Fractals and Fugues: How Self-Similarity Emerges in Musical Structure

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Abstract: ***Fractals refer to mathematical structures that exhibit self-similarity, i.e., the recursive repetition of identical structures, with complex patterns being created under simple parameters. This paper investigates the emergence of fractal patterns in the music of Johann Sebastian Bach, who was known for his repetitive modifica-tion of motifs, especially in his work The Art of Fugue. Although his compositions are not strictly fractal, they serve to highlight a unique relationship between math-ematics and music, exemplifying the use of recursive processes to create complexity in music.***

Keywords: Fractals, Self-similarity, Bach's music, Recursive patterns, Mathematics and Music, Pink Noise, Fugues, Power Series

1. Introduction

Before we can explore how fractals are related to fugues, we must first understand what they are. A fractal is any mathematical structure, be it a set, geometrical shape, or function, that exhibits self-similarity, has a non-integer Hausdorff dimension number (meaning it does not conform to any traditional integer dimension such as a 1D line or 2D plane), and cannot be simplified at any scale. In short, it is any structure that shows repetition across arbitrarily small scales.

As a common example, one can consider the cover of Pink Floyd's album *Um-magumma*, which features an image of the band members repeating itself in a mirror, getting smaller each time. Of course, this repetition on the album cover stops eventually, but in mathematical fractals, the recursion is infinite.

Possibly the most famous example of a proper fractal is the Mandelbrot set—a sub-set of the complex plane that exhibits a complex tendril-like pattern that repeats itself forever. Any Mandelbrot set can be defined as follows: M = c ∈C : sup$n$ |zn| < ∞, where zn+1 = z2 n+ c

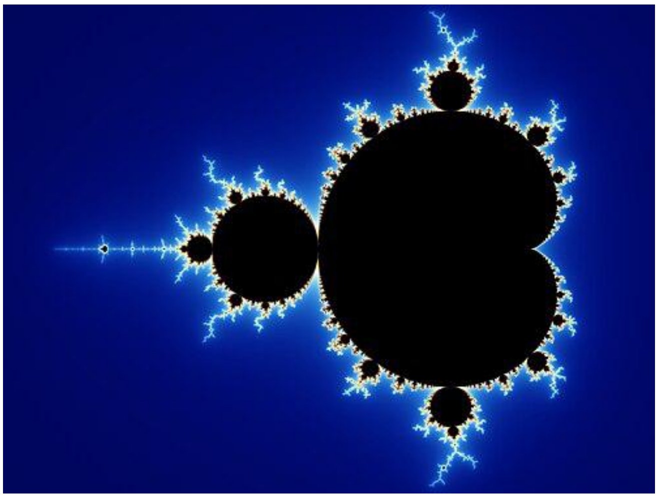

Figure 1: The Mandelbrot set generated by iterating zn+1 = z2 n+ c.

We can see in Figure 1 the basic shape of a Mandelbrot set. The boundary exhibits infinitely recursive self-similarity.

The fugue, on the other hand, is a musical structure used most prominently by the composer Johann Sebastian Bach in the Baroque era of Western classical music. It is categorized by a main subject melody, which is constantly reiterated and expanded upon in different contexts, whether by use of a different instrument, key change, or tempo change. The use of different independent voices to produce a complete sound is known as counterpoint, a technique Bach is considered a master of. A popular example of a fugue is Bach's *Little Fugue in G Minor* (BWV 578), which introduces a subject played on the right hand of the organ and then repeats it with the left hand, albeit in a different key.

Figure 2: Opening 12 bars of Bach's Little Fugue in G minor, adapted for recorders (BWV 578).

Figure 2 shows how the soprano line is mirrored in the alto but changed from G minor to D minor, shifting to the fifth degree. This trend is further continued by the tenor, which returns to G. In this way, fugues also show a "nesting" effect, repeating and expanding on the same subject using counterpoint.

We are now familiar with the basics of both fractals and fugues. Now we can dive into the main premise of this paper—does music, specifically the contrapuntal fugue, have traits similar to those of fractals? And if so, what does it reveal about the intertwined nature of these two contrasting subjects?

To understand the role of relatively new fractal math in Bach's 18th-century com-positions, we must review the existing research on the topic, its musical connection, and previous analysis of Bach's music in this context. The possibly fractal structure of Bach's music still remains a topic of debate, making it necessary to understand why this connection was made in the first place.

2. Literature Review

The fractal found a concrete start in mathematics with the French polymath Benoit Mandelbrot, who in 1975 coined the term to describe such infinitely recursive structures. The Hausdorff number, used to quantify non-integer dimensionality, and early geometric depictions of fractals such as the Koch snowflake were major innovations in the field.

The early history of fractals includes extensive research that had already developed a connection to musicology [9, 10]. The discovery of "pink noise" and its relation to fractals in time series was a key finding in this field.

Pink noise refers to sound that roughly follows an inverse power spectrum as follows: $S(f) \propto 1/f$

This follows from the general power law:

$$S(f) \propto 1/f\beta$$

Where β is a constant depending on the type of sound. Music usually follows a power law where $\beta \approx 1$, which is called "pink noise," a bridge between overly random white noise ($\beta \approx 0$) and overly calm brown noise ($\beta \approx 2$). This was a significant discovery, as the time series related to this $1/f$ relation also exhibits self-similar behavior, showing similar ups and downs no matter the time scale of measurement.

As the study of fractals advanced, the intersection of fractal math and classical music became more apparent. The father-and-son team of Kenneth J. Hsu and Andrew J. Hsu observed that the chord progressions and interval changes in the music of classical composers like Mozart implied self-similarity [3, 4], comparing them to the well-known hypothesis that the length of a coastline cannot be measured, as it also possesses a fractal self-replicating form, showing complexity regardless of scale.

In 2012, this idea of musical compositions obeying the $1/f$ law was pushed even further, with a large-scale study by D. J. Levitin, P. Chordia, and V. Menon [5]. This study analyzed a wide range of music from Bach to Joplin, ultimately reinforcing the fact that this "inverse frequency law" was obeyed by most music, and thus that music had fractal nature built into it.

Bach's music has been central in the discussion around fractal rule-forming musical structure, as his music possesses the most clear connection to the recursive repetition often observed in fractals. His works usually center around a small 4–5 note main theme, which is then expanded and often nested into itself, such as in the Little Fugue in G Minor (BWV 578). A PNAS news feature review by Stephen Ornes surveyed the already existing research on fractal geometry in music, with attention towards Bach specifically [8]. This review uncovered the scaling of motifs and demonstrated the existence of structural self-similarity. A clear case of this is seen in Bach's famous Cello Suites, where interval scaling is used [1].

However, it is important to note that Bach's music does not follow fractal geometry perfectly, nor do fractal patterns reveal themselves in all of his works. This fact is demonstrated by Henderson-Sellers, who revealed logical flaws in the argument that classical music possesses a fractal nature [2]. We can think of this comparison as closer to the repeating image in *Ummagumma* than to a Koch snowflake. The purpose of these studies was merely to analyze the emergence of fractal-like patterns in the field of music, and thus prove that these fields are related to each other on a fundamental level.

3. The Gap in Research

Although previous research has connected fractal geometry to musical structure, the focus has been on statistical self-similarity. This has been done by essentially reducing music to a simple power series that can then be compared to fractal-natured time series. The recursive development of motifs, especially those of Bach, has remained underexplored. This paper attempts to address this gap by analyzing individual motifs through the lens of fractal geometry, and thus goes beyond statistical analysis. It aims to demonstrate that fractal recursive self-similarity exists in the individual phrasings and motifs of Bach's music, and intends to highlight how this structure shapes the perception of his works.

a) Self-Similar Motifs

In music, the term "motif" refers to a short recurring idea. This can be a specific rhythm or a set of notes. In Western classical music, the motif usually acts as an anchor for the piece, providing a clear structure in longer pieces with higher musical complexity. Bach famously uses the B-A-C-H motif in his works, which acts as a musical signature in the German notation system, translating to $B\flat - A - C - B\natural$, as shown in Figure 3.

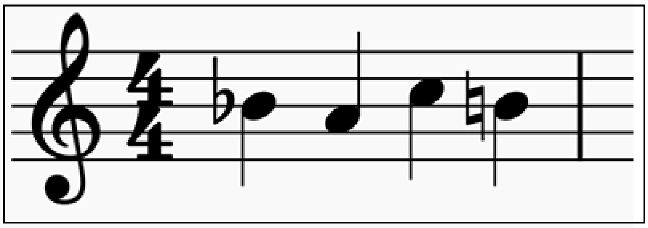

Figure 3: The BACH Motif

In fractal geometry, self-similarity refers to the repetition of a geometrical structure across arbitrarily large or small scales. Thus, the geometrical structure is preserved, but it may be distorted in some way, such as in size. Bach's fugues often have motifs that are stretched, shortened, or nested in a way that closely resembles the geometry of fractals. This correlation in the fugal structure and the geometry of fractals invites further investigation into the contrapuntal nature of these compositions.

In a fugue, the subject is often repeated entirely in each voice (*soprano →alto →tenor →bass*), often with modulation or interval alteration. Repetition is usually written in different pitch levels, most commonly in the relative minor (iv) or dominant (V), and the time values for each note may be augmented or diminished. This suggests an inherently fractal nature, as the entire composition often reduces to the repetition of a small number of motifs. Madden argues that this property of composition by altering repetition of a single theme makes Bach inherently fractal [6]. McDonough terms Bach's motifs as structurally fractal, claiming that they exhibit nesting [7].

This feature makes Bach's fugues efficient, as the vast complexity in his music is only generated from a minute, often 4–5 note phrasing. This correlation also provides a concrete basis for arguing that fugues are, in fact, fractal structures, as it shows that these fugues exhibit more than a mere statistical $S(f) = 1/f$ scaling.

b) Recursive Imitation

We have seen that the individual motifs follow to a great extent the basic rules that govern fractal geometry. We will now expand our view to the complete fugue structure, applying the same qualities of altered motif repetition and recursive imitation to the composition of a fugue in general. It is a well-known feature of the contrapuntal fugue that the individual melodic lines imitate each other. A large degree of nesting is seen here, as the soprano serves as a leading line that is imitated by the alto, tenor, and finally the bass.

This concept, pushed to its logical extreme, results in a fugue that is essentially a macrocosm of its subject. This shows similarity to the Mandelbrot condition of possess-ing a non-integer Hausdorff dimension. Although the Hausdorff dimension of a musical composition is naturally not a real value, we can interpret this as the composition ex-hibiting more complexity than is suggested by the individual motif.

An example of this motivic microcosm can be clearly seen in Bach's Passacaglia in C Minor (BWV 582). The piece is based on an 8-bar ostinato that persists throughout the composition. The recursive imitation of this theme serves to add further complexity, providing a deeper analysis of the motif. This piece demonstrates an exceptionally fractal nature, as the fugue itself is a reflection of the ostinato that forms it. The descending motif resembles the overall structure of the fugue, which further unifies this Passacaglia.

Figure 4: The ostinato of Bach's Passacaglia in C Minor (BWV 582)

c) Statistical Self-Similarity

This structural self-similarity has been largely based on external structural commonalities between Bach's fugues and fractal structures. This correlation is substantiated by statis-tical similarities between them, most clearly seen in the form of the series $S(f) = 1/f$.

Statistical self-similarity means that the system will retain a similar structure across different scales on average. Therefore, it allows us to reduce all music to a power law of the form:

$$S(f) \propto 1/f\beta$$

In this relation, β represents the degree of randomness exhibited by the sound, with a small value ($\beta \approx 0$) signifying complete randomness, observed in white noise, and higher values ($\beta \approx 2$) suggesting stable noise, with special reference to brown noise.

Music shows a middle ground between complete randomness and high stability, exist-ing as statistical "pink noise," which conforms to the relation:

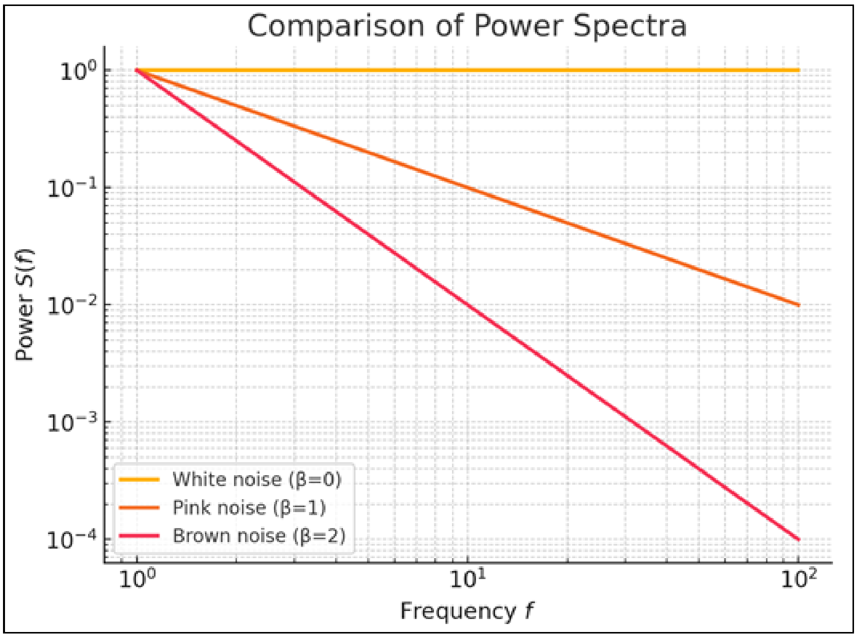

Figure 5: Comparison of power spectra for white, pink, and brown noise, illustrating statistical self-similarity.

Here, it is apparent that $\beta \approx 1$, as the frequency of sound has an inverse relationship to power. This is a very important relation, as it provides a direct mathematical correlation with fractals. In time series, a similar relation is seen which shows fractal self-similarity. This essentially proves that, on a statistical scale, all music, not just Bach, shows fractal nature.

d) Case Study: Contrapunctus VII

Contrapunctus VII is an important fugue from Bach's *Art of Fugue* (BWV 1080), as it is a comprehensive example for every previously discussed feature of fractal-resembling music. It is a complex composition that alters the subject in many unique ways, thus showing a large amount of fractal-like recursive repetition that can be thoroughly analyzed.

e) The Subject

The subject of this piece is a variation on the theme of the *Art of Fugue*, a simple starting point. However, Bach introduces alterations as soon as the subject is stated, using the lengthening of note values (augmentation) and flipping the intervals. The 4-note rhythm remains constant in the piece, being restated in ascending and descending forms, in different keys, and projected in different ways, mirroring affine transformation.

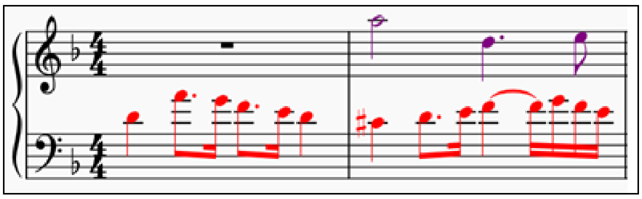

Figure 6: The subject of Bach's Contrapunctus VII (BWV 1080)

f) Recursive Imitation

This fugue is characterized by constant overlapping of the repeated subject, nesting aug-mented versions over inverted or diminished ones. The counterpoint has a large amount of complexity that is generated by only a small *seed motif*. This is a clear example of re-cursive imitation, where the overall fugal structure is based on the subject. This produces an effect quite similar to that of the Koch snowflake: nested layers of fractals.

4. Discussion

The connection of music to fractals reveals important insights into the nature of music and the close connection it has to mathematics. This self-similar fugal structure acts as an example of fractals existing in everyday phenomena, adding music to the already expansive list. It can also be argued that the apparent *fractality* of Bach's works and of music in general strengthens the idea that musical harmony is a natural phenomenon. Fractals were not a mathematical concept in Bach's time, nor was he a mathematician, so the fact that these Baroque-era compositions conform to such an extent to theories posed hundreds of years after them strengthens the fundamental nature of fractals.

That said, this does not mean that fractal math perfectly maps onto Baroque music. Bach's music does not have any of the mathematically rigorous properties of pure fractals (e.g., non-integer Hausdorff dimension), and the statistical relation of music to fractals also has many drawbacks [2]. Completely fractal music has never been made, though recent algorithmic developments make this a less distant possibility.

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5. Conclusion

We must now ask again our main question: Does Bach's music show fractal properties? Naturally, there is no single clear answer. Fractal geometry is certainly a useful lens for interpreting the fugue, but at the same time much of the connection remains analogical. There are as many strong supporting factors to this claim as there are limitations. We can, at most, term Bach's fugues *imperfectly fractal in nature*. While the resemblance they show to fractal shapes is undeniable and non-negligible, it still does not pass the rigor of mathematical proof.

So far, most of the research on fractality in music has used statistical similarities to support the claim. Power series and time series have been able to demonstrate a relation between statistical fractals and music in general, from Bach to Joplin. This paper has attempted to show the relation in a new light, focusing on structural similarity over statistics and taking J. S. Bach's music as a case study. The already close connection of fractal geometry to music opens up the possibility of creating truly fractal compositions in the future. Such music, generated by recursive algorithms, could embody genuine self-similarity across arbitrary scales rather than approximate it.

In this way, Bach's works take the concept of fractals beyond mathematics and nature and reveal that these structures also shape our perception of art. His fugues demonstrate that complexity can emerge from simplicity, that repetition and transformation can build vast architecture, and that mathematics and music are not merely parallel domains but deeply intertwined.

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