is made to run through its length for producing a line on the surface it continue until the whole surface is covered. This means that no part of surface is left without line and thus this surface is called a one-sided surface (Möbius surface / Möbius Band) and apprehensively the single twist i,e. a twist by an angle π radian is named as Möbius-twist. On the contrary similar test with pencil and a doubly twisted (twisting through an angle equals to 360° before pasting end-to-end) closed-loop band reveals that the resulting surface is a two-sided surface leaving one side completely untouched with pencil. This surface is termed as Non-Möbius surface (Non- Möbius band) and the antecedent twist is called a Non-Möbius twist. In this way quite a number of experiment has been performed one after another with so many similar paper-strips many times performing from no prior twist to seven prior twist without any gap in number of twist. The primary observation from all these experiments is that odd number of prior twist produces a Möbius-band while even number of prior twist produces a Non-Möbius band and therefore (2) every odd number of twist is a Möbius twist while every even number of twist is a Non-Möbius twist [Fig.1(d) - 2(d)].

If now 'prior-twist' is considered as an operation on band-surface-formation as a function then another operation is proposed to be a 'Longitudinal Midway-Cut' to operate on band-surface and let them be represented by '**τ**' and '$\mathbf{l}_{MC}$' respectively.

Principal Observations:

1) Two consecutive Möbius twists is equivalent to a Non-Möbius twist.
2) Two consecutive Non-Möbius twists is equivalent to Non-Möbius twist of next higher order. Order of a Möbius Twist and Band is {o} while the total angle of twist is '(2{o}- 1) times π radian ,{o} may be any real integer starting from '1'. Similarly the order of a Non-Möbius Twist and Band is the total angle of twist divided by2 π radian.
3) Möbius twist, while represented through an algebraic symbol is seen to have an odd parity while for similar expression Non-Möbius twist have even parity.
4) $\mathbf{l}_{MC}$ operation on a plane untwisted closed-loop paper-band produces two exactly similar bands of half width each and with basic two-sided surface [Fig.3(a)-3(c)].
5) $\mathbf{l}_{MC}$ operation on a Möbius-band adds up an additional twist to produce a Non- Möbius-band of same order, with half width and double length as that of Möbius-band [Fig.4(a) – 4(d),5]. In addition to those as mentioned above 'Knot' Formation' also take place as a consequence of $\mathbf{l}_{MC}$ operation on a Möbius-band of order higher than '1'.
6) $\mathbf{l}_{MC}$ operation on a Non-Möbius-band adds up two additional twist to produce a Non-Möbius- band of next higher order [Fig.5,6(a) – 6(b)]. This operation does produce also 'Interlocking' between two cut-parts but no knot.
7) 'Knot' and 'Interlocking' have also their order depending on how many times a Band goes round and round within itself or how many times a part of the cut duo binds rounding on the other respectively.

Brief Description of the Figures showing consequence of experiments along with examples:

It is consistent that if one twists the plane paper-strips before pasting in a direction opposite to that usually done in earlier cases the ultimate result will remain the very same but with a negative sign i,e. in reverse order. Fig.1(c) shows a Möbius-band first order and Non-Möbius-band of zero'th order or a non-twisted band side by side. (3)

This Möbius-band has been prepared with a 180° twist. Fig.1(d) shows three bands with 0°,180° and 360° twist referring to Non-twisted,180° twisted Möbius-band and a

360°-twisted Non-Möbius- band. Figs.2(a),2(b) show an ordinary Non-twisted two sided band while Fig.2(c) and Fig.2(d) shows singly twisted Möbius-band (twist-angle 180°) with only one side and doubly twisted Non-Möbius-band (twist-angle 360°) with two sides respectively. Fig.3(a, b, c) show result of Longitudinally

Midway Cut (LMC) of 0° twisted band. Fig.4(a,b,c and d) show the result of LMC of 180°-twisted band which is a two sided (360° twist-angle equivalent = 180° original twist + 180° twist-equivalent for a knot-embryo) Non-Möbius-band with double its parental length and half its parental width. Fig.5 shows a significantly interesting case where a 540°-twisted and a 720°-twisted bands are shown in comparison with each other. Fig.6(a,b) show the result of LMC of 360°-twisted Non-Möbius- band where two separate Non-Möbius-bands (twist-angle 360° each interlocked singly and with same as parental length and half their parental width) are produced. Fig.7(a) shows the result of LMC of only one of the two components produced in Fig.6 revealing that only the same process as has been taken place in Fig.6 with its antecedent takes place here also. Fig.7(b,c) show the consequence of LMC of 720°-twisted Non- Möbius-band that exhibit that two similar Non- Möbius-bands, doubly interlocked (twist-angle 720°), half their parental width and same length are produced. Fig.7(d) manifests the result of LMC of 540°-twisted Möbius-band showing that a total 720° twist-equivalent Non- Möbius-band (360°twist equivalent for a complete knot + 360° from original twist of 540°) with double its parental length and half its parental width is produced. This fact implies that a knot exploits partially the original twist so as to be fed by twist and here the original twist angle diminishes by 180° for a single knot-production. In this way prior-twist by up to 1260° angle and band-formation along with subsequent operation LMC has been done several times with plane untwisted paper-strips. As consequence some specific rules for results of proposed operations in producing knots and interlocking has been found out and presented here in terms of some symbolic representations as a new type of formalism. Figs.7(e-h) has been added with a view to clarify expressively the only purely knotted-bands without twist with a single knot to up to five knots undergoing LMC operation give no additional result except only the production of two (4) parallel exactly similar knotted bands with half the parental width and same length and having same number/order of knot-form.

2. A Brief preview of Mӧbius Strip Studies

Among quite a large number of works during a period as long as more than a century since its first discovery only a few are mentioned here along with very brief discussion on some major points regarding origin, physical form, corresponding mathematical representations on different fronts, application-avenues both from theoretical and practical points of view and beyond are discussed here.

Mӧbius-strip, having two fields of definitions namely topological and its corresponding physical counterparts, may be shown to consist of three planer and three cylindrical parts. It is abstract in topological space [2] which is constructed out of a rectangular open plane strip with a lower limit of ratio of length to breadth for its successful embedding identifying two opposite edges point by point (bijective mapping) with each other so that each vertex gets identified with one diagonally across. Expression for embedding through equations /functions showing direct correspondence between its topological definition and physical model. But this representation does not exactly gives the actual set of all properties of Mӧbius-strip. Moreover, the surface in embedding representation of Mӧbius-strip is never flat. Experimenting with Mӧbius-geometry is an entertaining activity which, along with its geometrical properties and algebraic representations can be ventured to make other important revelations to manifest [3]. Topological space and some operators could have rigour in the perspective of various applications. Topological equivalence between two different geometrical structure does not mean complete congruence in every respect. For example, doughnut is topologically equivalent to a sphere as both enclose a finite volume yet interlocking with space folded around a doughnut is not observable in case of a sphere [4]. Moreover Mӧbius-band is a space regarded as ‘non-orientable’ while the space we live in is said to be orientable where distinct sidedness does exist.

If one rotates the shorter edges of a rectangular paper-strip by an angle equal to ‘$n\pi'$’ , n being an integer one obtains a different embedding M_n of the strip in 3-space $\mathbb{R}^3$.If n' is even the resulting surface is a twisted cylinder-surface which is a non- Mӧbius surface while for n' being odd the it is n-twisted Mӧbius- surface. ‘α’, the angular twist-specification of a point on the Mӧbius- surface related to ‘θ’ the angular coordinate of the same point on the central line of the surface as [5];

$$\alpha = \frac{n\theta}{2}$$

With the semi explicit parameterization of a solid Mӧbius-strip $\mathbf{M}_n$ and with the help of coordinate-transformation the equation representing the embedding is of 2(n+2)-order {a polynomial of highest power 2(n+2)}. (5)

Application of the concept of Mӧbius-strip are many, both in theoretical and practical perspective. Some practical examples are mentioned here [6]; Mechanical belt wearing evenly on both sides, dual- track roller coaster where carriages alternate between two tracks, world-map printing showing antipoles appear opposite to each other, twisted Graphene-ribbon gains an additional electronic property including helical magnetism, Molecular orbitals’ alignment along a cycle in aromatic organic molecule replicating Mӧbius- strip, production of non-inductive electrical resistances with Mӧbius-Dielectric-strip. Besides all these there are also various other applications such as designing special type of resonator, q-plate polarization and social choice theory [6].

The Mӧbius-band can continuously be transformed into its center-line by decreasing its lateral width which actually be a twisted curve line. Mӧbius-surface is developable and can be bent and folded but not stretched. For embedded Mӧbius-band the minimal aspect ratio limit is ‘$\sqrt{3}$’ [7] and it is of-course for fully overlapped equilateral triangular area [For physically understanding please see Fig.(9)] where for partly overlapped triangle area with no gap/blank area in the middle of the triangle area the value of aspect ratio is $3\sqrt{3}$ {Fig.8}. Euler-characteristic value of Mӧbius- surface is equal to zero(0) [8].

As bending of a paper piece is more convenient than stretching it a strip can easily be made from a rectangular paper-piece and this type of deformation of strip barely affects its material properties and that's why it is the fastest accessible example of topological entity [9]. It has some potential applicability to some nanostructures. Absence of Gaussian curvature is a prerequisite for a surface to be developable. Parameterization of Mӧbius-surface is generally made through vector-algebraic expressions including classic and other dynamic energy and elastic torsion and also center-line torsion. Energy-minimization is thus turned into a *1*-dimensional variational problem represented by a form that remain invariant under Euclidian transformation.

It is possible to give a triangulation of the Mӧbius-band with only five vertices (one twist reduces the number of vertices for triangulation from that in cylindrical surface) and the boundary of the Mӧbius-band is a pentagon where every vertex is connected to every other vertex [10]. In early days of 'Topology' most of the object of interest were defined in terms of triangulation describing a topological space as a union of finitely many vertices, edges, triangles and higher dimensional simplexes identified in certain ways along their boundaries. A triangulation with relatively a small number of simplexes symmetrically placed could make computation easier and suggest new properties of object itself.

Normal curvature is defined in terms of local normal vector which reverses its direction after one orbit over Mӧbius-surface (that is moving around once through an angular displacement of 'π^c') [11]. (6)

Instability of Mӧbius-strip-minimal surface has certain important applications [12] and that's why it needs exploration. With the help of invariant variational bi-complex formalism Starostin and Heijden [13] formulate boundary value problem of the Mӧbius-strip and they solve it numerically. Application of this process could give new insight into energy-localization phenomena in unstretchable sheet that help predict points of onset of tearing or breaking apart. This formalism of theirs, as they claim, may help understanding relationship between geometry and physical properties of nano- and microscopic Mӧbius-strip structures. Among various expression regarding embedding Mӧbius-strip surface some vector-algebraic representations (parameterization) including physical properties such as torsion of the center-line, elastic properties such as Young's modulus, Poison-ratio, rigidity modulus etc. and related Lagrangian are considered in the paper. In this paper the author defines linking number L_k to k-twisted Mӧbius-strip where half-integral value of L_k means one sided surface. This formalism can proceed to solution for any L_k value including Penrose's Eschermatics with $L_k = (3/2)$[13].

In another interesting paper [14] Quantum-coherent states are constructed on Mӧbius-strip. The motions of a quantum-particle is constrained by equations representing Mӧbius-band and the corresponding Lagrangian and Hamiltonian are extensively considered to find out its ultimate dynamical states. It can be shown that Mӧbius-band is a natural phase-space for Fermionic-field. Neito and Perez-Enriquez [15] consider a spinning particle on a Mӧbius-strip. Associating Ortho-normal frame with spinning particle rotating along the center-line of a Mӧbius-strip and showing antisymmetric character of tensor related to rotational velocity they consider Lagrangian-Hamiltonian formalism and from there switching onto quantum-mechanical formalism of Schrӧdinger they find out by calculation values of some important parameters such as ground-state-energy, energy-eigen values, spectral constraints etc. concerned with microscopic particles in the quantum world.

3. A short mathematical prelude:

Set of coordinate transformation-equations in 3-dimensional real space that is frequently considered
is as follows {Fig.10};

$$x = R\cos\theta + l\cos\alpha\cos\theta$$
$$y = R\sin\theta + l\cos\alpha\sin\theta \qquad \text{Eq. (1)}$$
$$Z = l\sin\alpha$$

$0 \leqslant \alpha \leqslant \pi, 0 \leqslant \theta \leqslant 2\pi$, $\alpha \propto \theta$, P.C→ $(\frac{n}{2})$ (7)

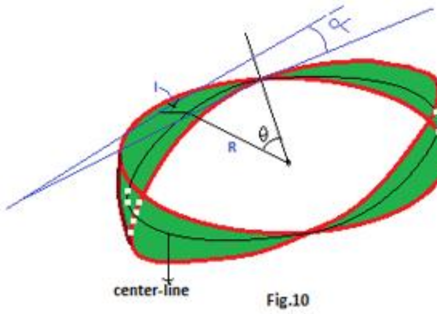

Fig.10

It is a transformation from (l, θ, α) to (x, y, z) with R as a parameter. For a normal one-twisted Mӧbius-strip $\alpha = \frac{\theta}{2}$ and for *n*-twisted Mӧbius-strip $\alpha = \frac{n\theta}{2}$. Angular position coordinate θ and twist-angular position coordinate α are orthogonal to each other and normally α and θ are considered to be strictly linearly proportional to each other throughout the band, *n* being a real integer. For single twisted normal Mӧbius-strip Eqns. (1) transform like the following;

$$x = R\cos\theta(1 + l\cos(\frac{\theta}{2}))$$
$$y = R\sin\theta(1 + l\cos(\frac{\theta}{2})) \qquad \text{Eqn. (2)}$$
$$Z = \sin(\frac{\theta}{2})$$

For partially overlapped area of a normal Mӧbius-band flattened symmetrically over a plane surface it looks like a perfectly symmetric irregular hexagon with three arms each equal to $(a+x)$ and three other arms each equal to (a) {Fig.11}. From the figure aside it is clear that the total length of the rectangular paper-piece from which it is made is 'L' given by

$L = \frac{3(2a+x)+3(a+x)}{2} = \frac{9a+6x}{2}$;

Hence $x = \frac{2L-9a}{6}$

For reducing x to be equal to zero (0) $L = (\frac{9}{2})a$.

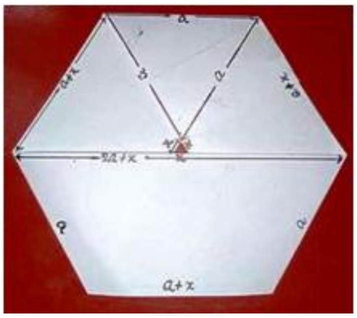

Figure 8

Figure 9 (a)

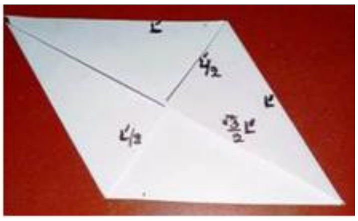

Figure 9 (b)

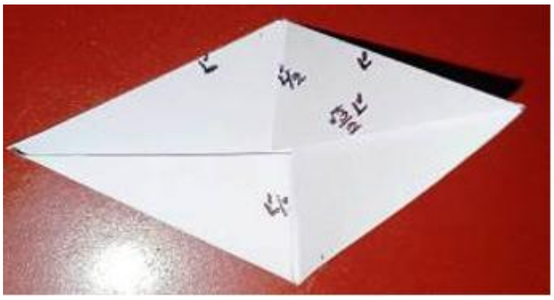

Figure 9 (c)

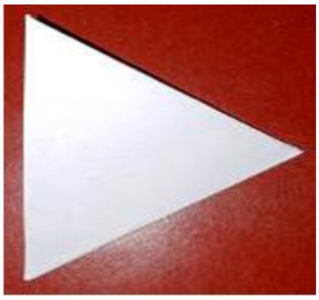

Figure 9 (d)

On the other side the width of the rectangular paper piece 'b' is given by $b = a\cos 30°$. Hence the aspect-ratio of the rectangular paper-piece required is $\left(\frac{L}{b}\right) = 3\sqrt{3}$.

Now for completely overlapped triangular area {Figs.12} the three triangles merges into one and the aspect ratio will then have the least value equal to $\sqrt{3}$. This can be shown physico-geometrically as follows;

From all the above figures it is very clear that the lower-bound value of the aspect-ratio for a Mӧbius-band to form from a rectangular paper-piece is $\left(\frac{L}{b}\right) = \frac{\frac{3L'}{2}}{\frac{\sqrt{3}L'}{2}} = \sqrt{3}$.

The differential distance between two close points on Mӧbius-band will then be given by
$d\zeta^2 = dx^2 + dy^2 + dz^2 = (R + l\cos\alpha)^2\, d\theta^2 + dl^2 + l^2 d\alpha^2 = \rho^2 d\varphi^2$ ….. Eqn.(3)

where ρ is the resulting equivalent radius of curvature and '$d\varphi$' is the equivalent differential angular distance. The resulting curvature will then be given by
$\chi = \frac{1}{\rho} = \frac{d\varphi}{d\zeta(l,\theta,\alpha)}$ where, $d\zeta(l,\theta,\alpha) = (\frac{\partial\zeta}{\partial l})dl + \left(\frac{\partial\zeta}{\partial\alpha}\right)d\alpha + \left(\frac{\partial\zeta}{\partial\theta}\right)d\theta$. ……. Eqn.(4)

From equation (2) one can write
$\zeta = \int_0^{2\pi} d\zeta = \int_0^{2\pi}[\{R + l\cos(\frac{\theta}{2})\}^2 + \frac{l^2}{4}]^{\frac{1}{2}}d\theta \;\; for\; l = constant.$ …….. Eqn.(5)

which means that ζ' is the length of a line that a point at a particular distance l from the center-line traces for a complete round traversing on a simple Mӧbius-strip. Eqn.(5) is reducible to an elliptic integral and has not any simple straight forward solution as such.

For minimum possible aspect-ratio the ratio $(\frac{R}{l}) = \frac{\sqrt{3}}{2\pi k}$ and for no blank space at the center of the triangle of Mӧbius-strip $(\frac{R}{l}) = \frac{3\sqrt{3}}{2\pi k}$, k being a real fraction having maximum value equal to 0.5. For 0.276<k<0.5 R< l for the first case but for the second and other cases R is always greater than l.

From Eqn. (1) and Eqn. (3) one can write, $\vec{\rho} = \hat{n}\,\rho = \hat{\imath}\,x + \hat{\jmath}\,y + \hat{k}\,z$ where $\rho = [R^2 + l^2 + 2Rl\cos\alpha]^{\frac{1}{2}}$ and $\hat{n} = i\cos u + j\cos v + k\cos w$, cos's are direction-cosines. Hence from Eqn.(3) it can be written that $d\varphi = \frac{[(R+l\cos\alpha)^2 d\theta^2 + dl^2 + l^2 d\alpha^2]^{\frac{1}{2}}}{[R^2+l^2+2Rl\cos\alpha]^{\frac{1}{2}}}$. (9)

Symbolic Representation of associated operators and parameters as defined (an operator formalism):
Let us first express the different parameters and operators through some symbols as the following;
$\boldsymbol{\tau}_{\boldsymbol{M}}^{[o]} \rightarrow$ *Mӧbius Twist of order[o]* , $\overline{\boldsymbol{\tau}}_{\boldsymbol{M}}^{[o]} \rightarrow$ *Non-Mӧbius Twist of order[o]*

International Journal of Science and Research (IJSR)
ISSN: 2319-7064
Impact Factor 2024: 7.101

$\boldsymbol{\tau}_{Ma}^{[o]}$ → *Möbius Anti-Twist of order[o]* (Twisted in opposite direction)
$\bar{\boldsymbol{\tau}}_{Ma}^{[o]}$ → *Non-Möbius Anti-Twist of order[o]* (Twisted in opposite direction),
χ_{CM} → *Longitudinal Midway Cut* , $\boldsymbol{\kappa}^{[f]}$ → *Knot Formation(fold-number f),*
$\boldsymbol{\iota}^{[r]}$ → *Interlocking of round number 'r'.*
$[\boldsymbol{o}]$, $[\boldsymbol{f}]$ and $[\boldsymbol{r}]$ may take any real integral value starting from '0'.
'0' value of the above-mentioned three quantities mean 'No Twist', 'No Fold' and 'No Rounding' respectively.

Following the principal observations mentioned earlier the following relations may be stated and be considered as rules for the said operations;

$$\boldsymbol{\tau}_M^{[o]} * \boldsymbol{\tau}_M^{[o']} \equiv \bar{\boldsymbol{\tau}}_M^{[(o+o')-1]} \equiv \boldsymbol{\tau}_M^{[o']} * \boldsymbol{\tau}_M^{[o]} \quad \text{...(6)}$$

$[\boldsymbol{o}] = \mathbf{1}$ means 'a twist by an angle equals to 180° for *Möbius Twist* and 360° for *Non-Möbius Twist.* Anti-twists will follow exactly the same rules differing only in the context of twist-angle being in the reverse direction as that of *Möbius Twist* and *Non-Möbius Twist.* Hence

$$\boldsymbol{\tau}_M^{[o]} * \boldsymbol{\tau}_{Ma}^{[o]} \equiv \mathbf{I} \text{ , \{I → Identity\} and also } \bar{\boldsymbol{\tau}}_M^{[o]} * \bar{\boldsymbol{\tau}}_{Ma}^{[o]} \equiv \mathbf{I} \quad \text{.......(7)}$$

$$\boldsymbol{\tau}_M^{[o]} * \bar{\boldsymbol{\tau}}_M^{[o']} \equiv \boldsymbol{\tau}_M^{[o+o']} \equiv \bar{\boldsymbol{\tau}}_M^{[o']} * \boldsymbol{\tau}_M^{[o]} \quad \text{..............(8)}$$

$$\bar{\boldsymbol{\tau}}_M^{[o]} * \bar{\boldsymbol{\tau}}_M^{[o']} \equiv \bar{\boldsymbol{\tau}}_M^{[o'+o]} \quad \text{............ (9)}$$

$$\boldsymbol{\tau}_M^{[o]} * (\boldsymbol{\tau}_M^{[o']} * \boldsymbol{\tau}_M^{[o'']}) \equiv \boldsymbol{\tau}_M^{[o]} * \bar{\boldsymbol{\tau}}_M^{[(o''+o')-1]} \equiv \boldsymbol{\tau}_M^{[(o+o'+o'')-1]} \equiv \boldsymbol{\tau}_M^{[o]} * \boldsymbol{\tau}_M^{[o']} * \boldsymbol{\tau}_M^{[o'']} \equiv (\boldsymbol{\tau}_M^{[o]} * \boldsymbol{\tau}_M^{[o']}) * \boldsymbol{\tau}_M^{[o'']} \quad \text{.....(10)}$$

$$\chi_{CM} * \boldsymbol{\tau}_M^{[o]} \equiv \bar{\boldsymbol{\tau}}_{M(eq)}^{[0]} \equiv \{ \boldsymbol{\kappa}^{[o-1]} + \bar{\boldsymbol{\tau}}_M^{[1]} \} \quad \text{..........(11)}$$

$$\chi_{CM} * \bar{\boldsymbol{\tau}}_M^{[o]} \equiv \{2\}\bar{\boldsymbol{\tau}}_M^{[o]} \equiv \{\boldsymbol{\iota}^{[o]} + 2\bar{\boldsymbol{\tau}}_M^{[o]}\} \quad \text{..........(12)}$$

$$\chi_{CM} * \boldsymbol{\kappa}^{[f]} \equiv \{2\}\, \boldsymbol{\kappa}^{[f]} \quad \text{...........(13)}$$

Now let us express a surface, especially in this context of one and two-sided surface in its parametric form so as to quantify a surface beyond coordinate-representation. A plane surface is implied with the symbol [Ψ (τ, κ, ι), λ, ω,ν] ≡ $\mathcal{S}$ [2(0, 0, 0), l, d, 1] where 'ν' refers to number of sides(1 or 2) comprising along with τ ,the number of twist, κ number of knots(folds), ι the number of interlocking(rounding) and λ the length, ω the width, ν the number of similar objective entities (1 or 2). In case of a surface which is somehow bound (longitudinally closed, Fig1(c)) is expressed by $\oint$ [Ψ(τ, κ, ι), λ, ω, ν] ≡ $\oint$ [2(0, 0, 0), l, d, 1]. Then the transformation of such closed surfaces undergoing the above-mentioned operations may be expressed as follows;

$\boldsymbol{\tau}_M^{[1]}$ $\oint$ [2(0, 0, 0),l,d,1] ≡ $\oint$ [1(1, 0, 0),l,d,1] ,
$\bar{\boldsymbol{\tau}}_M^{[1]}$ $\oint$ [2(0, 0, 0),l,d,1] ≡ $\oint$ [2(2, 0, 0),l,d,1]
χ_{CM} $\oint$ [2(0, 0, 0),l,d,1] ≡ $\oint$ [2(0, 0, 0),l,(d/2),2]
$\chi_{CM} * \boldsymbol{\tau}_M^{[1]}$ $\oint$ [2(0, 0, 0),l,d,1] ≡ $\oint$ [2(2, 0, 0),2l,d,1]
$\chi_{CM} * \bar{\boldsymbol{\tau}}_M^{[1]}$ $\oint$ [2(0, 0, 0),l,d,1] ≡ $\oint$ [2(2, 0, 1),l,(d/2),2]
$\boldsymbol{\tau}_M^{[o]}$ $\oint$ [2(0, 0, 0),l,d,1] ≡ $\oint$ [1((2[o]-1),0,0),l,d,1]
$\bar{\boldsymbol{\tau}}_M^{[o]}$ $\oint$ [2(0, 0, 0),l,d,1] ≡ $\oint$ [2((2[o]), 0, 0),l,d,1]
$\chi_{CM} * \boldsymbol{\tau}_M^{[o]}$ $\oint$ [2(0, 0, 0),l,d,1] ≡ $\oint$ [1((2[o]-1), ([o]-1), 0),2l, (d/2),1]
$\chi_{CM} * \bar{\boldsymbol{\tau}}_M^{[o]}$ $\oint$ [2(0, 0, 0),l,d,1] ≡ $\oint$ [2(2[o], 0, [o]),l,(d/2),2]

For surfaces like these the product of the last three parameters is conserved i, e. (λ. ω.ν) = constant.

A Brief Discussion: For singly twisted Möbius-band there is only a single edge-line untwisted parallel to its width in original form which remains vertical and erect forming an angle equal to 90° with the center-line and which is $2\pi^c$ away from its other end of the band. Central circular line of an untwisted cylinder remains in a plane and of a singly twisted Möbius-band while embedded remains in a plane but it should be called a twisted axis in regard of its material-content but not geometrically {Fig.11}. But any other line at a fixed distance from the central line gets displaced from its original position by a distance varying from (R – l) to (R + l) from the center of its central circular line and the Möbius-space gets transformed into one referenced at three dimensional E- space.

Figure 11

Any moving and spinning object/entity, moving on a Möbius-band longitudinally spontaneously changes its axial direction of rotation continuously and become opposite after moving through one circular perimeter and then regain again its original direction of motion after moving around twice the same perimeter. A spinning electron will thus experience a reversal of rotational axial up to down to up – direction as if in a single orbital rotation. If considered to manifest its effect in the context of physical spectroscopy *l-s* interaction of a spinning electron moving in such an orbit on a Möbius-surface continuously changes should be yielding radiation-spectrum of a continuously changing wavelength because then the spin-quantum number changes from ($+\frac{1}{2}$) ($-\frac{1}{2}$) or the vice-versa after an orbital rotation with the orbit transformed into a twisted line as axis.

On a Möbius-band the central circular line is considered to be the axial curve and a similar curve, parallel to it on right/up side of it goes exactly opposite after a complete round over the band i,e. to left/down side {Fig.12}.


Figure 11 (a) **Figure 11 (b)**

But the central line is uniquely placed at a fixed position though it may be regarded as a twisted curve-line. This is somehow Fig.12(a) Fig.12(b) equivalent, in an embedded form to a bent system of a flux of narrow tubes conforming a circular cylinder like narrow component-threads that constitute an ordinary physical string or rope keeping the central thread twisted at its original central geometric location.

Knots, Interlocking and Longitudinal Midway Cut of Mӧbius-band:

Most ordinarily a loop means physically a continuous line that encompass some area within a finite boundary and then goes beyond cross-passing antecedent part of its own. A loop basically has two parts three points of projected intersection {Fig.12}. Now while cross-passing each other one part of the loop encircle the other a 'knot' is produced.

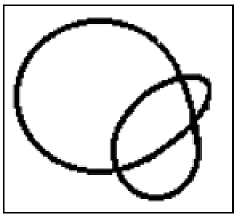
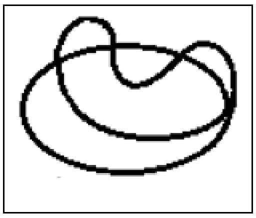

Figure 12 (a) **Figure 12 (b)**

In the above figures wavy structure around a curve-line component of the whole entity is seen to produce what may be called a 'knot'. In this perspective of wave-like formation one may associate wavelength that corresponds to knot number. One wavelength means one 'knot', two wavelengths means two knots, three wavelengths for three knots and so on. Knots materialistically need to be made mechanically in open loop while in case of closed loop LMC (Longitudinal Mid-way Cut) of Mӧbius-band produces knot. Knots, in general may both be 'open' and 'closed'. Open knot is conventionally known as 'over-hand' knot while a closed knot is called usually 'a Trefoil knot' [16]. A knot may thus be defined as a geometrical figure with two arms enclosing an area in space cross-passing each other at three points of distinction in a way up- down-up or conversely down-up-down alternately. Embedding a simple knot also requires similar three points named as vertices. Set of equations representing an embedded knot (single knot)




{Different Types of Knot-complex}is as follows;

$x = (R + A\cos\varphi)\cos\theta$, $y = (R + A\cos\varphi)\sin\theta$, $z = A\sin\varphi$ ….Eq.(13)

Set of equations representing an embedded knot (single knot) is as follows;

$x = (R + A\cos\varphi)\cos\theta$, $y = (R + A\cos\varphi)\sin\theta$, $z = A\sin\varphi$ …..Eq.(13)

This set appears exactly similar to Eq.(1) except the limit of values of 'φ' and 'θ' .

$0 \le \varphi \le 2\pi$ and $0 < \theta < 2\pi$ and $\varphi \ne \frac{\theta}{2}$.

A nice description of various types of knot in terms of number of crossings and of prime knots along with some alpha-numeric symbols is presented in tabular form by G.D. Santi [17].

The crossings mentioned here are the up-down-up like sequence mentioned earlier. We have seen that when an LMC is done to a Mӧbius-band 'Knots' are produced from Mӧbius-twist of second order onward. For one twist no knot, for three twists one knot, for five twists two knots and so on and while more than one knot is produced the knots become internally linked with each other.

Interlocking, as of the present context means some common set of values or points through which two separate set-entities are internally connected. Here when two closed loops are internally connected by mutually cross-passing each other at a point they are called to be interlocked and the occurrence is called an interlocking. When an ordinary double-twisted two-sided surface is cut longitudinally midway a simple interlocking is produced which is called an interlocking of first order and when a 4-twisted two-sided surface undergoes LMC operation an interlocking of second order is produced i, e. interlocking with two rounding. In general when an n – twisted non-Mӧbius-band is put to LMC operation interlocking of ($n/2$)th order is produced, n being an even number. Thus, LMC produces 'knots' in case of Mӧbius-band and 'interlocking' in case of non-Mӧbius-band.

A finite cylindrical surface is an untwisted two-sided surface which undergoing LMC produces no interlocking but only two separate fractal components of its own.

4. Conclusion

Mӧbius-strip or band concept, initiated more than a century ago as a magical fun-game with rectangular paper-strips has experienced a long journey through manifold of revelations from so many different angles both experimental and theoretical. Importance of considering Mӧbius-strip or band seems to appear to be an ever-growing ever-developing matter starting from its simple geometrical and topological significances ingenuous applications to theoretical and applied physics for understanding different yet-unknown dynamical properties in the microcosmic domain and applying to different engineering and technological and even to sociological fields. In the discussion part some such works have been very briefly mentioned.

In this paper the author, after experimenting with preparation of and operations on one-sided surface such as Mӧbius-band and with twisted two-sided surface as its natural immediate following up and repeated observation of consequences proposes to define and represent different operations as with some operators in a way ,abruptly different from that of so many other operators that are in frequent use ,usually defined in applied mathematics and physics.

Here operators are assigned with quite a number of parameters with some integral eigen values that changes within a set after operation.

A little abstraction of allied and related concepts necessitates in an introduction to embedding and immersion for understanding algebraic and coordinate-geometric signature of Mӧbius-band.

After all it is seen that this typical one-sided surface known by the name of Mӧbius-band or Mӧbius-strip involves too many mathematical and physical matters of substantial interests and importance and therefore it is quite obvious that anything new in the domain of revelation on Mӧbius-strip and subsequent successors will affect both mathematical and physical world huge. For an example if for an atomic electron orbiting around the nucleus it is imagined to move on a Mӧbius-surface then the surface may be looked upon as a constraint of electron’s motion with all its geometrical properties which may effectively be thought to be due to a new type of *Force-Field* or a new potential function.

For another example of Mӧbius-motion (higher order) a spinning electron moving under Mӧbius- surface-constraint may be thought to move along a helical path while its consequent spin-angular momentum is bound to change giving rise to a new type of interaction-scenario. Relativistic properties may also undergo a certain type of modification in case of such Mӧbius-motion. Many other geometric figures and forms like such one- sided surface other than Mӧbius-strip following the formation-procedure of Mӧbius-strip but in a little different way may also be explored in this context as a one sided surface yields a varieties of consequences in real world.

Plate- I

Figure 1 (a)

Figure 1 (b)

Figure 1 (c)

Figure 1 (d)

Figure 2 (a)

Figure 2 (b)

Figure 2 (c)

Figure 2 (d)

Figure 3 (a)

Figure 3 (b)

Figure 3 (c)

Plate-II

Figure 4 (a)

Figure 4 (b)

Figure 4 (c)

Figure 4 (d)

Figure 5

Figure 6 (a)

Figure 6 (b)

Figure 7 (a)

Figure 7 (b)

Figure 7 (c)

References

[1] Rabindranath Chattopadhyay, 'Möbius Twist and Subtriangles', Mathematical Education, October-December (1989)109-111

[2] Gideon E. Schwarz, 'The Dark side of the Möbius Strip', Hebrew University, Jerusalem, *http://about.jstor.org/terms*,pp.890-897,1990

[3] Peter Prevos,'The Möbius Strip in Magic [A Treatise on Afghan Bands] (2018) *magicperspective.net*, Third Hemisphere Publishing, Australia

[4] Ian Stewart, 'Concepts of Modern Mathematics', Penguin Books,1982

[5] Stephen Klaus, 'Solid Möbius Strip as algebraic Surface',2009, pp 1-10. https://www.imaginary.org , IMAGINARY, Open Mathematics

[6] Wikipedia, 'Möbius Strip'

[7] Richard Evan Schwartz, 'The optimal paper Möbius Band, Supported by N.S.F. Grant DMS-2102802, a Simons Sabbatical Fellowship, and a Mercator Fellowship.

[8] 'Example of Surfaces Lecture 4 [Topology I]',*https://old.mccme.ru>postscript>* TopologyI-TopI,MCCME

[9] Amal Joseph, Dr.Susheel George Joseph, Abhilas Mathew, Bibin Joseph, Anandhu M B, 'The Möbius Strip's shape', International Journal of Engineering Technology and Management Sciences', Vol.6, No.4, Aug.-Sept.2022

[10] W. K*ü*hnel, T. F. Banchoff , 'The 9-vertex complex projective plane', The Mathematical Intelligencer, Vol.5, No.3, 1983, Springer Verlag, NewYork.

[11] Ashwin Devraj, 'An overview of curvature', May 2020, (downloaded from Internet)

[12] Pesci, Adriana I, Goldstein Raymond E, Alexander Gareth P., Moffat M. Keith, 'Instability of a Möbius Strip Minimal Surface and a link with Systolic Geometry, Physical Review Letters ,114, 127801, http://wrap.warwick-ac.uk

[13] E.L.Starostin, G.H.M. van der Heijden, 'The Shape of a Möbius Strip', Center for Nonlinear Dynamics, Dept. of Civil and Environmental Engineering, Univ. College of London, London, WC1E,6B T, UK

[14] Diego Julio Cirilo-Lombardo, 'Quantum Particle on a Möbius Strip', arXiv:1204, 5784v2[Math-Ph],5 Jun,2012

[15] A. Nieto, R. P*é*rez Enriquez*ź* , 'A Spinning Particle in a Möbius Strip', arXiv:1102, 5748v1[quant-ph],28 Feb.,2011

[16] Wikipedia, "Knot (Mathematics) and Knot Theory", Internet Search Google

[17] Giovanni De Santi, "An Introduction to the Theory of Knots", December 11, 2002 Internet Upload.

Author Profile

Rabindranath Chattopadhyay, *MTA, IAPT, IPS, ASI, ISEC, IACS, ICSP.*
Email: *rchattopadhyay03[at]yahoo.co.in*