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A Quantile Based Study of Residual Income

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Abstract: ***The concept of residual incomes helps the study of poverty and affluence. In the present work we define α -percentile residual income and in particular median residual income for the poor and the rich. We have also derived these measures for some of the income distributions having explicit quantile functions.***

Keywords: Residual income, percentile, quantile function

1. Introduction

The concept of residual income is useful in constructing poverty and affluence indices. The left and right proportional residual incomes have been considered by [1], in terms of which they defined some classes of distributions and showed the connection between these classes and some other measures of income inequality. [1] presented the left and right proportional residual incomes using the distribution function approach. In the present paper analogous results for quantile function of incomes have been discussed. The distribution function and quantile function are two alternative ways to define a probability distribution. Let X be a real-valued and continuous random variable with distribution function $F(x)$ which is continuous from the right. Then, the quantile function $Q(u)$ of X is defined as

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\},\ 0 \leq u \leq 1.$$

The term quantiles was introduced by [2], but the ideal of quantiles seems to have originated in the work "Statistics by inter comparison with remarks on the Law of Frequency of Error" by [3]. The work on exploratory data analysis by [4] and the work [5] sparked the development of the quantile function as an essential tool instead of the distribution function in statistical analysis. A comprehensive overview of statistical modeling with quantile functions was provided by [6]. Quantile based reliability analysis has done much in literature but few have used the quantile function approach to model the income data ([7], [8], [9]).

The α -percentile residual life function of the non-negative random variable T is defined by [10]. Using this approach we have defined α -percentile residual income and in particular median residual income for constructing both poverty and affluence indices. Although percentile residual life and in particular median residual life are existing in literature, the concepts of percentile or median residual incomes do not exist. So, we give formal definitions for these concepts in Section 3. Also we have converted mean left proportional residual income and mean right proportional residual income into quantile forms.

After this introduction, in Section 2 basic definitions in terms of distribution function are given. In Section 3 analogous definitions in terms of quantile functions are discussed. Some results on quantile based residual incomes are given in Section 4 and in Section 5 concluding remarks are discussed.

2. Definitions in terms of Distribution Function

Given a continuous, non-negative random variable X, which represents the income of a society or community, with distribution function $F_X(x)$, a poverty line ω and an affluence line θ, where $0 < \omega < \theta < \infty$ are considered, such that $F_X(\omega)$ represents the proportion of the poor and $1 - F_X(\theta)$ represents the proportion of the rich. For details one can refer [11], [12], [13] and [14].

The income distribution of the poor for a poverty line ω, is the right truncated income distribution at ω, $X_r(\omega) = \{X | X \leq \omega\}$ and its distribution function is given by

$$F_{X_r(\omega)}(x) = \begin{cases} \dfrac{F_X(x)}{F_X(\omega)}, & x \leq \omega \\ 1 \quad, & x \geq \omega \end{cases}$$

For the income distribution of the affluent people we get the left truncated distribution at θ,

$X_l(\theta) = \{X | X > \theta\}$ with distribution function given by

$$F_{X_l(\theta)}(x) = \begin{cases} 0, & x \leq \theta \\ \dfrac{F_X(x) - F_X(\theta)}{1 - F_X(\theta)} & , x \geq \theta. \end{cases}$$

The random variable,

$$X_l''(\theta) = \frac{X_l(\theta)}{\theta} \tag{2.1}$$

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has been considered by [1] and in the context of income distribution it represents the proportional income to θ for incomes greater than θ. This random variable is called the left proportional residual income at level θ. They also defined the mean left proportional residual income (MLPRI) as

$$e_{X_l''}(\theta)=E\left[X_l''(\theta)\right]=E\left[\frac{X}{\theta}\middle|X>\theta\right] \qquad (2.2)$$

In a similar way, the mean right proportional residual income (MRPRI) is

$$e_{X_r''}(\omega)=E\left[X_r''(\omega)\right]=E\left[\frac{X}{\omega}\middle|X\le\omega\right] \qquad (2.3)$$

3. Definitions in Terms of Quantile Function

The $\alpha-$percentile $(0<\alpha<1)$ residual life function at time t is defined as the $\alpha-$percentile of the remaining life given survived up to time t ([15]). For $0<\alpha<1$, the $\alpha-$percentile residual life function $q_{\alpha,F}(t)$ of F is defined as

$$q_{\alpha,F}(t)=Q(1-(1-\alpha)\bar{F}(t))-t,\quad 0\le t<T_F \qquad (3.1)$$

where $T_F=Q(1)=\sup\{x|F(x)<1\}$. Motivated by this we defined $\alpha-$percentile left (right) proportional residual income. The $\alpha-$percentile left proportional residual income is defined as

$$r_{X_l''}(\theta)=\frac{Q\left[1-(1-\alpha)\bar{F}(\theta)\right]}{\theta},\text{ where } 0<\theta<T_F \qquad (3.2)$$

In particular, median left proportional residual income is defined as

$$M_{X_l''}(\theta)=\frac{Q\left[1-0.5\bar{F}(\theta)\right]}{\theta} \qquad (3.3)$$

The $\alpha-$percentile right proportional residual income is defined as

$$r_{X_r''}(\omega)=\frac{Q(\alpha F(\omega))}{\omega},\text{ where } T_F<\omega<\infty \qquad (3.4)$$

In particular, median right proportional residual income is defined as

$$M_{X_r''}(\omega)=\frac{Q(0.5F(\omega))}{\omega} \qquad (3.5)$$

The $\alpha-$percentile left proportional and right proportional residual income for some commonly used quantile functions are given in Table 1.

Table 1: $\alpha-$percentile left proportional and right proportional residual income for some quantile functions

Distribution	Quantile Function	$\alpha-$percentile left proportional residual income	$\alpha-$percentile right proportional residual income
Pareto	$\sigma(1-u)^{-\frac{1}{\beta}}$	$(1-\alpha)^{-\frac{1}{\beta}}$	$\left(\frac{1-\alpha p}{1-p}\right)^{-\frac{1}{\beta}}$
Power	$ku^{\frac{1}{\beta}}$	$\left[\frac{1-(1-\alpha)(1-p)}{p}\right]^{\frac{1}{\beta}}$	$\alpha^{\frac{1}{\beta}}$
Exponential	$\frac{1}{\lambda}(-\ln(1-u))$	$1+\frac{\ln(1-\alpha)}{\ln(1-p)}$	$\frac{\ln(1-\alpha p)}{\ln(1-p)}$
Weibull	$\sigma(-\log(1-u))^{\frac{1}{\lambda}}$	$\left[\frac{\log(1-\alpha)(1-p)}{\log(1-p)}\right]^{\frac{1}{\lambda}}$	$\left[\frac{\ln(1-\alpha p)}{\ln(1-p)}\right]^{\frac{1}{\lambda}}$
Pareto Type II	$\beta\left[(1-u)^{-\frac{1}{c}}-1\right]$	$\frac{\left[(1-\alpha)(1-p)\right]^{-\frac{1}{c}}-1}{(1-p)^{-\frac{1}{c}}-1}$	$\frac{(1-\alpha p)^{-\frac{1}{c}}-1}{(1-p)^{-\frac{1}{c}}-1}$

4. Characterizations

Theorem 4.1: Pareto distribution is the only distribution for which $\alpha-$percentile left proportional residual income is a constant.

Proof: The $\alpha-$percentile left proportional residual income for Pareto distribution is

$r_{X_l''}(p)=(1-\alpha)^{-\frac{1}{\beta}}$, constant. Now suppose $r_{X_l''}(p)=k$; a constant. That is,

$$\frac{Q\left[1-(1-\alpha)(1-p)\right]}{Q(p)}=k.$$

Solving this functional equation, we get $Q(p) = C(1-p)^{\frac{\ln k}{\ln(1-\alpha)}}$ which is the quantile function of the Pareto type I distribution with $\sigma = C$ and $\beta = \frac{-\ln(1-\alpha)}{\ln k}$. So we can say that the only distribution for which $\alpha-$percentile left proportional residual income is a constant is the Pareto distribution.

Theorem 4.2: Power distribution is the only distribution for which $\alpha-$percentile right proportional residual income is a constant.

Proof: The $\alpha-$percentile right proportional residual income for power distribution is $r_{X_r''}(p) = \alpha^{\frac{1}{\beta}}$, constant. Now suppose $r_{X_r''}(p) = m$, a constant. That is, $\frac{Q(\alpha p)}{Q(p)} = m$. Solving this functional equation, we get $Q(p) = Cp^{\frac{\ln m}{\ln \alpha}}$ which is the quantile function of a power distribution with $k = C$ and $\beta = \frac{\ln \alpha}{\ln m}$. So we can say that the only distribution for which $\alpha-$percentile right proportional residual income is a constant is the power distribution.

5. Concluding Remarks

In this paper we have defined $\alpha-$percentile right (left) proportional residual incomes which will be useful for the study of poverty (affluence) measures of income distributions which have explicit quantile functions but not distribution functions. We have computed these measures for some quantile functions and found that constant right (left) proportional residual incomes are characteristic property of power (Pareto) distribution.

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Competing Interest Declaration

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Author Contribution Declaration

Haritha N Haridas wrote the main manuscript text, Ashlin Varkey prepared table and Anilkumar P. proved a theorem. All authors reviewed the manuscript.

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