International Journal of Science and Research (IJSR)
ISSN: 2319-7064
Impact Factor 2024: 7.101

Degeneracy Pressure and Its Role in Formation of Compact Stars

Prosad Bhattacharya

Email: *prosad3012[at]gmail.com*

Abstract: ***Under the influence of excessively high gravitational pressure, identical electrons, i.e., electrons with same set of quantum numbers, in the core of an hugely compressed red giant star are forced to occupy the same energy states at the same time violating the Pauli's Exclusion Principle. Such degenerated electrons create and exert Electron Degeneracy Pressure in the star's core that acts against the gravity and halts the core collapse of a cold star. When inwardly acting gravitational pressure becomes equal to the newly created outwardly acting electron degeneracy pressure, core-collapse of the cold star stops and the star becomes a stable White Dwarf star. Stars having mass maximum up to the Chandrasekhar limit (1.44 times the solar mass) can only become white dwarfs. Electron degeneracy pressure, however, is unable to restrict the core collapse of cold giant / super giant stars of mass heavier than the Chandrasekhar limit. As its radius reduces, the gravitational pressure of the star increases exponentially. Incredibly high gravity forces the electrons of a star's core to combine with the protons in their nuclei and form neutrons. The cold star becomes a sphere made of neutrons. Core collapse continues, gravitational pressure increases and in due course gravitational pressure compels the identical neutrons occupy the same energy states at the same time defying the Pauli's Exclusion Principle. These degenerated neutrons, like the degenerate electrons, create another type of quantum pressure inside the core of a cold star called neutron degeneracy pressure. Neutron degeneracy pressure restricts the core collapse of the red giant / supergiant stars and finally helps to create Neutron Stars when it becomes equal to the gravitational pressure of the stars. Stars having masses in between the Chandrasekhar limit (1.44 MΘ) and the TOV limit (2.928MΘ) will become only neutron stars. At and beyond the TOV limit, a star, upon core collapse, will become a Black Hole first as its escape velocity will become equal to the velocity of light. But newly formed black holes are not pressure balanced and up to a certain mass limit, the black holes eventually will become neutron stars also. Above that mass limit, a black hole will explode as a supernova before becoming a neutron star.***

Keywords: Electron Degeneracy Pressure; Neutron Degeneracy Pressure, Pauli's Exclusion Principle; Gravitational Energy; Quantum Chromo-dynamics Binding Energy

1. Introduction

1.1 Degeneracy pressures are quantum mechanical pressures that develop inside the core of a compact star. While electron degeneracy pressure is responsible for formation of white dwarfs, neutron degeneracy pressure helps to create neutron stars and black holes. White dwarfs are formed when the gravitational pressure of a cold star of mass up to 1.44 solar mass becomes equal to its electron degeneracy pressure. Similarly, more massive cold stars become neutron stars when their respective gravitational pressure equals the neutron degeneracy pressure. Like a neutron star, neutrons are also the main constituent of a stellar black hole and ultimately this black hole becomes a neutron star. More massive black holes explode as supernovae before becoming neutron stars.

1.2 Compact stars are those cold or dead stars which are formed at the end of star's life. The giant stars normally become white dwarfs stars while supergiant stars end up either as neutron stars or a black hole. Cores of the compact stars are hugely dense as excessively high gravitational pressures act on them.

2. Formation of Compact Stars

2.1 During its main sequence phase, a low mass star (mass up to the Chandrasekhar limit, i.e., 1.44 times the solar mass) burns mainly hydrogen atoms as nuclear fuel. The fusion process converts hydrogen to helium and generates thermal energy that exerts outward pressure to counter the gravity of the star and finally balances the gravitational pressure of the star. When hydrogen atoms get exhausted, the core of the star cools down and contracts. The contraction of the core generates huge temperature that starts fusing helium atoms to carbon atoms. When all the helium atoms get consumed, again, in absence of the thermal pressure, the star contracts and consequently generates pressure that begins to fuse carbon atoms which is the next heavier element. Fusion process continues in this way and the star in its main sequence stage remains balanced by the inwardly acting gravitational pressure and the outwardly acting thermal pressure generated by the fusion process. A low mass star may take more than a billion of years to consume all the hydrogen at its core in its main sequence phase.

2.2 For a low mass star, when all the nuclear fuels, mainly hydrogen but also helium, carbon and oxygen, get exhausted at its core, no more thermonuclear pressure is generated and the main sequence phase of the star ends. The star cools down and starts contracting. At that stage, the carbon or oxygen atoms present in the core cannot fuse heavier elements; however, the hydrogen atoms present in the outer gaseous shell of the star get compressed and start fusing. The heat generated in this process expands the outer shell of the star to a great extent. The low mass star becomes a red giant star. The core of a red giant star mostly comprises carbon or oxygen or a combination of both. Usually, the star does not have enough energy left in its core to fuse elements heavier than oxygen.

2.3 It may take millions of years for a star, depending on its initial mass, to become a red giant star from its main sequence phase. Presently our Sun is at its main sequence phase. It will take more than a billion years to fuse all the hydrogen atoms at its core and ultimately the Sun will become a red giant star. Its volume will become so huge that its outer shell may swallow many of its inner planets including this Earth.

2.4 As there is no more fusion of nuclear fuel, the giant star will be cooled and the volume of its core will be reduced. The hot gas from its outer shell will be blown away, by the influence of cosmic wind or magnetic field, forming a planetary nebula. The core will continue to collapse as it is subjected to inwardly acting gravitational pressure only. The star will become a dead or cold compact star, either a white dwarf star or a neutron star or a black hole, depending on its mass.

2.5 After the lapse of a considerable time the star will not be a voluminous giant star anymore. As its radius shrinks drastically, its gravitational pressure increases enormously. This huge gravitational pressure begins to compact the core of the star. The core becomes heavily dense. The gravity then pressurizes the electrons of the core to occupy and fill the higher energy levels. This process degenerate the electrons. Normally electrons occupy the orbital of an atom as per their energy states, lowest level being the Fermi energy level which is occupied by the low energy state electrons initially. After occupying the lowest energy level, electrons start filling the higher energy states obeying the Pauli's Exclusion Principle. This principle stipulates that no two identical fermions (+/- ½ spin subatomic particles like electron, proton, neutron, etc.) can occupy the same quantum state at the same time. Fermions have four quantum numbers; principal quantum number, angular momentum (azimuthal) quantum number, magnetic quantum number and spin quantum number. So, as per the Pauli's exclusion principle two electrons cannot have the same set of four quantum numbers. At normal condition electrons of different combinations of quantum numbers only occupy the same energy state in the orbital of the atoms. But the extraordinary high pressure exerted by the gravity of the star compels the identical electrons (having the same combination of quantum numbers) in the core of a star to occupy the same energy states violating the Pauli's exclusion principle. This causes degeneration of electrons at the star's core. Degenerate state is a type of quantum state of matter. These degenerate electrons exert a quantum mechanical pressure which is called electron degeneracy pressure. This newly created electron degeneracy pressure acts outwardly, i.e., against the gravitational pressure, restricts the core collapse of the star and finally balances the gravitational pressure of the star.

2.6 Electron degeneracy pressure, as explained above, is generated at the subatomic level inside the core of the star. The degeneracy pressure increases as the gravitational pressure increases with reduction in the core volume of the star. Rate of increase of the degeneracy pressure is higher than the rate of increase of the gravitational pressure and finally at a particular reduced radius of the core, both the pressures equalize. This stops the core collapse and a balanced cold (dead) star called 'White Dwarf' is formed. At the end of its red giant phase, our Sun will become a white dwarf only. It cannot become a neutron star or a black hole. While the mass of a white dwarf is around the mass of the Sun (maximum is 1.44 MΘ) its radius is comparable to the radius of Earth.

2.7 More massive stars, mass higher than the Chandrasekhar limit, after their main sequence phase become red supergiant stars. Unlike a low mass star, a high mass star is capable of fusing elements heavier than carbon and oxygen. Broadly, the fusion process of such massive stars continues starting from hydrogen, then helium, and then sequentially carbon, neon, oxygen, silicon till iron is produced. All these fusion processes up to the fusion of silicon are exothermic and so continue to produce energy needed for the thermonuclear combustion of the successive heavier elements. But iron is not fused as fusion of iron is an endothermic reaction that absorbs heat (energy). As no energy is produced the thermonuclear fusion of the star stops and its iron core starts cooling down. The main sequence phase of the high mass star ends when iron is produced in the core.

2.8 Like the formation process of red giant, the iron core of the star along with its gaseous outer shell contracts as it cools. The remaining hydrogen atoms in its outer shell get heated up and then start fusing. The outer shell expands in a huge proportion as it happens in case of a red giant star. Thermonuclear fusion of the outer shell continues till all the residual elements (fuels) are exhausted. A huge red supergiant star is formed. This process continues for millions of years depending on the mass of the star. However, the lifespan of a supergiant star is much smaller than that of a giant star.

2.9 While it cools, the volume of the supergiant star is reduced. At this stage only gravity is acting on the star and so the volume of the super giant gets further squeezed. As the red supergiant star gets incredibly compacted, the star is heated up enormously. Presently it is believed that heated materials of the outer shell strike its dense core and rebound at a great speed. This causes formation of powerful shock waves that set off the type-II supernova explosion of the red supergiant star. The explosion leaves a compact core, either a neutron star or a black hole, as a supernova remnant.

2.10 The existing theory on the supernova explosion outlined above does not take into account the role of degeneracy pressure. The shock waves are not powerful enough to trigger supernova explosions of the ultra dense core of the star which involve a tremendous amount of energy. It may, however, cause partial blow up of the outer shell and in that process the star loses a considerable chunk of its mass. It can be cited here that in a recent study by a group of researchers of the Kavli Institute for Astrophysics and Space Research of the MIT, the supergiant star M31-2014-DS1, which had a mass 20 times greater than the sun and located 2.5 million light-years away in the neighboring Andromeda galaxy, it was reported that though initially the mass of the M31 supergiant star was 20 times the solar mass, the star attained a terminal mass of 6.7

times the solar mass. Astronomers were expecting a supernova explosion of the star. But no evidence could be obtained for a supernova. Ultimately the star faded away as it was converted to a black hole.

2.11 In case of low mass star we've seen how the electron degeneracy pressure is developed at the subatomic level inside the core of the star. For stars having mass higher than the Chandrasekhar limit, i.e., more than 1,44 MΘ, even the electron degeneracy pressure is unable to restrict the core collapse of the star. The core collapse of the supergiant star continues; its volume becomes much smaller than the volume of a white dwarf. At that stage under the incredibly high gravitational pressure the electrons start combining with the protons to form electrically neutral neutrons. This process is called 'electron capture' and due to the enormous gravitational pressure the core of the star ultimately becomes a sphere constituted by neutrons only. During electron capture neutrinos are also produced. Neutrinos escape the star and travel in the space unimpeded by any cosmic objects.

$$e^- + p^+ = n + \upsilon_e$$

2.12 Reduction in the radius of the star causes further increase in the gravitational pressure which is more than million times higher than the electron degeneracy pressure. This amazingly high pressure degenerates the neutrons. Like degenerate electrons, the neutrons having identical quantum numbers are forced by gravity to occupy the same energy states at the same time violating Pauli's exclusion principle and become degenerate neutrons. That creates a new type of quantum pressure called neutron degeneracy pressure inside the star's core. The neutron degeneracy pressure is analogous to the electron degeneracy pressure, it acts against the gravity and it starts countering the gravitational pressure of the star.

2.13 After thousands to millions of years, volume of the erstwhile red supergiant gets drastically reduced under the influence of the huge gravitational pressure. The radius of the star reduces down to in the range of tens of kilometers only. The rate of increase in gravitational pressure will be restricted by the newly generated neutron degeneracy pressure which also increases with the increase in gravity of the collapsing star. At a particular radius the neutron degeneracy pressure built up in the core of the star will become equal to its gravitational pressure and the star will become a stable Neutron Star. Neutron stars are immensely compact celestial objects. Though the mass of neutron stars is multiple of the solar mass, their radii are around tens of kilometers only. Calculations presented here below in tabular form show that the stars having mass more than the Chandrasekhar limit and maximum up to the Tolman, Oppenheimer and Volkoff (TOV) limit will become neutron stars.

2.14 According to the present concept the neutron stars are formed by the stars having maximum mass up to the TOV limit. Beyond the TOV limit the star will collapse to form a black hole. Present theory considers a black hole as warped space-time that does not even have any physical surface. However, the calculations taking into account the degeneracy pressure reveal that (stellar) black holes are rather similar to the neutron stars. Basic constituents of both neutron stars and black holes are neutrons and their densities are comparable. Basic difference is that in between the mass range of the Chandrasekhar limit and the TOV limit only neutron stars are formed whereas the massive stars at and beyond the TOV limit will become black holes first and then upon further core collapse the black holes will become neutron stars also. A collapsing star becomes a black hole only when its escape velocity reaches or exceeds the velocity of light. Neutron degeneracy pressure plays a crucial role in formation of these ultra dense cosmic objects by restricting and finally nullifying the increasing gravitational pressure of neutron stars as well as black holes.

2.15 The calculated value of the TOV limit is around 2.928 MΘ. The calculations clearly show that at the TOV limit the star is a neutron star and at the same time it is also a stellar black hole. This concurrent formation of a neutron star and a black hole happens as at the TOV limit not only the gravitational pressure equals the neutron degeneracy pressure of the star, its escape velocity also attains the speed of light. As even light and any other electromagnetic radiation cannot escape the event horizon of the star, it becomes a black hole. Therefore, at the TOV limit, the star is a neutron star as well as a black hole. This is a new idea formulated based on the concepts of degeneracy pressure which is conspicuously absent in the existing theories on formation of a neutron star or a black hole.

2.16 For the stars having masses beyond the TOV limit that is of mass nearly 3 solar-mass and above a star will become a black hole first as its escape velocity attains the velocity of light. But the newly formed black hole is not pressure balanced and therefore will continue to collapse as the gravitational pressure is still more than its neutron degeneracy pressure (NDP). It will cease to collapse only when the NDP becomes equal to its gravitational pressure at a reduced radius. At that particular radius it is also a pressure balanced two-in-one star, a black hole and a neutron star, concurrently. The escape velocity of a black hole having mass above the TOV limit is more than the velocity of light. The particular radius at which the escape velocity of a cold star attains the speed of light is its 'Schwarzschild radius'.

2.17 There is, however, an upper limit of mass up to which the stars with such dual characteristics are formed. That limit is 7.14 MΘ. Black holes of mass up to 7.14 MΘ will become neutron stars and will remain balanced. But, the calculations show, a black hole of mass 7.15 MΘ and above will explode as supernova before becoming stable neutron stars. It is not that the neutron degeneracy pressure is failed to counter the gravitational pressure, it is because at that reduced radius the gravitational energy of a black hole attains such a great value that it equals (or exceeds) to the quantum chromo-dynamic (QCD) binding energy of the neutrons at the core of the black holes and triggers its supernova explosion before the black holes become pressure-balanced neutron stars. QCD binding energy is constant for a particular star and the steadily increasing gravitational energy blows up the imploding black hole when at reduced radius the gravitational energy of the black hole equals the QCD binding energy. The black holes cum neutron stars are formed up to the mass of 7.14 MΘ, and those stars cannot explode as supernovae as their gravitational energy cannot attain their respective QCD binding energy.

2.18 The Quantum Chromo-dynamic binding energy of the neutrons is the energy of the field of 'strong force' which binds two elementary particles 'quarks' into neutrons (hadrons), mediated by the gluons (bosons). Quarks are the elementary constituents of the subatomic particle neutrons (and also protons). When gravitational energy of a core collapsing black hole reaches a value equal to or more than the QCD binding energy of neutrons, the neutrons which are the only constituent of the core of the star, disintegrate into quarks and the black hole explodes as core collapse (type II) supernova. QCD binding energy of a neutron is around 927.7 MeV per nucleon (approx.). [2]

2.19 After the supernova explosion an enormous amount of energy is released into space that propagates as gravitational wave. In most of the cases, a neutron star or a black hole is created as the supernova remnant. Post supernovae huge amounts of gas and dust also get scattered in the interstellar space creating a nebula. Nebula is the birthplace of star(s).

3. Theoretical Presentation of Degeneracy Pressure [1], [3]

3.1 It is clear from the preceding paragraphs that degeneracy pressures play vital roles behind formation of white dwarfs, neutron stars and even black holes. The formula for theoretical computation of the degeneracy pressure has been derived from Schrodinger's wave equation which is based on quantum mechanics and its application in 'infinite square well' or 'particle in a box' theory. This helps to ascertain the Fermi energy as well as the total energy inside the core of a cold star. Max Plank proposed that light with frequency υ is emitted in quantized lumps of energy,

$$\mathbf{E = h\,\upsilon = (h/2\pi)\ (2\pi\upsilon) = \hbar\,\omega},$$

h = Planck constant = 6.63*10^-34 J.s
Ђ = reduced Plank constant = 1.055*10^-34 J.s
ω = angular frequency (rad. s^-1)
Einstein's equation for energy, $\mathbf{E = m.c^2}$;
momentum $\mathbf{p = m\,c}$, then $\mathbf{E = p\,c}$ …………(1)

For a light wave, $\boldsymbol{\omega = c\,k}$ where $\boldsymbol{k}$ is the wavenumber (1/λ, λ is wavelength)
Now, $\mathbf{E = \hbar\,\omega = \hbar\,(c\,\mathit{k}) \Longrightarrow p\,c = \hbar\,(c\,\mathit{k}) \Longrightarrow p = \hbar\,\mathit{k}}$ ……… .. ...(2)

This relates the momentum p of a photon to the wavenumber k of the wave it is associated with.
The time-independent Schrodinger energy equation is the basis of calculating the degeneracy pressure which is a quantum mechanical pressure that develops inside the core of a cold (dead) star due to its own gravity.

Classical non-relativistic expression for the energy of a particle is

$$\mathbf{E = K + V = (½)\,m\,v^2} + V(x) = \mathbf{(m^2\,v^2)/(2*m)} + V(x) = \mathbf{p^2/(2m)} + V(x) \quad ..(3)$$

$$\text{Now, } E = \hbar\,\omega = (\hbar^2\,k^2)/(2m) + V(x) \quad …………….(4)$$

A wave with frequency ω and wavenumber k can be expressed as
$\boldsymbol{\varphi(x,t) = Ae^\wedge i(kx - \omega t)}$, in 3-D , $\varphi(r,t) = Ae^\wedge i(kr - \omega t)$

$$\mathbf{\delta\varphi/\delta t = -\mathit{i}\omega\varphi \Longrightarrow \omega\varphi = \mathit{i}\,\delta\varphi/\delta t} \quad ……………….(5)$$

$$\mathbf{\delta^2\varphi/\delta x^2 = -\mathit{k}^2\,\varphi \Longrightarrow \mathit{k}^2\varphi = -\,\delta^2\varphi/\delta x^2} \quad …………… (6)$$

Multiplying energy equation (4) by φ,
$\hbar\,(\omega\,\varphi) = (\hbar^2/2.m)\,(k^2\,\varphi) + V(x)\,\varphi \Longrightarrow \mathbf{\mathit{i}\,\hbar\,\delta\varphi/\delta t = -\,(\hbar^2/2\,m)\,\delta^2\varphi/\delta x^2 + V\varphi}$ ..(7)

This is time-dependent Schrodinger equation.

If x and t are put back, the equation takes the form

$$\mathbf{\mathit{i}\,\hbar\,\delta\varphi(x,t)/\delta t = (-\hbar^2/2m)\,\delta^2\varphi(x,t)/\delta x^2 + V(x)\,\varphi(x,t)} \quad …(8)$$

In 3-D, the x dependence turn into dependence on all three co-ordinates (x, y, z) and the δ²φ/δx² term becomes ∇²φ (the sum of the second derivatives).

In quantum mechanics the whole complex wavefunction is relevant. The theory is structured in such a way that anything we want to measure (position, momentum, energy, etc.) will always turn out to be a real quantity.

In exponential wave form, $\boldsymbol{\varphi(x,t) = e^{-i\omega t} f(x)}$

Plugging this into equation (8) and cancelling $e^{-i\omega t}$ yeilds

$$\hbar\omega f(x) = (-\hbar^2/2m)\,\delta^2 f(x)/\delta x^2 + V(x) f(x) \quad …………(9)$$

$$\mathbf{E\varphi(x) = (-\hbar^2/2m)\,\delta^2\varphi(x)/\delta x^2 + V(x)\,\varphi(x)} \quad ……(10)$$

This is time-independent Schrodinger equation.

To solve for the φ(x) in the time-independent Schrodinger equation, we need to know the potential energy V(x). There are different applications of the Schrödinger equation, like:

1. Constant potential
2. Infinite square well
3. Finite square well

As we intend to calculate the degeneracy pressure inside a cold star, we'll consider the most analogous example. We'll follow the theory on 'infinite square well' which is a special case of 'constant potential'. This particular concept is also called a 'particle in a box' because the particles are positioned inside a confined region; in our case, the electrons and neutrons in the core of a cold star and have zero probability of leaving the star, just as a particle trapped inside a box.

Mathematically, a particle has zero chance of being found outside the region $0 \leq x \leq L$.

So, the potential energy of 'infinite square well' i.e., inside the box, in our case, inside the star,

$$\mathbf{V(x) = 0\ (0 \leq x \leq L\ ).}$$

The equation for constant potential is the basis of 'infinite square well' theory also and therefore, we may consider V(x) = 0 in a given region. Plugging $\varphi(x) = A\, e^{ikx}$ into the time-independent Schrodinger equation (eq. 9), we get

$$\boldsymbol{E} = \frac{\hbar^2 \mathbf{k^2}}{\mathbf{2}\boldsymbol{m}} + \mathbf{V0} \quad ==> \boldsymbol{k} = \frac{\sqrt{\mathbf{2m\,(E-V0)}}}{\hbar} \text{ …….(11)}$$

k will be real when E > V0, so we have solution,

$$\boldsymbol{\varphi(x) = A\, e^{kx} + B\, e^{-kx}} \text{ ……………(12)}$$

The full wave function (including the time dependence) for a particle with a specific value of E is given by

$$\boldsymbol{\varphi(x,t) = e^{-iwt}\ \varphi(x) = e^{\left(\frac{-iEt}{\hbar}\right)}\ \varphi(x)} \text{ …… (13)}$$

So, as inside a cold star E of a particle (i.e., electron or neutron) is always > 0, and V(x) = 0, infinite square well theory stipulates,

$$\varphi(x) = A \cos kx + B \sin kx, \text{ where } E = \frac{\hbar^2 k^2}{2m} \quad ==> \boldsymbol{k} = \frac{\sqrt{\mathbf{2\,m\,E}}}{\hbar} \text{ …………………....(14)}$$

The coefficients A and B may be complex and φ must be continuous at the boundaries at x=0 and x=L.

Since φ(x) = 0, continuity at x = 0 gives A cos(0) + B sin(0) = 0 ==> A = 0.

Continuity at x = L gives B sin(kL) =0 ==> kL = nπ , where n is an integer. So, $k = \frac{\mathbf{n\,\pi}}{\mathbf{L}}$ and the solution for φ(x) is, $\boldsymbol{\varphi(x) = B \sin(\frac{n\,\pi\,x}{L})}$. The full solution is (including time dependence)

$$\Phi(x,t) = B\, e^{-\frac{iEt}{\hbar}} \text{ Sin } (n\pi x/L)$$

$$\text{Where, Energy, } \mathbf{E} = \frac{\hbar^2\,\mathbf{k^2}}{\mathbf{2}\,\boldsymbol{m}} = \frac{\hbar^2 \boldsymbol{\pi}^2 \mathbf{n}^2}{\mathbf{2\,m\,L^2}} \text{ ……………… (15)}$$

The energy equation (15) as derived from the Schrodinger equation and its application in 'infinite square well' (particle in a box) theory, can be used to calculate the Fermi energy of the fermions (electrons / neutrons) of a cold star when all its lowest energy levels are filled. We consider the Fermi energy in 'n' space as

$$\text{EF} = \frac{\pi^2 \hbar^2}{2m\,L}\,(n^2x + n^2y + n^2z) \text{ ……………………(16)}$$

$$\text{Or, } \mathbf{EF} = \frac{\boldsymbol{\pi}^2 \hbar^2}{\mathbf{2}\boldsymbol{m}\,\boldsymbol{L}}\ \mathbf{r^2 n} \text{ ……………………………..(17)}$$

where, **rn** is the radius in n-space and nx, ny & nz are the principal quantum numbers in 3-D.

Assuming cold stars are in uniform spherical shape, total energy can be obtained by integrating the Fermi energy over the volume of one eighth of the sphere

$$\mathbf{Etotal} = \mathbf{2\ (⅛\ )} \int_{\mathbf{0}}^{\boldsymbol{rn}} \frac{\pi^2 \hbar^2}{2mL^2}\ r^2 n\ (4\pi r^2 n)\ \text{drn……… (18)}$$

In the eq.(18) above, since nx, ny & nz are positive integers, only positive values have been taken which is one eighth (1/8) of the sphere and 2 has been considered for 2-spin states for the fermions.

$$\text{Or, } \mathbf{E\ total = (2 * (1/8) *} \frac{\pi^2 \hbar^2}{2mL^2} \mathbf{* 4 * \pi) * (1/5) * r^5 n} = \frac{\pi^3 \hbar^2}{10mL^2}\ \mathbf{r^5 n} \text{ ……….….(19)}$$

Now, the total number of states (N) available to the fermions within the radius of the Fermi energy (sphere of radius rn),

$$\mathbf{N = 2 * (1/8) * (4/3 * \pi *} \boldsymbol{r^3 n}\mathbf{)} \text{ ………………(20)}$$

$$\text{N} = (1/3) * \pi * \boldsymbol{r^3 n}\ ; \quad \textbf{or, rn} = \left(\frac{\boldsymbol{3N}}{\boldsymbol{\pi}}\right)^{\frac{1}{3}} \text{ ....(21)}$$

Putting the value of rn in eq.18 & eq.16, we get

$$\mathbf{E\ total} = \frac{\pi^3 \hbar^2}{10mL^2} \left(\frac{\boldsymbol{3N}}{\boldsymbol{\pi}}\right)^{\frac{1}{3}} \text{ …………………… (22)}$$

$$\&\quad \mathbf{EF} = \frac{\boldsymbol{\pi}^2 \hbar^2}{\mathbf{2}\boldsymbol{m}} \left(\frac{\boldsymbol{3N}}{\boldsymbol{\pi L^3}}\right)^{\frac{2}{3}} \text{ ……………………...(23)}$$

Replacing L³ from the square well theory to V for a star (sphere), we get,

$$\text{Fermi energy, } \mathbf{EF} = \frac{\boldsymbol{\pi}^2 \hbar^2}{\mathbf{2}\boldsymbol{m}} \left(\frac{\boldsymbol{3N}}{\boldsymbol{\pi V}}\right)^{\frac{2}{3}} \text{ ………….. (24)}$$

$$\text{And, Total energy, } \mathbf{E\ total} = \frac{\pi^3 \hbar^2}{10m} \left(\frac{\boldsymbol{3N}}{\boldsymbol{\pi V}}\right)^{\frac{1}{3}} \text{ …….(25)}$$

Now, we know, E = F. ds = P .A.ds = P. dV

So, assuming constant entropy (S) and constant particle number (N),

We get, P total = -(δE total/δV) S, N

$$\text{or, } \mathbf{P\ total} = \frac{\boldsymbol{\pi}^3 \hbar^2}{\mathbf{15}\boldsymbol{m}} \cdot \left(\frac{\boldsymbol{3N}}{\boldsymbol{\pi V}}\right)^{\frac{5}{3}} \text{ ……………..…(26)}$$

Equation (26) is the general form of the degeneracy pressure. If we put mass of electron as me and number of nucleons as Ne, we get the electron degeneracy pressure for white dwarf stars.,

$$\textbf{Electron degeneracy pressure} = \frac{\boldsymbol{\pi}^3 \hbar^2}{\mathbf{15}\boldsymbol{me}} \cdot \left(\frac{\boldsymbol{3Ne}}{\boldsymbol{\pi V}}\right)^{\frac{5}{3}} \text{ ……(27)}$$

Similarly, putting the mass of neutron, Mn, and the number of nucleons (neutron for a neutron star)), Nn, in the equation (26), we get

$$\textbf{Neutron degeneracy pressure} = \frac{\boldsymbol{\pi}^3 \hbar^2}{\mathbf{15}\boldsymbol{Mn}} \cdot \left(\frac{\boldsymbol{3Nn}}{\boldsymbol{\pi V}}\right)^{\frac{5}{3}} \text{ …….(28)}$$

Paper ID: SR251016190731 DOI: https://dx.doi.org/10.21275/SR251016190731

3.2 Derivation of gravitational pressure

Degeneracy pressure is created inside the core of a cold star by the effect of gravitational pressure that is exerted on a star due to its own mass. To derive the gravitational pressure we should calculate the gravitational energy of the star. This can be done by considering two masses in a star. First we take one strip of unit length in the shape of a spherical shell in the star with mass, ms= $4\pi r^2$ ρ where ρ is the density of the shell. This shell has a gravitational attraction to the entire mass of the star M = (4/3) r^3 ρ . These two masses can be plugged into the formula for the gravitational potential energy and assuming constant ρ we integrate over every shell in the star's radius. We get,

$$Eg = -G \int_0^R \frac{ms\ M}{r}\ dr \quad \text{……………………………(29)}$$

$$= -G \int_0^R \left(\frac{(4\pi r^2 \rho)\left(\frac{4}{3}\pi r^3 \rho\right)}{r}\right) dr$$

$$= -\frac{16 G \pi^2 \rho^2 R^5}{15}$$

$$\text{or, } Eg = -(3/5)\ GM^2/R \quad \text{……………………..…(30)}$$

where, M = $\left(\frac{4}{3}\pi R^3 \rho\right)$ is the mass of the star.

Replacing the R by V in the eq. (29),

$$\mathbf{Eg = -(3/5)\ G}\ \boldsymbol{M^2} \left(\frac{4\pi}{3V}\right)^{\frac{1}{3}} \quad \text{……………………..….(31)}$$

Then, differentiating w.r.t. volume V,

$$\mathbf{Pg = -(1/5)\ G}\ \boldsymbol{M^2} \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \boldsymbol{V^{-\frac{4}{3}}} \quad \text{……………...…(32)}$$

In the above equations following constants & symbols are used

G = Universal gravitational constant (6.67E-11 m^3.kg^-1.s^-2)

M = Mass of the star

V = Volume of the star

ħ = reduced Planck constant (1.055E-34 J.s)

Me = Mass of electrons

Ne = Number of electrons

Mn = Mass of neutron (1.675E-27 kg)

Nn = Number of neutrons

4. Calculations and Discussions

4.1 White dwarfs are formed when the electron degeneracy pressure of a collapsing star becomes equal to its gravitational pressure. Mass of a white dwarf is similar to the solar mass whereas its radius is comparable to the radius of the Earth. Escape velocity of a white dwarf is much less than the velocity of light. Though a white dwarf is a stable cold star, eventually, after considerable time, it may accumulate mass from a companion star in accretion process and become heavier than the Chandrasekhar limit and explode as type-I supernova. At the end of its life, our Sun will become a white dwarf only; it can neither become a neutron star nor a black hole.

Table 1_

f	Mass (kg)	Radius (m)	Gravitational Pr. (kg.m^-2)	E.D.P (kg.m^-2)	Escape Velocity (m.s^-1)	Remarks
0.8	1.591E+30	7716000	**2.274E+21**	**2.274E+21**	5244984	**White Dwarf**
1.0	1.989E+30	7163600	**4.782E+21**	**4.782E+21**	6085968	**White Dwarf**
1.44	2.864+30	6347500	**1.609E+22**	**1.608E+22**	7758454	**White Dwarf**

White dwarfs are formed when the gravitational pressure and the electron degeneracy pressure of the star become equal. Escape velocities of the white dwarfs are less than the velocity of light.

4.2 Formation mechanism of both neutron star and black hole is same. Both are balls (spheres) comprising neutrons only. Following are the main differences.

4.3 While collapsing its core a cold star of mass greater than the Chandrasekhar limit becomes a neutron star when its neutron degeneracy pressure becomes equal to its gravitational pressure. With the decrease in radius of a star, both pressures increase, however, NDP increases at a higher rate and finally catches up with the gravitational pressure.

Table 2

f	Mass (kg)	Radius (m)	Gr. Pr. (kg.m^-2)	N. D.P (kg.m^-2)	EV (m..s^-1)	Gr. Energy (J)	Remarks
2.5	4.973E+30	**9100**	**1.148E+34**	**1.148E+34**	269988095	1.295E+47	**Neutron Star**
2.92	5.808E+30	**8648.4**	**1.919E+34**	**1.919E+34**	299308360	1.858E+47	**Neutron Star**

Neutron stars are formed when the gravitational pressure and the neutron degeneracy pressure of the star become equal. Escape velocities of the neutron stars are also less than the velocity of light.

4.4 For smaller mass stars (up to the TOV limit) the escape velocity remains less than the velocity of light when they become neutron stars. These stars cannot become black holes.

4.5 At the TOV limit (2.928 times the solar mass) a cold star becomes a neutron star and also a black hole concurrently.

Table 3

f	Mass (kg)	Radius (m)	Gr. Pr. (kg.m^-2)	N. D.P (kg.m^-2)	EV (m.s^-1)	Gr. Energy (J)	Remarks
2.9287	5.825E+30	8634. 218	**1.944E+34**	**1.944E+34**	**3E+8**	1.872E+47	**NS & BH TOV limit**

At the TOV Limit, a star is a neutron star and simultaneously it is a black hole also, as its escape velocity is equal to the velocity of light.

Its radius 8634.218 m is the Schwarzschild radius of the star.

4.6 Beyond the TOV mass limit, a star becomes a black hole first and then at reduced radius the black hole becomes a neutron star also.

Table 4

f	Radius (m)	Gr. Pr. (kg.m^-2)	N. D.P (kg.m^-2)	EV (m.s^-1)	Gr. Energy (J)	QCD Binding Energy (J)	Remarks
2.94	8667.53	1.929E+34	1.92E+34	**3E+8**	1.879E+47	5.189E+47	**Black Hole**
2.94	8625	**1.967E+34**	**1.967E+34**	**300738770**	1.889+47	5.189E+47	**N S also B H**
3.0	8844.42	1.852E+34	1.795E+34	**3E+8**	1.918E+47	5.294E+47	**Black Hole**
3.0	8568	**2.103E+34**	**2.103E+34**	**304800872**	1.98E+47	5.294E+47	**N S also B H**

Beyond the TOV Limit, a star will become a black hole first and then it will turn into a neutron star.

4.7 But there is an upper mass limit for a black hole that converts to a neutron star. Theoretical calculations suggest the upper limit as 7.14 times the solar mass. At the 7.15 MΘ (and beyond) a black hole will explode as a core-collapse (Type-II) supernova before it stabilises as a neutron star. At that condition the gravitational energy of the black hole will equal (or marginally exceed) its quantum chromo-dynamic (QCD) energy.

Table 5

f	M (kg)	R (m)	E.V (m.s^-1)	Gr. B. Energy (Joules)	QCD B Energy (Joules)	Remarks
7.14	1.42E+31	21049.72	**300000000**	3.834E+47	1.26E+48	**Black Hole**
7.14	1.42E+31	6417	543348400	1.258E+48	1.26E+48	**NS & BH***
7.15	1.422E+31	21079.201	**300000000**	3.84E+47	1.262E+48	**Black Hole**
7.15	1.422E+31	6414.5	543847427	**1.262E+48**	**1.262E+48**	**Supernova**
10	1.989E+31	29481.4	**300000000**	5.37E+47	1.765E+48	**Black Hole**
10	1.989E+31	8971.33	543856797	**1.765E+48**	**1.765E+48**	**Supernova**

**** At 7.14 MΘ both the gravitational pressure and the neutron degeneracy pressures are same, 3.786E+35 kg.m^-2. So, it is a stable neutron star as well as a black hole and its gravitational energy is almost nearing its QCD binding energy. But there will be no supernova explosion. A mass of 7.15 MΘ, is the least mass of a black hole that will be exploded as supernova.***

5. Conclusions

5.1 White dwarfs, neutron stars and black holes all are compact cold stars. While electron degeneracy pressure is responsible for formation of white dwarfs, neutron degeneracy pressure is active behind the creation of neutron stars and black holes. Degeneracy pressures are a sort of quantum pressure that counter the gravitational pressure of cold stars and prevent core collapse of the stars.

5.2 The dilemma on mass gap between the neutron stars and black holes has been adequately cleared by specifying the precise limits for masses of all the three types of compact stars.

5.3 In the existing model, the concept of degeneracy pressure is totally absent. The black hole has been mystified as a warped space-time. Black holes are enormously compact celestial objects, very similar to the neutron stars and not anything out of the universe.

5.4 Criterion defined for supernova explosion correlating the gravitational energy of the collapsing star with its quantum chromo-dynamic binding energy of neutrons is a new idea. Existing theory on the core-collapse supernova (Type-II) is not based on the concept of QCD binding energy.

References

[1] Drexel, https://www.physics.drexel.edu > John Timlin
[2] Wikipedia, https://en.wikipedia.org / wiki/quantum-chromo-dynamics – binding energy
[3] Introduction to quantum mechanics, David Morin, morin@physics.harvard.edu