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Understanding Compactness: The Harmony of Boundedness, Closedness, and Convergence in Real Analysis

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Abstract: ***One of the distinguishing properties of a bounded closed interval [a, b] is that every sequence in it has a subsequence converging to a limit in the interval. This need not happen with an unbounded interval such as [0, 1) or a bounded non closed interval such as (0,1]; the former contains the sequence {n}n≥1, which has no convergent subsequence, and the latter contains the sequence {1/n} n≥1, which has no subsequence converging to a limit belonging to the interval. In fact, it is true of any bounded closed subset of R that any sequence in it has a subsequence converging to a limit belonging to the subset. To see why, we first note that any sequence in a bounded subset must, by the Bolzano-Weierstrass theorem have a convergent subsequence with limit in R; this limit must then be in the closed subset by the definition of a closed subset. A compact set in R is a set E satisfying the property that if U is a collection of open sets in R whose union contains E, then there is a finite sub collection V of U whose union contains E. Recall that such a collection is called an open cover and V is called a finite sub cover of U for E. In terms of this, a set E in R is compact if every open cover of the set E has a finite sub cover for E. Because of these criteria, compact sets are also viewed as a generalization of finite sets.***

Keywords: Open cover, compactness, bounded interval, Bolzano-Weierstrass theorem, convergent subsequence and closed subset

1. Introduction and Concept

A compact set in a metric space is a set that resembles a closed and bounded subset of R, it is "small" in a certain sense and "contains" all its adherent points. One of the main reasons for studying compact sets is that they are in some ways very similar to finite sets. In other words, there are many results which are easy to show for finite sets, the formulations as well as the proofs of which carry over with minimal changes to compact sets. It is often said that "compactness is the next best thing to finiteness".

Definition: A compact set in R is a set E satisfying the property that if U is a collection of open sets in R whose union contains E, then there is a finite sub collection V of U whose union contains E. Recall that such a collection is called an open cover and V is called a finite sub cover of U for E. In terms of this, a set E in R is compact if every open cover of the set E has a finite sub cover for E. Because of these criteria, compact sets are also viewed as a generalization of finite sets. Here we try to extend this idea to metric spaces. For that we first introduce the concept of an indexed family of subsets of a set. A thorough knowledge of this will be needed to understand open covers in a metric space.

Example: Let (X, d) be a metric space. For any

$$r_o > 0, \{B(x, r_o) : x \in X)$$

is an open cover for X. If we fix an $x_o \in X$, then $\{B(x_o, r): r$ E $[0, \infty)\}$ is also an open cover for X.

Example: Consider the interval] 0,1 [⊂ R. The family

$$U = \{] 1/n, 1[: n \in N, n \geq 2\}$$

is an open cover for]0, 1 [.

Example: The family {] - n, n[: n ∈ N) is an open cover for R. Further, {(-2n, 2n): n ∈ N) is a sub cover.

Proposition: A finite subset of a metric space is compact.

Proof: Let (X, d) be a metric space and E be a non-empty finite subset of X. We denote the elements of E by $X_I, \ldots\ldots\ldots, X_N$.

Let U = $\{A_i\}_{i \notin I}$ be an open cover of E. That is, E ⊂ $\bigcup_{i \in I} A_i$. This implies that each element x_j, j = 1,………… N, belongs i∈l to ⊂ $\bigcup_{i \in I} A_i$. That means for each j, $x_j \in A_i$ for some i ∈I. Let us denote these A_i's by A_{i_k} k, = 1, . . . , m.

Let $U_o = \{A_{i_k} : 1 \leq k \leq m)$. Then U_0 is a finite sub cover of for E. This shows that every open cover of E admits a finite sub cover for E.

Hence E is compact.

Example: R^2 with the usual metric is not a compact space. In fact U = {B(0, n) : n ∈ N) is an open cover for R^2 which has no finite sub cover.

Theorem: Every compact set in a metric space is closed and bounded.

Proof: Let X be a metric space and E be a compact subset. To show that E is closed, it is enough to show that E^c is open.

Let $x_o \in E^c$. Now we apply Hausdorff property to each element y ∈ E.

Then we get that for each y ∈ E, there exists open sets U_y and V_y of the points x_o and y respectively such that

$$U_y \cap V_y = \varnothing.$$

Then the collection

$$V = \{V_y : y \in E\}$$

is an open cover for E. Since E is compact, V admits a finite subcover for E. Then there exists sets $V_{y1}, \ldots, V_{yn}$ such that

$$E \subset \bigcup_{i=1}^{n} V_{y_i}$$

let $V = \bigcup_{i=1}^{n} V_{y_i}$ and $U = \bigcap_{i=1}^{n} U_{y_i}$

where U_{y_i}'s are the neighborhoods of x_o corresponding to $V_{y1}, \ldots, V_{yn}$. Since U is a intersection of finite open sets, U is open and $x_o \in U$. Also

$$U \cap V = \varnothing \text{ (by the choice of } U_{y_i}\text{'s and } V_{y_i}\text{'s)}$$

which implies that

$$U \cap E = \varnothing.$$

Thus U is a neighborhood of xo which is fully contained in E^c. Hence E^c is open.

Now we have to show that E is bounded.

Fix any $x_o \in E$. Consider the open cover

$$U = \{B(x_o, n) : n \in N) \text{ of } E.$$

This admits a finite sub cover for E, say,

$$\{B(x_o, n_j) : 1 \le j \le p),$$

Since $B(x_o, m) \subset B(x_o, n)$ for $n \ge m$,

It follows that

$$E \subset B(x_o, M)$$

for $M = \max\{n_j : 1 \le j \le p)$.

Thus, E is bounded.

Remark: But the converse of the above theorem is not true. For example, let X be an infinite set with discrete metric space. Then every subset of X is closed and bounded, and we have seen in earlier example that only the finite subsets of X are compact. This shows that a closed and bounded set need not be compact in a general metric space.

Theorem: Closed subsets of compact sets in a metric space are compact.

Proof: Let X be a metric space and K is a compact set in X. Suppose F is a closed subset of K. Since K is compact it is closed in X. Since F is closed in K, it is closed in X. Let $\{V_o\}$ be an open cover of E. If F^c is adjoined to $\{V_o\}$, we obtain an open cover G of K. Since K is compact, there is a finite sub collection $\varnothing$ of G which covers K, and hence F. If F^c is a member of $\varnothing$, we may remove it from $\emptyset$ and still retain an open cover of F. We have, thus, shown that a finite sub collection of $\{V_o\}$ covers F. Therefore, F is compact. Hence, we get the result.

Theorem: If A and B are compact sets in a metric space X, then $A \cup B$ and $A \cap B$ are compact sets in X.

Proof: We shall first consider A U B. Let U be an open cover for $A \cup B$. Then U is an open cover for A as well as for B. Since A is compact, U admits a finite cover, say U_l. for A. Since B is compact, U admits a finite sub cover, say U_2, for B. Then the collection obtained by adjoining the sets in U_2 to U_l, becomes a finite sub cover for $A \cup B$. This shows that $A \cup B$ is compact.

Since A and B are compact sets, by Theorem state earlier, they are closed. So, $A \cap B$ is closed, and it is a subset of A. Since any closed subset of a compact set is compact, it follows that $A \cap B$ is compact. Hence, we get the result.

Theorem: If S is an infinite subset of a compact metric space X, then S has a limit point in X.

Proof: We prove this by a contradiction argument. Let, if possible, S has no limit point in X. Then given any point x in X, there exists U_x open in X such that $x \in U_x$ and U_x contains no points of S except possibly x itself. The family $\{U_x : x \in X)$ is an open cover for X and, since X is compact, has a finite subcover, say

$$\{U_{x_1}, U_{x_2}, \ldots\ldots\ldots\ldots\ldots U_{x_n}\}$$

So,

$$S = S \cap X \subseteq S \cap \left(\bigcup_{i=1}^{n} U_{x_i}\right) \subseteq \bigcup_{i=1}^{n} (S \cap U_{x_i}) \ldots. \quad (1)$$

Each of the sets U_{x_i} contains almost one point of S. So the R. H. S. of equation (1) has almost n point which in turns implies that S is finite set. This is contradiction. Therefore, S has limit point in X.

Theorem: Let X be a metric space. Then the following are equivalent.

1) X is compact.
2) Every sequence in X has a convergent subsequence.

Proof: We shall first show that $1 \Rightarrow 2$

Suppose X is compact. We have to show that every sequence in X has a convergent subsequence. On the contrary, assume that it is not so. Then there exists a sequence $\{x_n\}$ such that it has no subsequence which converges in X. This implies that each $x \in X$, there is some $r_x > 0$ and a positive integer n_x such that

$$x_n \notin B(x, r_x) \text{ for } n \ge n_x .$$

To see this let us assume that it is not true.

Then, $\exists\ x \in X$, $\forall\ r > 0$ and $j \in N$ $\exists\ m_{j,r} \ge j$ with $x_{m_{j,r}} \in B(x, r)$

Let us take $r_1 = 1$ and $j_1 = 1$, Then $\exists\ n_1 \ge j$ with $x_{n_1} \in B(x,1)$, Note $n_1 \ge 1$, Now we take $r_2 = ½$, $j_2 = n_1 + 1$. Note that $j_2 > n_1$. Then $\exists\ n_2 \ge j_2$ with $x_{n_2} \in B(x,1/2)$. Note that $n_2 \ge n_1$

Thus, we take $r_3 = 1/3$, $j_3 = n_2 + 1$. Note that $j_3 > n_2$. Then $\exists$

$n_3 \ge j_3$ with $x_{n_3} \in B(x,1/3)$. Note that $n_3 \ge n_2$ Proceeding like this we get an increasing sequence $\{n_i\}_{i=1}^{\infty}$ for which

$$x_{n_i} \in B(x, 1/i) \ \forall \ i \in \mathbb{N}. \ \{x_{n_i}\}_{i=1}^{\infty}$$

is subsequence of the sequence $\{x_n\}$ and $x_{n_i} \to x$ as $i \to \infty$. This is not possible by our assumption. Therefore, we got that for each x∈X, there is some $r_x > 0$ and a positive integer n, such that $x_n \notin B(x, r_x)$ for $n \geq n_x$.

Then $\{B(x, r_x): x \in X)$ is an open cover for X. Since X is compact, there exist $y_1, y_2, y_3, \ldots\ldots\ldots\ldots y_m \in X$ such that

$$X \subset \bigcup_{i=1}^{m} B(y_i, r_{x_j})$$

Let, $n_o = \max\,[n_{x_1}, n_{x_2}, n_{x_3} \ldots, n_{x_j}]\ B(y_i, r_{x_j})$

Then $x_{n_o} \notin \bigcup_{i=1}^{m} B(y_i, r_{x_j})$ so that $x_{n_o} \notin X$. which is impossible. Therefore, our assumption is wrong. Hence we get the claim.

Next, we shall show that (ii) ⇒ (i)

Suppose that every sequence in X has a convergent subsequence we have to show that X is compact. We shall prove this in three steps.

Step 1: We shall first claim that given any open cover U of X, there exists a number $\delta > 0$ such that for each subset B of X having diameter less than δ, there exists an element of U containing B. We shall refer to such a number δ as a Lebesgue number for U.

Let, if possible, there be no such δ for some open cover U. That means for any $\delta > 0$, there exists a subset, whose diameter is less than δ, and this subset does not lie inside any element of U. In particular, for each $n \in \mathbb{N}$, we can choose a set C_n, having diameter less than 1/n which is not contained in any element of U. Now for each n, choose an element $x_n \in C_n$. Then $\{x_n\}$ so obtained has a convergent subsequence $\{x_{n_j}\}$ converging to x, say, in view of the fact that (ii) holds. Now, x belongs to some element A of U, and because A is open, there is an $r > 0$ such that $B(x, r) \subset A$. Choose i large enough that satisfies

$$d(x_{n_j}, x) < r/2 \text{ and } 1/n_i < r/2$$

Because the diameter of C_{n_j} is less than $1/n_i$ we have

$$C_{n_j} \subset B(x_{n_j}, \tfrac{1}{n_j})$$

which in turn is contained in $B(x_{n_j}, \frac{r}{2})$. It follows then, that

$$C_{n_j} \subset B(x, r)$$

Therefore, $C_{n_j} \subset A$ this is not so. Hence, we get the claim.

This establishes the claim made in the beginning of Step 1.

Step 2: We now claim that for every $\epsilon > 0$, there exists a finite covering of X by ϵ - balls.

Let, if possible, there be an $\epsilon > 0$ for which such a covering does not exist. Now, we construct a sequence $\{x_n\}$ in X. Choose an element $x_l \in X$. Since $B(x_l, \epsilon)$ is not all of X (otherwise X could be covered by a single ϵ ball), ∃ an $x_2 \in X$ such that $x_2 \notin B(x_l, \epsilon)$. So $d(x_2, x_l) \geq \epsilon$. Then $B(x_l, r) \cup B(x_2, \epsilon)$ is not all of X. So there exists $x_3 \in X$ with

$$x_3 \notin B(x_l, \epsilon) \cup B(x_2, \epsilon).$$

Therefore,

$$d(x_2, x_l),\ d(x_3, x_2) \geq \epsilon.$$

We already have $d(x_l, x_2) \geq \epsilon$.

Proceeding similarly, we get that given $x_1, x_2, x_3, \ldots\ldots x_n \in X$ with $d(x_j, x_k) \geq \epsilon$. for $1 \leq j \neq k \leq n$, there exists $x_{n+1} \in X$ such that

$$x_{n+1} \notin \bigcup_{i=1}^{m} B(x_i, \epsilon)$$

Therefore, $d(x_j, x_{n-1}) \geq \epsilon$. for $1 \leq j \leq n$,

Thus by induction we get a sequence (x_n) for which

$$d(x_j, x_k) \geq \epsilon \text{ for } j \neq k.$$

Thus $\{x_n\}$ can have no convergent subsequence. This is a contradiction because (ii) holds. Hence we get the claim.

Step 3: Now we claim that X is compact.

Let U be an open covering of X. Then the covering U has a Lebesgue number δ as specified in Step 1. Corresponding to this δ, using Step 2 $\epsilon = \frac{\delta}{3}$ there exists using with $\epsilon = \frac{\delta}{3}$ a finite covering. F of X by balls of radius $\frac{\delta}{3}$. Then each of these balls has diameter at most $\frac{2\delta}{3}$. So, we can choose, for each of these balls, t an element of V containing it using the fact that $\frac{2\delta}{3} < \delta$ a Lebesnue number for V, as in Step 1. If U' denote this collection, then U' becomes a finite sub cover of U for X. Thus, we get that every open covering of X has a finite sub cover. Hence X is compact.

2. Conclusion

The property of compactness holds significant importance as it extends the concepts of boundedness and closedness, ensuring that continuous functions attain their maximum and minimum values within a compact domain. This characteristic facilitates the proof of theorems, aids in solving optimization problems and offers a method for managing infinite sets by transforming them into finite, more manageable subsets, as demonstrated by the existence of finite sub covers. The concept of compactness permits the reduction of infinite collections of open sets to finite ones, thereby simplifying numerous proofs in topology and analysis, making them considerably more "constructive".

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