Double Weighted Exponential-Exponential Distribution: Theory, Properties and Application to Survival Analysis of Cancer Data

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Abstract: ***In recent years so many statistical distributions have been derived in the literature. The weighted distributions are widely used in many real-life fields such as medicine, ecology, reliability, etc., for the development of proper statistical model. The concept of double weighted distribution is applied in Exponential-exponential distribution and thus we propose a new double weighted distribution. The statistical properties like mean, variance, coefficient of variation, moments, mode, reliability function, hazard function and the reverse hazard function of double weighted exponential-exponential distribution (DWEED) are explored. The parameters of this distribution are estimated by the maximum likelihood estimation method. The plots of survival function and hazard function of the proposed distribution are also presented. The parameter criterion such as AIC, BIC, AICC, and Kolmogorov- Smirnov test has been used to choose a better fitted probability model. Two cancer data sets are used to demonstrate the superiority and adaptability of the suggested model by comparing it with other models using the above model criterions.***

Keywords: Double weighted distributions, exponential-exponential distribution, order statistics, simulation, maximum likelihood estimation.

1. Introduction

The concept of weighted distributions was first given by Fisher [7]. In that Fisher, examined the way in which ascertainment techniques can affect the form of the distribution of recorded observations when the standard distributions were unable to accurately represent these observations with equal probabilities in the statistical data model, Rao [11] later developed this idea in a unified manner. In the study explained in Patil and Rao [9] examined length biased (size biased) sampling in relation to animal populations and human families.

They examined specific generic models that produced weighted distributions with weighted functions that were not always bound by unity. This concept has become an essential tool in statistical theory particularly in scenarios where observations are derived from non-experimental, non-replicated and non-random conditions. The length and area-biased weighted Maxwell distribution and its properties were discussed by Sharma et al. [15]. A novel area-biased Aradhana distribution was obtained by Elangovan et al. [12] and applied to cancer data. Perveen [10] studied the Area biased weighted Weibull distribution with applications. The size biased Zeghdoudi distribution and discuss its various statistical properties and application is discussed by Chouia [6]. The theory of double weighted distribution was first discovered by Khadim and Hantoosh [1], they applied it to the exponential distribution and explained the statistical properties. The characterization and estimation of the Double Weighted Rayleigh Distribution is introduced by Rishwan [13]. The double weighted inverse Weibull distribution and its statistical properties were discussed by Al-Khadim and Hantoosh [2]. Double Weighted Weibull Distribution Properties and Application in engineering survival field is discussed by Saghir and Saleem [3]. In this paper we present the Double Weighted distribution (DWD), taking one type of weight function w(x) = x and using the Exponential-Exponential distribution Ayeni [4] as original distribution, we discuss the pdf, cdf and some useful properties of Double Weighted Exponential-Exponential distribution.

2. Double Weighted Distribution

Definition (2.1): The double weighted distribution is given by Fisher (1934).

$$f_w(x;c) = \frac{w(x)f(x)F(cx)}{W_D},\ x \geq 0,\ c \geq 0 \tag{1}$$

Where $W_D = \int_0 w(x)f(x)F(cx)\,dx$, the first weight is w(x) and the second weight is cx. F(cx) is the cdf of original distribution f(x).

3. Double Weighted Exponential-Exponential Distribution (DWEED)

Consider the first weight w(x) = x and the pdf of exponential-exponential distribution Ayeni [4] is $f(x;\lambda) = \lambda^2 e^{-\lambda^2 x}$, $F(cx) = 1 - e^{-\lambda^2 cx}$ and $W_D = \frac{c(c+2)}{\lambda^2(c+1)^2}$ by equation (1).

Then the pdf of Double Weighted Exponential-Exponential Distribution (DWEED) is:

$f_w(x;\lambda,c) = \frac{\lambda^4 (c+1)^2}{c(2+c)} xe^{-\lambda^2 x}$ (1- $e^{-\lambda^2 cx}$), x ≥ 0, λ, c ≥ 0. (2)

The cdf of DWEED is

$F_w(x;\lambda,c) = \frac{(c+1)^2}{c(c+2)} \left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right) - \frac{1}{(c+1)^2}\left(1-(1+\lambda^2(c+1)x)e^{-\lambda^2(c+1)x}\right)\right]$ (3)

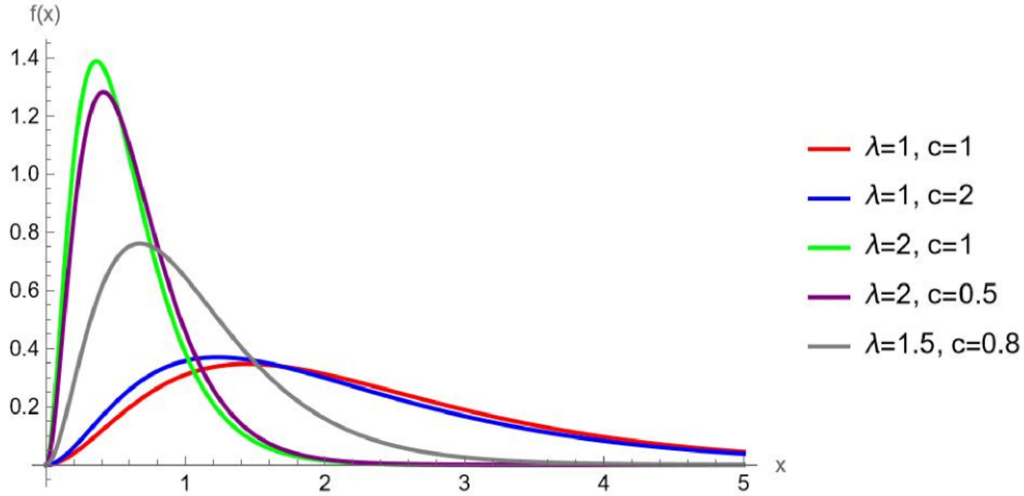

Figure 1: The plot of probability density function of DWEED for different parameter values

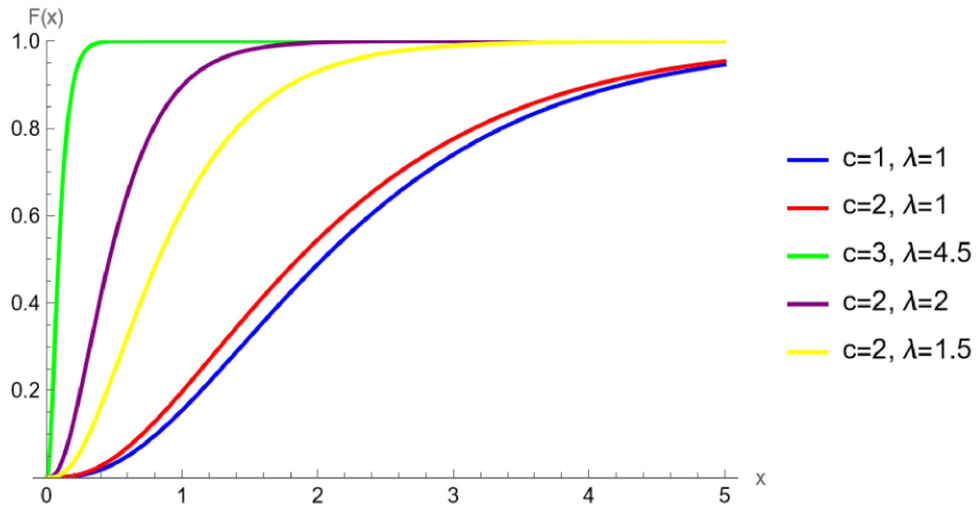

Figure 2: The plot of cumulative density function of DWEED for different parameter values

4. Reliability Analysis

The survival function or reliability function of the DWEE distribution is given by

$$S(x) = 1 - F_w(x;\lambda,c)$$

$$S(x) = 1 - \left\{\frac{(c+1)^2}{c(c+2)}\left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right) - \frac{1}{(c+1)^2}\left(1-(1+\lambda^2(c+1)x)e^{-\lambda^2(c+1)x}\right)\right]\right\}$$

The hazard function of the DWEE distribution is defined to be

$$h(x) = \frac{f_w(x;\lambda,c)}{S(x)}$$

$$= \frac{\frac{\lambda^4 (c+1)^2}{c(2+c)} xe^{-\lambda^2 x}\left(1-e^{-\lambda^2 cx}\right)}{1-\left\{\frac{(c+1)^2}{c(c+2)}\left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right) - \frac{1}{(c+1)^2}\left(1-(1+\lambda^2(c+1)x)e^{-\lambda^2(c+1)x}\right)\right]\right\}}$$

The reverse hazard function of the DWEE distribution is

$$h_r(x) = \frac{f_w(x;\lambda,c)}{F_w(x;\lambda,c)}$$

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$$= \frac{\frac{\lambda^4 (c+1)^2}{c(2+c)} x e^{-\lambda^2 x} (1- e^{-\lambda^2 c x})}{\left\{\frac{(c+1)^2}{c(c+2)} \left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right) - \frac{1}{(c+1)^2}\left(1-(1+\lambda^2 (c+1)x)e^{-\lambda^2 (c+1)x}\right)\right]\right\}}$$

The Mills Ratio of the DWEE distribution is

$$\frac{1}{h_r(x)} = \frac{1-\left\{\frac{(c+1)^2}{c(c+2)}\left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right)-\frac{1}{(c+1)^2}\left(1-(1+\lambda^2(c+1)x)e^{-\lambda^2(c+1)x}\right)\right]\right\}}{\frac{\lambda^4 (c+1)^2}{c(2+c)} x e^{-\lambda^2 x} (1- e^{-\lambda^2 c x})}$$

The cumulative hazard function of the DWEE distribution is defined as

$$H(x) = -\ln(1 - F_w(x;\lambda,c))$$

$$= -\ln\left\{1 - \left\{\frac{(c+1)^2}{c(c+2)}\left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right) - \frac{1}{(c+1)^2}\left(1-(1+\lambda^2(c+1)x)e^{-\lambda^2(c+1)x}\right)\right]\right\}\right\}$$

The odds rate function of the DWEE distribution is defined as

$$O(x) = \frac{F_w(x;\lambda,c)}{1-F_w(x;\lambda,c)}$$

$$= \frac{\left\{\frac{(c+1)^2}{c(c+2)}\left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right) - \frac{1}{(c+1)^2}\left(1-(1+\lambda^2(c+1)x)e^{-\lambda^2(c+1)x}\right)\right]\right\}}{1-\left\{\frac{(c+1)^2}{c(c+2)}\left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right) - \frac{1}{(c+1)^2}\left(1-(1+\lambda^2(c+1)x)e^{-\lambda^2(c+1)x}\right)\right]\right\}}$$

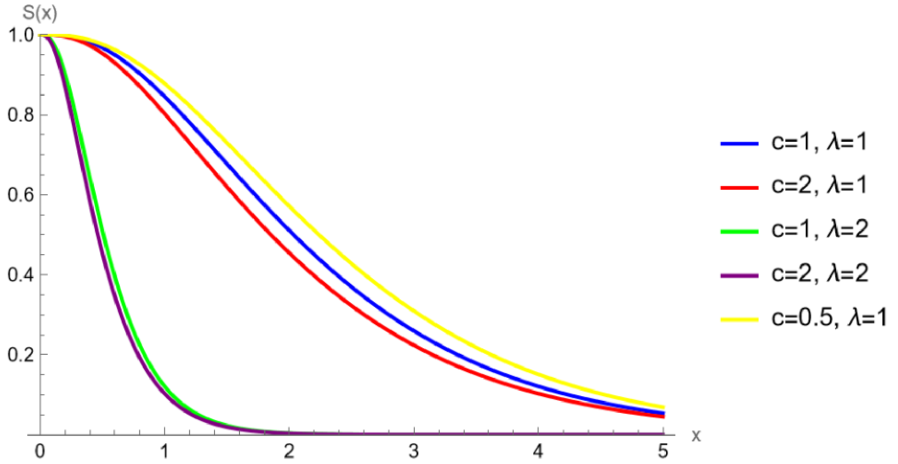

Figure 3: The survival function plot of DWEED for different parameter values

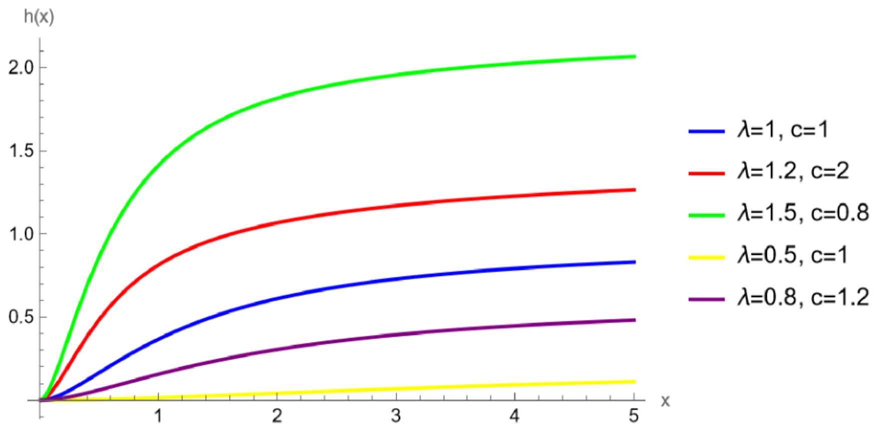

Figure 4: The hazard function plot of DWEED for different parameter values

5. Moments and Related Measures

In this section, we derive the structural properties, moments, moment-generating function, characteristic function, and the r^{th} moment for the DWEE distribution. We also investigate the mean, variance, coefficient of variation, skewness and kurtosis (numerical values) of the proposed distribution using R code.

Theorem 5.1. The r^{th} moment about the origin of DWEE distribution is given by

$$E(X^r) = \mu_r' = \frac{(c+1)^2\Gamma(r+2)}{c(2+c)\lambda^{2r}}\left[1 - \frac{1}{(c+1)^{r+2}}\right], r = 1, 2, 3, \ldots$$

Proof:
using equation (2), r^th moment about the origin of DWEE distribution is given by:

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_w(x;\lambda,c)\, dx$$

$$E(X^r) = \mu_r' = \int_0^\infty x^r \frac{\lambda^4 (c+1)^2}{c(2+c)}\, xe^{-\lambda^2 x}\,(1 - e^{-\lambda^2 cx})\, dx$$

$$E(X^r) = \mu_r' = \frac{\lambda^4 (c+1)^2}{c(2+c)} \int_0^\infty x^{r+1}\, e^{-\lambda^2 x}\,(1 - e^{-\lambda^2 cx})\, dx \quad (4)$$

where $\int_0^\infty x^{r+1}\, e^{-\lambda^2 x}\,(1 - e^{-\lambda^2 cx})\, dx = \frac{\Gamma(r+2)}{\lambda^{2(r+2)}}\left[1 - \frac{1}{(c+1)^{r+2}}\right]$, by substituting this in equation (4), we get

$$E(X^r) = \mu_r' = \frac{(c+1)^2\Gamma(r+2)}{c(2+c)\lambda^{2r}}\left[1 - \frac{1}{(c+1)^{r+2}}\right] \quad (5)$$

Now by using (5) the expressions for the first four raw moments were obtained by putting $r = 1, 2, 3, 4$ respectively.

$$\mu_1' = \frac{2(c+1)^2}{c(2+c)\lambda^2}\left[1 - \frac{1}{(c+1)^3}\right] \quad (6)$$

$$\mu_2' = \frac{6(c+1)^2}{c(2+c)\lambda^4}\left[1 - \frac{1}{(c+1)^4}\right] \quad (7)$$

$$\mu_3' = \frac{24(c+1)^2}{c(2+c)\lambda^6}\left[1 - \frac{1}{(c+1)^5}\right] \quad (8)$$

$$\mu_4' = \frac{120(c+1)^2}{c(2+c)\lambda^8}\left[1 - \frac{1}{(c+1)^6}\right] \quad (9)$$

From equations (6) and (9) we can find variance, coefficient of variation, skewness and kurtosis is as follows:

Mean $= \frac{2(c+1)^2}{c(2+c)\lambda^2}\left[1 - \frac{1}{(c+1)^3}\right]$

The **variance** is given by

$$Var\ (x) = \frac{1}{\lambda^4}\left\{\frac{6(c+1)^2}{c(2+c)}\left[1 - \frac{1}{(c+1)^4}\right] - \left[\frac{2(c+1)^2}{c(2+c)}\left[1 - \frac{1}{(c+1)^3}\right]\right]^2\right\} \quad (10)$$

The **coefficient of Variation** is given by

$$CV = \frac{Var\ (x)}{Mean} = \frac{\frac{1}{\lambda^4}\left\{\frac{6(c+1)^2}{c(2+c)}\left[1-\frac{1}{(c+1)^4}\right] - \left[\frac{2(c+1)^2}{c(2+c)}\left[1-\frac{1}{(c+1)^3}\right]\right]^2\right\}}{\frac{2(c+1)^2}{c(2+c)\lambda^2}\left[1-\frac{1}{(c+1)^3}\right]} \quad (11)$$

Table 1: Mean, Variance, CV, Skewness and Kurtosis for various parameter values

λ	c	Mean	Variance	CV	Skewness	Kurtosis
0.5	0.5	10.1333	35.9822	0.5919	1.2338	5.3554
0.5	1.0	9.3333	32.8888	0.6144	1.3258	5.7370
0.5	2.0	8.6666	31.5555	0.6481	1.4074	6.0267
1.0	0.5	2.5333	2.2488	0.5919	1.2338	5.3554
1.0	1.0	2.3333	2.0555	0.6144	1.3258	5.7370
1.0	2.0	2.1666	1.9722	0.6481	1.4074	6.0267
1.5	0.5	1.1259	0.4442	0.5919	1.2338	5.3554
1.5	1.0	1.0370	0.4060	0.6144	1.3258	5.7370
1.5	2.0	0.9629	0.3895	0.6481	1.4074	6.0267

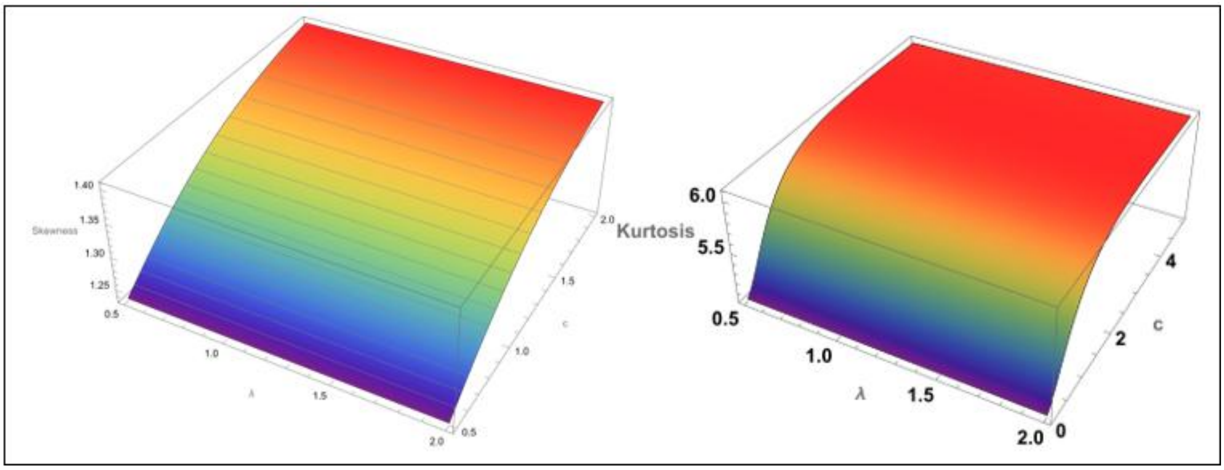

Figure 5: The 3D plot of skewness and Kurtosis of DWEED for different parameter values

Preposition1: The pdf $f_w(x;\lambda,c) = \frac{\lambda^4 (c+1)^2}{c(2+c)}\, xe^{-\lambda^2 x}$ (1- $e^{-\lambda^2 cx}$), is log concave and unimodal.

Proof: The pdf of Double Weighted Exponential-Exponential Distribution (DWEED) is:

$$f_w(x;\lambda,c) = \frac{\lambda^4 (c+1)^2}{c(2+c)}\, xe^{-\lambda^2 x}\,(1 - e^{-\lambda^2 cx})$$

$$g(x) = \log f_w(x;\lambda,c) = \log A + \log x - \lambda^2 x + \log(1 - e^{-\lambda^2 cx}) \quad (12)$$

where $A = \frac{\lambda^4 (c+1)^2}{c(2+c)}$

The first derivative of equation (12) is

$$\frac{\partial}{\partial x} g(x) = \frac{1}{x} - \lambda^2 + \frac{\lambda^2 Ce^{-\lambda^2 x}}{1 - e^{-\lambda^2 x}} \quad (13)$$

Take the second derivative of equation (13)

$$\frac{d^2}{dx^2} g(x) = -\frac{1}{x^2} - \frac{\lambda^4 c^2 e^{-\lambda^2 x}}{\left(1 - e^{-\lambda^2 x}\right)^2} < 0, \text{ for all } x > 0.$$

Therefore, $\log f_w(x;\lambda,c)$ is strictly concave, which implies that the pdf $f_w(x;\lambda,c)$ is log concave and unimodal.

Corollary 1:
Now equating equation (13) to zero, we obtain,

$$\frac{1}{x} - \lambda^2 + \frac{\lambda^2 Ce^{-\lambda^2 x}}{1 - e^{-\lambda^2 x}} = 0 \quad (14)$$

By solving the above non-linear equation (14), we get the mode of DWEE distribution.

Theorem 5.2. The Moment Generating Function and characteristic function of DWEE distribution is given by $M_X(t) = \frac{\lambda^4 (c+1)^2}{c(2+c)} \left[\frac{1}{(\lambda^2-t)^2} - \frac{1}{(\lambda^2(1+c)-t)^2}\right]$ for $t \leq \lambda^2$ *and* $\phi_x(t) = E\,(e^{itx}) = \frac{\lambda^4 (c+1)^2}{c(2+c)} \left[\frac{1}{(\lambda^2-it)^2} - \frac{1}{(\lambda^2(1+c)-it)^2}\right]$.

Proof:
The moment generating function of DWEE distribution is given by:

$$M_x(t) = E\,(e^{tx})$$
$$M_x(t) = \int_0^\infty e^{tx} \frac{\lambda^4 (c+1)^2}{c(2+c)} x\, e^{-\lambda^2 x} (1 - e^{-\lambda^2 cx})\, dx$$
$$M_x(t) = \frac{\lambda^4 (c+1)^2}{c(2+c)} \int_0^\infty x\, e^{(t-\lambda^2)x} (1 - e^{-\lambda^2 cx})\, dx \quad (15)$$

By splitting the integral and applying standard Laplace integral transforms,
$\int_0^\infty x e^{-ax}\, dx = \frac{1}{a^2}$ *, for* $Re(a)>0$ we get,

$$M_x(t) = \frac{\lambda^4 (c+1)^2}{c(2+c)} \left[\frac{1}{(\lambda^2-t)^2} - \frac{1}{(\lambda^2(1+c)-t)^2}\right]$$

The corresponding characteristic function is given by

$$\phi_x(t) = E\,(e^{itx}) = \frac{\lambda^4 (c+1)^2}{c(2+c)} \left[\frac{1}{(\lambda^2-it)^2} - \frac{1}{(\lambda^2(1+c)-it)^2}\right] \quad (16)$$

6. Parameter Estimation

In this section, we consider maximum likelihood estimation of the unknown parameters λ and c of DWEE distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from DWEE distribution, then the likelihood function is,

$$L(x;\ \alpha,\beta,\lambda) = \prod_{i=1}^{n} f_w(x;\lambda,c)$$
$$L(x;\ \alpha,\beta,\lambda) = \prod_{i=1}^{n} \frac{\lambda^4 (c+1)^2}{c(2+c)} x e^{-\lambda^2 x_i} (1 - e^{-\lambda^2 c x_i})$$
$$L(x;\ \alpha,\beta,\lambda) = \frac{\lambda^4 (c+1)^2}{c(2+c)} \prod_{i=1}^{n} x e^{-\lambda^2 x_i} (1 - e^{-\lambda^2 c x_i})$$

The log likelihood function is given by

$$l(x;\ \alpha,\beta,\lambda) = \log L(x;\ \alpha,\beta,\lambda)$$
$$l(x;\ \alpha,\beta,\lambda) = n \log \frac{\lambda^4 (c+1)^2}{c(2+c)} + \sum_{i=1}^{n} \log x_i - \lambda^2 \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log(1 - e^{-\lambda^2 c x_i}) \quad (17)$$

Differentiating the above equation (17) partially w.r.to λ and c, and equating to zero, we get the following system of normal equations.

$$\frac{\partial l}{\partial \lambda} = \frac{4n}{\lambda} - 2\lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{2cx_i\lambda e^{-\lambda^2 c x_i}}{1-e^{-\lambda^2 c x_i}} = 0 \quad (18)$$
$$\frac{\partial l}{\partial c} = \frac{2n}{c+1} - \frac{n}{c} - \frac{n}{2+c} + \sum_{i=1}^{n} \frac{\lambda^2 x_i e^{-\lambda^2 c x_i}}{1-e^{-\lambda^2 c x_i}} = 0 \quad (19)$$

The maximum likelihood estimators of λ and c say $\hat{\lambda}$ *and* $\hat{c}$, are the simultaneous solutions of the above equations (18) and (19). Since we don't have any explicit form of MLE's, we need to use numerical methods to get estimates of parameter.

7. Distribution of Order Statistics

Order statistics make their appearance in many statistical theory and practice. We know that if $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a continuous population with

cdf $F_x(x)$ and pdf $f_x(x)$, then the pdf of rth order statistics $X_{(r)}$ is

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} . f_x(x). [F_x(x)]^{r-1} [1 - F_x(x)]^{n-r} \quad (20)$$

The first order statistic $X_{(1)}$ is

$$f_{X(1)}(x) = n\,[1 - F(x)]^{n-1} f(x)$$

$$f_{X(1)}(x) = n \left[1 - \left\{\frac{(c+1)^2}{c(c+2)} \left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right) - \frac{1}{(c+1)^2}\left(1-(1+\lambda^2(c+1)x)e^{-\lambda^2(c+1)x}\right)\right]\right\}\right]^{n-1} \frac{\lambda^4 (c+1)^2}{c(2+c)} x e^{-\lambda^2 x} (1 - e^{-\lambda^2 cx}) \quad (21)$$

Also, the nth order statistic $X_{(n)}$ *is*

$$f_{X(n)}(x) = n\,[F(x)]^{n-1} f(x)$$

$$f_{X(n)}(x) = n \left[\left\{\frac{(c+1)^2}{c(c+2)} \left[\left(1-(1+\lambda^2 x)e^{-\lambda^2 x}\right) - \frac{1}{(c+1)^2}\left(1-(1+\lambda^2(c+1)x)e^{-\lambda^2(c+1)x}\right)\right]\right\}\right]^{n-1} \frac{\lambda^4 (c+1)^2}{c(2+c)} x e^{-\lambda^2 x} (1 - e^{-\lambda^2 cx}) \quad (22)$$

8. Stochastic Ordering

The stochastic ordering of a non-negative continuous random variable is a tool for comparing the behavior of components. Let X and Y be two random variables. If the random variable X is said to be smaller than another random variable Y in the

1) Stochastic order ($X \leq_{st}$ Y), if $F_X(x) \geq F_Y(x)$ for all x.
2) Hazard rate order ($X \leq_{hr}$ Y), if $F_X(x) \geq F_Y(x)$ for all x.
3) Likelihood ratio order ($X \leq_{lr}$ Y), if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following stochastic ordering relationships due to Shaked and Shanthi Kumar [14] are well known for establishing stochastic ordering for continuous distributions.
$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$

Theorem 1: Let $X \sim DWEE\,(\lambda_1, c_1)$ and $Y \sim DWEE\,(\lambda_2, c_2)$), then for $c > 0$ *and* $0 \prec \lambda_1 \prec \lambda_2$ the densities of DWEE distribution satisfy the monotone likelihood ratio (MLR) property: $\frac{f_w(x;\lambda_1,c)}{f_w(x;\lambda_2,c)}$ is a decreasing function of x € (0, ∞), then $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st}$.

Proof: Let $c_1 = c_2 = c$,

$$\frac{f_w(x;\lambda_1,c)}{f_w(x;\lambda_2,c)} = \frac{\frac{\lambda_1^4\,(c+1)^2}{c(2+c)}\, x e^{-\lambda_1^2 x}\,(1 - e^{-\lambda_1^2 cx}),}{\frac{\lambda_2^4\,(c+1)^2}{c(2+c)}\, x e^{-\lambda_2^2 x}\,(1 - e^{-\lambda_2^2 cx}),}$$

$$= \frac{\lambda_1^4\, e^{-\lambda_1^2 x}\,(1 - e^{-\lambda_1^2 cx})}{\lambda_2^4\, e^{-\lambda_2^2 x}\,(1 - e^{-\lambda_2^2 cx})}$$

$$\frac{f_w(x;\lambda_1,c)}{f_w(x;\lambda_2,c)} = \frac{\lambda_1^4}{\lambda_2^4}\, e^{-(\lambda_1^2 - \lambda_2^2)\,x}\, \frac{(1 - e^{-\lambda_1^2 cx})}{(1 - e^{-\lambda_2^2 cx})}$$

Taking log on both sides

$$\ln \frac{f_w(x;\lambda_1,c)}{f_w(x;\lambda_2,c)} = \ln \lambda_1^4 - \ln \lambda_2^4 - (\lambda_1^2 - \lambda_2^2)\,x + \ln\,(1 - e^{-\lambda_1^2 cx}) - \ln\,(1 - e^{-\lambda_2^2 cx})$$

Differentiating the above ratio w.r.to x and yields

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}} \ln \frac{f_w(x;\lambda_1,c)}{f_w(x;\lambda_2,c)} = -\,(\lambda_1^2 - \lambda_2^2) + \frac{\lambda_1^2 c e^{-\lambda_1^2 cx}}{(1 - e^{-\lambda_1^2 cx})} - \frac{\lambda_2^2 c e^{-\lambda_2^2 cx}}{(1 - e^{-\lambda_2^2 cx})}$$

Since $\lambda_1 \prec \lambda_2$

$\frac{d}{dx} \ln \frac{f_w(x;\lambda_1,c)}{f_w(x;\lambda_2,c)} < 0 \Rightarrow \frac{f_w(x;\lambda_1,c)}{f_w(x;\lambda_2,c)}$ is decreasing in x.

This means that $X \leq_{lr} Y$, and hence, $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$.

9. Simulation Study

In this section, a simulation study is performed to test the performance of MLEs. We generate a random sample of size n = 30, 50,100,150, 200 and 500 from DWEE distribution for the values of $\lambda = 1.5$, $c = 2$ and $\lambda = 1, c = 0.5$ with replication 1000 times using R software. It is observed that the mean squared error (MSE) and average bias decreases when the sample size increases. The simulation results for different parameter values of DWEE distribution are presented in table 2.

Table 2: Average values of the Bias and MSE's for the DWEE distribution for $\lambda = 1.5, c = 2$.

n	Parameter Estimates	Bias	MSE
n = 30	$\hat{\lambda}$ =1.6096 $\hat{c}$ = 1.7982	0.1096 -0.2017	0.0371 59.2078
n = 50	$\hat{\lambda}$ =1.6035 $\hat{c}$ = 1.2643	0.1035 -0.0301	0.0301 23.5375
N = 100	$\hat{\lambda}$ =1.5915 $\hat{c}$ = 0.8723	0.0915 -1.1276	0.0220 12.7221
N = 200	$\hat{\lambda}$ =1.5919 $\hat{c}$ =0.5618	0.09198 -1.4381	0.01810 2.3678
N = 500	$\hat{\lambda}$ =1.5874 $\hat{c}$ =0.5126	0.08744 -1.4873	0.0138 2.3611

The result obtained in Table 1 shows that as the sample size increases, biases and MSEs of the parameters become smaller. This result aligns with the first-order asymptotic theory.

10. Application

To check the real-life application of DWEE distribution, we used cancer data. To find the best model, we consider some performance criteria such as log-likelihood (-LL), AIC (Akaike information criterion), AICC (corrected Akaike information criterion), BIC (Bayesian information criterion) and HQIC (Hannan-Quinn Information Criterion) and goodness of fit test statistic Kolmogorov-Smirnov (KS) statistic and p-value related to KS also calculated. The formulae for calculation of AIC, BIC, AICC and HQIC are; AIC $= 2k - 2\log L$, AICC $= AIC + \frac{2k\,(k+1)}{n-k-1}$, BIC $= k\log n - 2\log L$ and HQIC $= 2k\log\,(\log(n)) - 2\log L$.

where n is the sample size, K is the number of parameters, and L denotes the likelihood function. Any probability model having smaller value of AIC, BIC and $-LogL$ being the best model to fit the data set.

The data are fitted to Double weighted Exponential-Exponential distribution (DWEED), Double weighted Weibull distribution (DWWD), Double Weighted Rayleigh (DWRD) and Exponential-exponential distribution (EED). The pdfs of these distributions are as follows:

$$f(x) = \frac{\lambda^4\,(c+1)^2}{c(2+c)}\, x e^{-\lambda^2 x}\,(1\text{-}\, e^{-\lambda^2 cx})$$

$$f_w(x;c,\lambda) = \frac{\lambda\,(1 + c^\lambda)^{1+\frac{1}{\lambda}}\, x^\lambda\, e^{-x^\lambda}(1 - e^{-c^\lambda x^\lambda})}{\Gamma\left(1 + \frac{1}{\lambda}\right)\left((1 + c^\lambda)^{1+c^\lambda} - 1\right)}$$

$$g_w\,(\text{x;λ,c}) = \sqrt{\frac{2}{\pi}}\, \frac{(1+c^2)^{\frac{3}{2}}}{\left((1+c^2)^{\frac{3}{2}} - 1\right)}\, \frac{x^2}{\theta^3} \exp\left(-\frac{x^2}{2\theta^2}\right) \left\{1 - exp\left(\frac{(cx)^2}{2\theta^2}\right)\right\}$$

$$f(x) = \lambda^2 e^{-\lambda^2 x}$$

Following are the two real data sets and its statistical analysis:

Dataset 1:

The data set given in table 1 represents the non-censored data represents the survival times (in months) of 32 patients of melanoma (non-censored data) has been taken from Susarla et. al. [16] later used by Kayid et. al. [8]. 3.25, 3.5, 4.75, 4.75, 5, 5.25, 5.75, 5.75, 6.25, 6.5, 6.5, 6.75, 6.75, 7.78, 8, 8.5, 8.5, 9.25, 9.5, 9.5, 10, 11.5, 12.5, 13.25, 13.5, 14.25, 14.5, 14.75, 15, 16.25, 16.25, 16.

Table 3: Descriptive Statistics for data set 1

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
3.250	6.125	8.500	9.360	13.312	16.250

Table 4: ML estimates of the parameters with measures for model selection for data set 1

Models	Parameter	-LL	AIC	BIC	AICC	HQIC
DWEED	$\hat{\lambda} = .565656$ $\hat{c} = 1e\text{-}05$	-90.74705	185.4941	188.4256	185.9079	186.268
DWWD	$\hat{\lambda} = 0.6197$ $\hat{c} = 0$	-98.2419	200.4838	203.4152	200.8975	201.4555
DWRD	$\hat{\theta} = 4.5682$ $\hat{c} = 5e\text{-}04$	-91.7824	187.5649	190.4963	187.9787	188.5366
EED	$\hat{\lambda} = .3265$	-103.6207	209.2414	210.7071	209.3747	209.7272

Dataset 2

Data regarding the remission time (in months) of 50 breast cancer women reported by university cancer registry department of Benin teaching hospital given in Alice and Jose [5]. 50, 74, 35, 39, 21, 37, 27, 35, 30, 35, 26, 28, 34, 34, 26, 41, 61, 33, 33, 26, 25, 41, 35, 34, 34, 33, 60, 61, 42, 30, 80, 31, 24, 49, 26, 31, 28, 41, 37, 41, 61, 33, 26, 34, 50, 73, 45, 80, 39, 21.

Table 5: Descriptive Statistics for data set 2

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
21.00	30.00	34.50	39.40	41.75	80.00

Table 6: ML estimates of the parameters with measures for model selection for data set 2

Models	Parameter	-LL	AIC	BIC	AICC	HQIC
DWEED	$\hat{\lambda} = 0.2759$ $\hat{c} = 1e\text{-}05$	-209.539	423.0796	426.9037	423.3349	424.5358
DWWD	$\hat{\lambda} = 0.15931$ $\hat{c} = 0$	-233.688	471.3766	475.2006	471.6319	472.8328
DWRD	$\hat{\theta} = 18.823$ $\hat{c} = 0.001$	-232.214	468.4282	462.2522	458.6835	469.8844
EED	$\hat{\lambda} = 0.1593$	-232.688	469.3766	471.2886	469.4599	470.1047

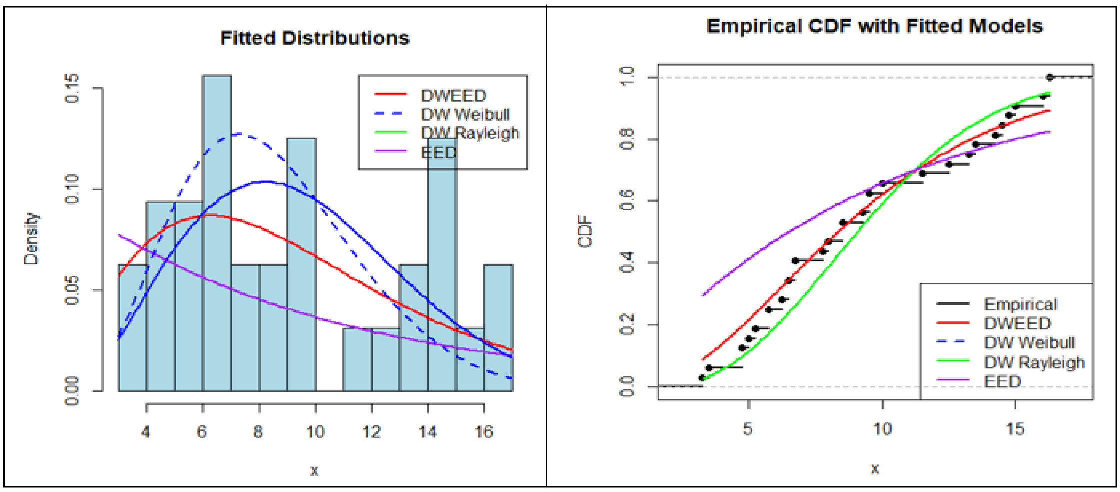

Figure 6: Histogram and Fitted pdfs, Empirical CDF and Fitted CDFs for data set 1

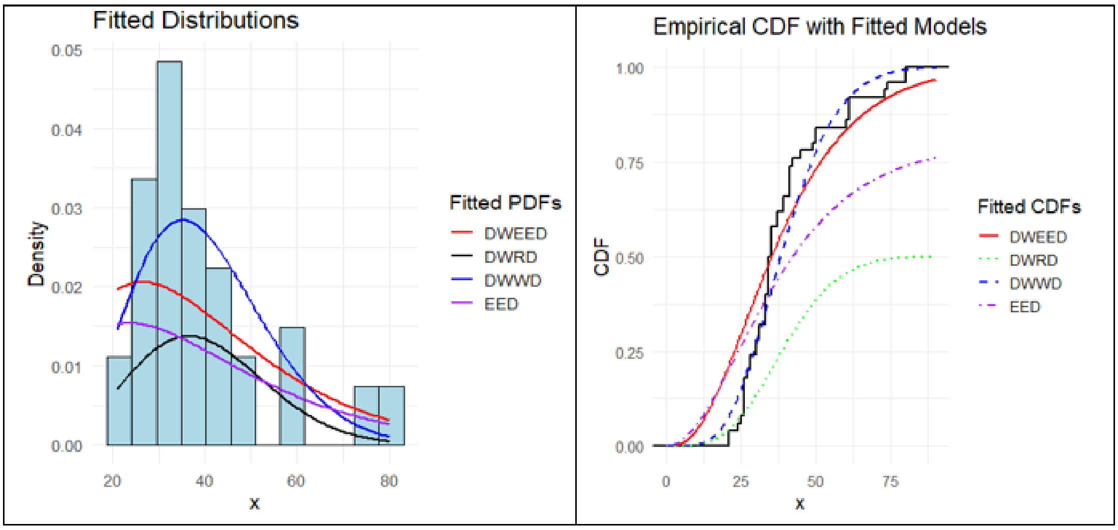

Figure 7: Histogram and Fitted pdfs, Empirical CDF and Fitted CDFs for data set 2

11. Conclusion

In this paper we proposed a new double weighted distribution named as double weighted exponential-exponential distribution (DWEED). Some statistical and reliability concepts have been studied and plotted. The maximum likelihood approach is used to estimate the parameter. The results of the data analysis indicate that the DWEE distribution has the lowest AIC, BIC, AICC and HQIC values. Therefore, the proposed model is more superior and flexible match than the other competing models. Hence, the proposed distribution can be seen as an important distribution in modelling real-life data.

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