EXPLORING NONTRIVIAL SOLUTIONS FOR STEKLOV BOUNDARY VALUE PROBLEMS WITH $(p(x), q(x))$-LAPLACIAN

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ABSTRACT. In this study, we demonstrate that a Steklov boundary value problem involving $(p(x), q(x))$-Laplacian has nontrivial weak solutions in diverse situations. The mountain pass theorem and a conventional Weierstrass type theorem are the basis for the existence results. We used the symmetric mountain pass theorem to derive our multiplicity results.

Keywords: $(p(x), q(x))$-Laplacian operator, Mountain Pass Theorem, Steklov eigenvalue problem, Weierstrass type theorem.

1. INTRODUCTION

We consider the following nonhomogeneous Steklov eigenvalue problem

$$\begin{cases} \triangle_{p(x)}u + \triangle_{q(x)}u = 0 & \text{in } \Omega, \\ (|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2})\frac{\partial u}{\partial \nu} = g(x,u) - |u|^{p(x)-2}u - |u|^{q(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$, $\frac{\partial u}{\partial \nu}$ is the normal derivative of the outer unit on $\partial\Omega$, $p, q \in C_+(\overline{\Omega}) := \{m \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} m(x) > 1\}$ such that $q(.) \leq p(.)$ and g is a function fulfilling appropriate conditions.

For $m \in C^+(\overline{\Omega})$, we denote

$$m^- = \min_{x\in\overline{\Omega}} m(x); \quad m^+ = \max_{x\in\overline{\Omega}} m(x);$$

and

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

The study of differential equations and variational problems with variable exponents has garnered attention due to both their intriguing mathematical properties and their significant applications in fields such as fluid mechanics, material science, and biological systems. Understanding the behavior of solutions to these problems is essential for accurate modeling in these areas.

One notable example is the model for electrorheological fluids with $p(x)$-growth developed by Rajagopal and Růžička (see [27, 28]). These fluids change their mechanical properties in response to an electric field $E(x)$. In the steady-state case, the model is described by the equation

$$-\operatorname{div} S(x, \mathcal{E}(v)) = g(x, v, Dv), \quad \operatorname{div} v = 0,$$

where v is the fluid velocity, $\mathcal{E}(v)$ is the symmetric part of the gradient Dv, and the "extra stress" tensor S satisfies

$$D^2S(x,z) \geq v(1 + |z|^2)^{(p(x)-2)/2}\text{Id},$$

with $p(x) \equiv p(|E|^2)$ and E given.

Similarly, models have been developed for thermo-rheological fluids, where viscosity is affected by temperature (see [3, 36, 37]). For example, the differential system modeling the "thermistor problem" includes equations such as

$$-\operatorname{div}(p(x)|Du|^{p(x)-2}Du) = 0.$$

These models are crucial for the design of advanced materials and systems in engineering. In image processing, the $p(x)$-Laplacian operator is used to restore images by smoothing while preserving edges, which is important for noise reduction (see [6]):

$$-\Delta_{p(x)}u = I(x) - u,$$

where $I(x)$ is the original noisy image and u is the restored image. This technique is vital for improving the quality of visual data in various technologies. Other applications include elasticity problems (see [26, 35]), variational integrals with nonstandard growth (see [1, 23]), and fluid flow in porous media (see [4]).

Recently, there has been significant interest in nonhomogeneous eigenvalue problems involving operators with variable exponent growth conditions, such as the $p(x)$-Laplacian. These operators present unique challenges due to their nonhomogeneous nature, which renders many analytical techniques that work for constant exponent cases ineffective. For example, the Lagrange multiplier theorem does not apply to many problems involving these operators, indicating greater complexity compared to problems involving standard p-Laplacian operators.

Our research is also motivated by the challenges in obtaining multiple solutions in the superlinear case and other complexities in applying variational methods. Specifically, we must demonstrate that the operator Φ_i' (see Lemma 3.3) has the (S_+) property, a compactness condition crucial for establishing other important properties, such as the Palais-Smale condition, within a variational framework.

In the following, we provide a concise yet comprehensive overview of the current state of research in this field.

The constant case where $p(x) = p$ and $q(x) = q$ (p and q are constants) has been extensively investigated, as noted by the authors in the survey [22], which outlines several key applications of these results. Furthermore, in [30], variational methods combined with critical point theorems are utilized to explore these problems.

Regarding the case $p(x) = q(x)$, Numerous researchers have investigated elliptic equations with $p(x)$-Laplacian subjected to different types of boundary conditions in both bounded domains as discussed in [12, 19, 33], and in unbounded domains as detailed in [11, 15, 34]. Some studies establish the uniqueness of solutions, while others confirm the existence of two, three, multiple, or even infinitely many solutions. Particularly, in [19], A. Zerouali et al. studied the problem

$$\begin{cases} \triangle_{p(x)}u = |u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

and established, under suitable assumptions on g, the existence of weak solutions.

For further information and specific insights regarding variable exponent problems, we recommend interested readers to consult the references [11, 12, 24, 25].

To our knowledge, no articles have been published on the existence results for this class of Steklov problems with the $(p(x), q(x))$-Laplacian. The authors employ a variational approach to investigate the existence and multiplicity results of Steklov elliptic equations involving the $(p(x), q(x))$-Laplacian, incorporating concepts from the previously mentioned literature.

Our aim is to establish conditions on the function g that are sufficient for the existence of nontrivial weak solutions to problem (1.1) under the given cases :

i) $g(x,u) = f(x,u) + \lambda|u|^{\alpha(x)-2}u$, $\lambda \in \mathbb{R}^+$ and $\alpha \in C_+(\overline{\Omega})$;
ii) $g(x,u) = \lambda f(x,u)$, $\lambda > 0$,

The following conditions on f are enumerated, but it is important to note that these conditions do not have to be satisfied simultaneously.

(H0) $f : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ is continuous.

(H1) There exists $d > 0$ and $s \in C_+(\overline{\Omega})$ with $p^+ < s^- < s^+ < p^\partial(x)$, such that

$$|f(x,t)| \le d|t|^{s(x)-1},$$

for a.e. $x \in \partial\Omega$ and all $t \in \mathbb{R}$.

(H'1) There exists $d > 0$ and $s \in C_+(\overline{\Omega})$ with $s^+ < p^-$, such that

$$|f(x,t)| \le d|t|^{s(x)-1},$$

for a.e. $x \in \partial\Omega$ and all $t \in \mathbb{R}$.

(H2) There exists $\mu > p^+$ and $l > 0$ such that

$$0 < \mu F(x,t) \le f(x,t)t,$$

for a.e. $x \in \partial\Omega$ and all $|t| > l$, where $F(x,t) = \int_0^t f(x,s)\,ds$.

(H3) For a.e. $x \in \partial\Omega$ and all $t \in \mathbb{R}$, $f(x,-t) = -f(x,t)$.

(H4) There exists $t_0 > 0$ such that $F(x,t_0) > 0$ a.e. $x \in \partial\Omega$.

(H5) For all $x \in \partial\Omega$, $\displaystyle\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p(x)-1}} = 0$.

(H6) For all $x \in \partial\Omega$, $\displaystyle\lim_{|t|\to 0} \frac{f(x,t)}{|t|^{p(x)-1}} = 0$.

In this paper, our first and last results rely on the utilization of the mountain pass theorem (see, [2]). The determination of critical points of functionals has become an essential approach for solving elliptic

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equations and variational problems due to its substantial intuitive appeal and practical significance. The second and third ones are based on a variant of symmetric mountain pass theorem.

This paper comprises four sections. The second section present basic properties related to the Lebesgue and Sobolev spaces with variable exponents. Additionally, we compile the essential components of our proofs. The proofs for our primary outcomes under case (i) are detailed in section three, while section four is dedicated to addressing case (ii).

2. Preliminaries

In this section, it is necessary to explore the theory related to generalized Lebesgue-Sobolev spaces. However, for the sake of convenience, we will focus on only few aspects within the theory.

We introduce the generalized Lebesgue space

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \subset \mathbb{R}^N \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\},$$

possessing the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} := \inf \left\{ \alpha > 0 \; ; \; \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\},$$

which is separable and reflexive Banach space (see, [13]).

Let us define the space

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) \; / \; |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_\Omega := \inf \left\{ \alpha > 0 \; ; \; \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx + \int_\Omega \left| \frac{\nabla u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}; \quad \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 2.1. [9] *Let $u \in W^{1,p(x)}(\Omega)$.*
Let $\|u\| := |\nabla u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\partial\Omega)}$. Then the norm $\|u\|$ is a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|u\|_\Omega$.

Proposition 2.2. [11, 20, 10]

(1) $W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space;
(2) If $h \in C_+(\bar{\Omega})$ and $h(x) < p^\partial(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{h(x)}(\partial\Omega)$ is compact and continuous.

The mapping $\rho_p : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_p(u) := \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} |u|^{p(x)} d\sigma,$$

plays a crucial role in handling the Lebesgue-Sobolev spaces with variable exponents.

Proposition 2.3. [9] *For $u, u_k \in W^{1,p(x)}(\Omega)$; $\quad k = 1, 2, \dots,$ we have*

(1) $||u|| \geq 1$ implies $||u||^{p^-} \leq \rho_p(u) \leq ||u||^{p^+}$;
(2) $||u|| \leq 1$ implies $||u||^{p^+} \leq \rho_p(u) \leq ||u||^{p^-}$.

To prove Theorems 3.2 and 4.4, we will utilize the subsequent mountain pass theorem.

Theorem 2.4 ([17]). *Let X endowed with the norm $\|.\|_X$, be a Banach space. Assume that $\phi \in C^1(X; \mathbb{R})$ satisfies the Palais − Smale condition. Also, assume that ϕ has a mountain pass geometry, that is,*

(1) there exists two constants $\eta > 0$ and $\rho \in \mathbb{R}$ such that $\phi(u) \geq \rho$ if $\|u\|_X = \eta$;
(2) $\phi(0) < \rho$ and there exists $e \in X$ such that $\|e\|_X > \eta$ and $\phi(e) < \rho$.

Then, ϕ has a critical point $u_0 \in X$ such that $u_0 \neq 0$ and $u_0 \neq e$ with critical value

$$\phi(u_0) = \inf_{\gamma \in P} \sup_{u \in \gamma} \phi(u) \geq \rho > 0,$$

where P denotes the class of the paths $\gamma \in C([0, 1]; X)$ joining 0 to e.

The key element in demonstrating Theorem 3.5 lies in introducing a modified version of the symmetric pass theorem.

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Theorem 2.5 ([18]). *Let X be an infinite dimensional Banach space and $I \in C^1(X, \mathbb{R})$ satisfy the following two assumptions:*

(A_1). *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS);*
(A_2). *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$, Where Γ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$ with*

$$\gamma(A) := \inf\{k \in \mathbb{N}; \exists h : A \to \mathbb{R}^k \backslash \{0\} \text{such that } h \text{ is continuous and odd}\}$$

is the genus of A.

Then $I(u)$ admits a sequence of critical points u_k such that $I(u_k) < 0$; $u_k \neq 0$ and $u_k \to 0$ as $k \to \infty$.

Lastly, we bring to attention the Weierstrass type theorem, which will be employed in the demonstration of theorem 4.1.

Theorem 2.6 ([7]). *Assume that X is a reflexive Banach space and the function $\Phi : X \mapsto \mathbb{R}$ is coercive and (sequentially) weakly lower semicontinuous on X. Then, Φ is bounded from below on X and attains its infimum on X.*

3. The superlinear case

In this section, we discuss the case (i) and establish, under certain conditions on the number λ and the function f, the existence and multiplicity of weak solutions.

In this case, the Euler-Lagrange functional related to problem (1.1) is given by

$$\Phi_i(u) = \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx + \int_{\partial\Omega} \left(\frac{|u|^{p(x)}}{p(x)} + \frac{|u|^{q(x)}}{q(x)} \right) d\sigma$$
$$- \lambda \int_{\partial\Omega} \frac{|u|^{\alpha(x)}}{\alpha(x)} d\sigma - \int_{\partial\Omega} F(x, u) d\sigma, \quad \text{for all } u \in W^{1,p(x)}(\Omega).$$

Definition 3.1. *We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1) in the case (i) if :*

$$\int_\Omega \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2} \right) \nabla u \nabla \varphi dx + \int_{\partial\Omega} \left(|u|^{p(x)-2} + |u|^{q(x)-2} \right) u \varphi d\sigma$$
$$= \lambda \int_{\partial\Omega} |u|^{\alpha(x)-2} u \varphi d\sigma + \int_{\partial\Omega} f(x, u) \varphi d\sigma, \qquad \text{for all } \varphi \in W^{1,p(x)}(\Omega).$$

It is clear that the critical points of Φ_i are the weak solutions of our problem.

3.1. The existence results.

Theorem 3.2. *Assume that (H0), (H1) and (H2) are satisfied. If $\alpha^+ < p^-$, then there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ the problem (1.1) has at least a nontrivial weak solution.*

The proof of Theorem 3.2 will be based on the mountain pass theorem (refer to Theorem 2.4). To accomplish this, we have structured our proof in the following manner:

Throughout this subsection, we operate under the framework of Theorem 3.2

Lemma 3.3. *The functional Φ_i satisfies the Palais-Smale condition.*

Proof. Let $c \geq 0$ and (u_n) be a sequence in $W^{1,p(x)}(\Omega)$ such that $|\Phi_i(u_n)| < c$ and $\Phi_i'(u_n) \to 0$ as $n \to \infty$. First, let's demonstrate the boundedness of $(u_n)_n$. To do this, we use contradiction to argue and suppose that $\|u_n\| \to \infty$, up to a subsequence.Then, applying (H2), for sufficiently big n we obtain

$$\begin{aligned}
1 + c + \|u_n\| &\geq \Phi_i(u) - \frac{1}{\theta} \langle \Phi_i'(u_n), u_n \rangle \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \rho_p(u_n) + \left(\frac{1}{q^+} - \frac{1}{\theta} \right) \rho_q(u_n) - \lambda \left(\frac{1}{r^-} - \frac{1}{\theta} \right) \int_{\partial\Omega} |u_n|^{\alpha(x)} d\sigma \\
&\quad - \int_{\partial\Omega} \left(F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right) d\sigma \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \rho_p(u_n) - \lambda \left(\frac{1}{r^-} - \frac{1}{\theta} \right) \int_{\partial\Omega} |u_n|^{\alpha(x)} d\sigma \\
&\quad - \int_{\{x \in \partial\Omega; |u_n(x)| > l\}} \left(F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right) d\sigma \\
&\quad - |\partial\Omega| \sup\{|F(x, t) - \frac{1}{\theta} f(x, t) t|; x \in \partial\Omega, |t| \leq l\}.
\end{aligned}$$

According to the fact that

$$|u(x)|^{\alpha(x)} \leq |u(x)|^{\alpha^+} + |u(x)|^{\alpha^-}; \quad \forall x \in \bar{\Omega},$$

we deduce that for all $u \in W^{1,p(x)}(\Omega)$, we have

$$\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma \leq \int_{\partial\Omega} |u|^{\alpha^+} d\sigma + \int_{\partial\Omega} |u|^{\alpha^-} d\sigma.$$

Since $\alpha^+ < p^- < p^\partial(x)$ for any $x \in \bar{\Omega}$, then by Proposition 2.2, $W^{1,p(x)}(\Omega)$ is continuously and compactly embedded in $L^{\alpha^+}(\partial\Omega)$ and in $L^{\alpha^-}(\partial\Omega)$. It follows that there exists two positive constants C_1 and C_2 such that

$$\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma \leq C_1 \|u\|^{\alpha^+} + C_2 \|u\|^{\alpha^-}, \quad \forall u \in W^{1,p(x)}(\Omega),$$

Thus

$$\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma \leq c_1 \|u\|^{\alpha^-} \quad \text{if } \|u\| \leq 1,$$

and

$$\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma \leq c_1 \|u\|^{\alpha^+} \quad \text{if } \|u\| \geq 1.$$

Where c_1 is a positive constant. Now using Proposition 2.3 and (H2), we deduce that, for sufficiently large n,

$$\begin{aligned} c + 1 + \|u_n\| \geq & \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|u_n\|^{p^-} - \lambda c_1' \|u_n\|^{\alpha^+} \\ & - |\partial\Omega| \sup\{|F(x,t) - \frac{1}{\theta} f(x,t)t|; x \in \partial\Omega, |t| \leq l\}. \end{aligned}$$

Dividing by $\|u_n\|^{p^-}$ and letting $n \to \infty$ in the above inequality, since $\alpha^+ < p^-$, then we obtain a contradiction. This proves that $(u_n)_n$ is bounded in $W^{1,p(x)}(\Omega)$. For a subsequence still denoted $(u_n)_n$, we have $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$, $u_n \to u$ strongly in $L^{p(x)}(\partial\Omega)$, in $L^{q(x)}(\partial\Omega)$ and in $L^{\alpha(x)}(\partial\Omega)$. Therefore, $\langle \Phi_i'(u_n), u_n - u \rangle \to 0$, $\int_{\partial\Omega} |u_n|^{p(x)-2} u_n (u_n - u) d\sigma \to 0$, $\int_{\partial\Omega} |u_n|^{q(x)-2} u_n (u_n - u) d\sigma \to 0$, $\int_{\partial\Omega} |u_n|^{\alpha(x)-2} u_n (u_n - u) d\sigma \to 0$ and by (H1), we have $\int_{\partial\Omega} f(x, u_n)(u_n - u) d\sigma \to 0$. Thus,

$$\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \left(\nabla u_n - \nabla u\right) dx + \int_\Omega |\nabla u_n|^{q(x)-2} \nabla u_n \left(\nabla u_n - \nabla u\right) dx \to 0.$$

Since $\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx$ and $\int_\Omega |\nabla u_n|^{q(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx$ have the same sign, then each term converges to 0. Then

$$\int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \to 0.$$

According to the fact that the mapping $\Delta_{p(x)}$ is of type (S_+) (see, [12]). We deduce that $u_n \to u$ strongly in $W^{1,p(x)}(\Omega)$. Which completes the proof. □

Lemma 3.4. *With the same assumptions as in Lemma 3.3, we obtain the following results:*

(1) There exists a constant $\lambda_* > 0$ such that for every $\lambda \in (0, \lambda_*)$, there are $R > 0$ and $\rho > 0$ satisfying $\Phi_i(u) \geq R > 0$ for all $u \in W^{1,p(x)}(\Omega)$ where $\|u\| = \rho$.

(2) There exists a function $\varphi \in W^{1,p(x)}(\Omega)$ such that $\varphi > 0$ and $\Phi_i(t\varphi) \to -\infty$ as $t \to +\infty$.

Proof.

(1) We have

$$\begin{aligned} \Phi_i(u) = & \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx + \int_{\partial\Omega} \left(\frac{|u|^{p(x)}}{p(x)} + \frac{|u|^{q(x)}}{q(x)} \right) d\sigma \\ & - \lambda \int_{\partial\Omega} \frac{|u|^{\alpha(x)}}{\alpha(x)} d\sigma - \int_{\partial\Omega} F(x,u) d\sigma. \end{aligned}$$

By applying (H1), for every $\|u\| < 1$, we have

$$\Phi_i(u) \geq \frac{\|u\|^{p^+}}{p^+} - \frac{\lambda c_1}{\alpha^-} \|u\|^{\alpha^-} - k_1 \|u\|^{s^-},$$

where $c_1, k_1 > 0$. Consequently, we have

$$\Phi_i(u) \geq \|u\|^{p^+} \left(\frac{1}{p^+} - \frac{\lambda c_1}{\alpha^-} \|u\|^{\alpha^- - p^+} - k_1 \|u\|^{s^- - p^+} \right). \tag{3.1}$$

Let $\lambda > 0$ be given. We define the function $\psi_\lambda : (0, +\infty) \to \mathbb{R}$ by the formula

$$\psi_\lambda(t) = \frac{\lambda c_1}{\alpha^-} t^{\alpha^- - p^+} + k_1 t^{s^- - p^+}. \tag{3.2}$$

One can easily show that $\psi_\lambda(t)$ is continuous on the interval $(0, +\infty)$. Given that $s^- > p^+ \geq p^- > \alpha^+ \geq \alpha^- > 1$, we conclude that

$$\lim_{t \to 0^+} \psi_\lambda(t) = \lim_{t \to +\infty} \psi_\lambda(t) = +\infty.$$

Therefore, there exists a $t_* > 0$ such that $0 < \psi_\lambda(t_*) = \min_{t \in (0,+\infty)} \psi_\lambda(t)$, where t_* is given by the equation

$$\psi'_\lambda(t_*) = \frac{\lambda c_1}{\alpha^-}(\alpha^- - p^+) t_*^{\alpha^- - p^+ - 1} + k_1 (s^- - p^+) t_*^{s^- - p^+ - 1} = 0.$$

Hence, $t_* = \left(\frac{\lambda c_1 (p^+ - \alpha^-)}{\alpha^- k_1 (s^- - p^+)} \right)^{\frac{1}{s^- - \alpha^-}}$. An easy computation shows that

$$\psi_\lambda(t_*) = K.\lambda^{\frac{s^- - p^+}{s^- - \alpha^-}} \to 0 \text{ as } \lambda \to 0, \text{ where } K > 0. \tag{3.3}$$

From (3.1), (3.2), and (3.3), we deduce that there is a positive constant λ_* such that for every $\lambda \in (0, \lambda_*)$, one can choose $R > 0$ and $\rho > 0$ such that $\Phi_i(u) \geq R > 0$ for all $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = \rho$.

(2) By the condition (H2), we can conclude that there exists a positive function $a(x)$ such that for all $|\tau| > l$, $F(x, \tau) \geq a(x)|\tau|^\theta$ for a.e. $x \in \partial\Omega$. Consequently, for $t > 1$, we get

$$\begin{aligned}\Phi_i(tu) \leq\ & \frac{t^{p^+}}{p^-}\rho_p(u) + \frac{t^{q^+}}{q^-}\rho_q(u) - \frac{\lambda t^{\alpha^-}}{\alpha^+}\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma \\ & - |\partial\Omega| \inf\{|F(x,t)|; x \in \partial\Omega, |t| \leq l\} \\ & - t^\theta \int_{\{x \in \partial\Omega; |u(x)| > l\}} a(x)|u|^\theta d\sigma.\end{aligned}$$

Since $\theta > p^+ \geq p^- > q^+$, it follows that $\Phi_i(t\varphi) \to -\infty$ as $t \to +\infty$ with $\varphi(x) = |u(x)| + l + \epsilon$, for all $x \in \bar{\Omega}$, where $\epsilon > 0$. Therefor, for a fixed $u \not\equiv 0$, we can choose $e = t\varphi$ such that $\|e\| > \rho$ and $\Phi_i(e) < 0$.

□

Proof of Theorem 3.2. It is clear that $\Phi_i(0) = 0 < R$. Therefore, by Lemmas 3.3-3.4 and Theorem 2.4, we deduce that, for every $\lambda \in (0, \lambda_*)$, problem (1.1) admits at least one nontrivial weak solution in $W^{1,p(x)}(\Omega)$. □

Example 1.
For $N = 2$, let $\Omega = (0, \pi)^2$, take $p(x) = 3 + \frac{1}{2}\sin(x_1 + x_2)$, $q(x) = 2 + \frac{1}{4}\sin(x_1 - x_2)$. Note that $q(x) \leq p(x)$ for all $x \in \overline{\Omega}$. Let's define:

$$f(x, u) = |u|^{m(x)-2}u \text{ where } m(x) = 4 + \frac{1}{3}\sin(x_1 x_2).$$

Define $\alpha(.)$ as a continuous function such that $\alpha^+ < p^-$. For example:

$$\alpha(x) = 2 + \frac{1}{4}\cos(x_1 + x_2).$$

In this case, the problem (1.1) becomes :

$$\begin{cases} \triangle_{p(x)}u + \triangle_{q(x)}u = 0 & \text{in } \Omega, \\ (|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2})\frac{\partial u}{\partial \nu} = |u|^{m(x)-2}u + \lambda|u|^{\alpha(x)-2}u - |u|^{p(x)-2}u - |u|^{q(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

The function $f(.,.)$ is continuous on $\partial\Omega \times \mathbb{R}$ because $(x, u) \to |u|^{m(x)-2}u$ is continuous in both x and u, which means that $(H0)$ is satisfied.
Given $f(x, u) = |u|^{m(x)-2}u$, we have:

$$|f(x, t)| = |t|^{m(x)-1}.$$

This can be bounded by $d|t|^{s(x)-1}$ with $d = 1$ and $s(x) = m(x)$. We can easily see that $p(x)$ varies between 3 and 3.5, and $m(x)$ varies between 4 and 4.33, then $p^+ = 3.5$ and $s^- = 4$, ensuring the

condition $p^+ < s^- < s^+ < p^\partial(x)$ *is met. Consequently,* $(H1)$ *is also satisfied.*
For $f(x,u) = |u|^{m(x)-2}u$*, we have*

$$F(x,t) = \int_0^t |s|^{m(x)-2} s\, ds = \frac{1}{m(x)}|t|^{m(x)}.$$

Then,

$$f(x,t)t = |t|^{m(x)}.$$

To satisfy the condition $(H2)$*, let* $\mu = m^-$*. We get:*

$$\mu F(x,t) = \frac{m^-}{m(x)}|t|^{m(x)}.$$

Since $m^- < m(x)$*, we have:*

$$\frac{m^-}{m(x)} < 1,$$

thus:

$$\mu F(x,t) = \frac{m^-}{m(x)}|t|^{m(x)} \leq |t|^{m(x)} = f(x,t)t,$$

ensuring that the condition is satisfied for $|t| > l$ *with an appropriate choice of* l*.*
By Theorem 3.2, there exists $\lambda_* > 0$ *such that for any* $\lambda \in (0, \lambda_*)$ *the problem* (3.4) *has at least a nontrivial weak solution.*

3.2. **The multiplicity results.**

Theorem 3.5. *Assume that* $f : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ *is a Carathéodory function satisfying the conditions (H'1), (H2) and (H3). If* $\alpha^+ < q^- \leq q^+ < p^-$*. Then, there exists* $\lambda_0 > 0$*, such that for any* $\lambda > \lambda_0$ *there exists a sequence* (u_k) *of nontrivial weak solutions for the problem* (1.1) *in the case* (i)*. Moreover,* $u_k \to 0$*, as* $k \to \infty$*.*

We show that the symmetric mountain-pass theorem (see Theorem 2.5) can be applied. We have divided the proof into a sequence of lemmas.

Lemma 3.6. *Given the assumptions stated in Theorem 3.5, the functional* Φ_i *is an even function, bounded from below, satisfies the Palais-Smale (PS) condition, and we have* $\Phi_i(0) = 0$*.*

Proof. By the properties of f, it is obvious that $\Phi_i \in C^1$, Φ_i is even and $\Phi_i(0) = 0$.
Using (H'1), we obtain

$$\begin{aligned}
\Phi_i(u) \geq& \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)}\right) dx + \int_{\partial\Omega} \left(\frac{|u|^{p(x)}}{p(x)} + \frac{|u|^{q(x)}}{q(x)}\right) d\sigma \\
&- \lambda \int_{\partial\Omega} \frac{|u|^{\alpha(x)}}{\alpha(x)} d\sigma - k \int_{\partial\Omega} \frac{|u|^{s(x)}}{s(x)} d\sigma \\
&\geq \frac{1}{p^+}\rho_p(u) + \frac{1}{q^+}\rho_q(u) - \frac{\lambda}{\alpha^-}\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma - \frac{k}{s^-}\int_{\partial\Omega} |u|^{s(x)} d\sigma \\
&\geq \frac{1}{p^+}\rho_p(u) - \frac{\lambda}{\alpha^-}\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma - \frac{k}{s^-}\int_{\partial\Omega} |u|^{s(x)} d\sigma.
\end{aligned}$$

Since $\alpha^+ < p^- < p^\partial(x)$ and $s^+ < p^- < p^\partial(x)$, then by Proposition 2.2, there exist two positive constants c_1 and c_2 such that

$$\Phi_i(u) \geq \frac{1}{p^+}\rho_p(u) - \frac{c_1\lambda}{\alpha^-}\|u\|^{\alpha^+} - \frac{c_2 k}{s^-}\|u\|^{s^+}, \quad \text{if } \|u\| > 1,$$

and

$$\Phi_i(u) \geq \frac{1}{p^+}\rho_p(u) - \frac{c_1\lambda}{\alpha^-}\|u\|^{\alpha^-} - \frac{c_2 k}{s^-}\|u\|^{s^-}, \quad \text{if } \|u\| \leq 1,$$

Now using Proposition 2.3, we have

$$\Phi_i(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{c_1\lambda}{\alpha^-}\|u\|^{\alpha^+} - \frac{c_2 k}{s^-}\|u\|^{s^+}, \quad \text{if } \|u\| > 1,$$

and

$$\Phi_i(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{c_1\lambda}{\alpha^-}\|u\|^{\alpha^-} - \frac{c_2 k}{s^-}\|u\|^{s^-}, \quad \text{if } \|u\| \leq 1.$$

As $p^- > \alpha^+$ and $p^- > s^+$, Φ_i is bounded from below. This finishes the proof.

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Lemma 3.7. *Assume that* $\alpha^+ < q^- \leq q^+ < p^-$. *Then for each* $k \in \mathbb{N}^*$, *there exists an* $H_k \in \Gamma_k$ *such that:* $\sup\limits_{u\in H_k} \Phi_i(u) < 0$.

Proof. Let $v_1, v_2, \cdots, v_k \in C^\infty(\mathbb{R}^N)$ such that

$$\overline{\{x \in \partial\Omega;\ v_i(x) \neq 0\}} \cap \overline{\{x \in \partial\Omega;\ v_j(x) \neq 0\}} = \emptyset$$

if $i \neq j$ and

$$|\{x \in \partial\Omega;\ v_i(x) \neq 0\}| > 0,\ \forall i, j \in \{1, 2, \cdots, k\}.$$

Take $F_k = span\{v_1, v_2, \cdots, v_k\}$; we have $\dim F_k = k$. Denote $S = \{v \in W^{1,p(x)}(\Omega);\ \|v\| = 1\}$ and for $0 < t \leq 1$, $H_k(t) = t(F_k \cap S)$. For all $t \in]0, 1]$, we have $\gamma(H_k(t)) = k$. We claim that for any $k \in \mathbb{N}^*$, there exists $t_k \in]0, 1]$ such that $\sup\limits_{u\in H_k(t_k)} \Phi_i(u) < 0$. Indeed, for $k \in \mathbb{N}^*$, $0 < t \leq 1$, and using $(H'1)$, we have

$$\begin{aligned}
\sup_{u\in H_k(t)} \Phi_i(u) \leq & \sup_{v\in F_k\cap S} \Phi_i(tv) \\
\leq & \sup_{v\in F_k\cap S} \left\{ \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla v|^{p(x)} dx + \int_\Omega \frac{t^{q(x)}}{q(x)} |\nabla v|^{q(x)} dx + \int_{\partial\Omega} \frac{t^{p(x)}}{p(x)} |v|^{p(x)} d\sigma \right. \\
& \left. + \int_{\partial\Omega} \frac{t^{q(x)}}{q(x)} |v|^{q(x)} d\sigma - \int_{\partial\Omega} F(x, tv) d\sigma - \lambda \int_{\partial\Omega} \frac{t^{\alpha(x)}}{\alpha(x)} |v|^{\alpha(x)} d\sigma \right\} \\
\leq & \sup_{v\in F_k\cap S} \left\{ \frac{t^{p^-}}{p^-} \rho_p(v) + \frac{t^{q^-}}{q^-} \rho_q(v) - \lambda \frac{t^{\alpha^+}}{\alpha^+} \int_{\partial\Omega} |v|^{\alpha(x)} d\sigma \right\} \\
\leq & \sup_{v\in F_k\cap S} \left\{ \frac{t^{p^-}}{p^-} + \frac{t^{q^-}}{q^-} \rho_q(v) - \lambda \frac{t^{\alpha^+}}{\alpha^+} \int_{\partial\Omega} |v|^{\alpha(x)} d\sigma \right\}.
\end{aligned}$$

Since $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,q(x)}(\Omega)$, then there exists a constant $c_0 > 0$ such that $\|v\|_{q(x)} \leq c_0 \|v\|$, $\forall u \in W^{1,p(x)}(\Omega)$ and since $v \in S$, then $\|v\| = 1$ which means that $\|v\|_{q(x)} \leq c_0$. Thus $\left\|\frac{v}{c_0}\right\|_{q(x)} \leq 1$. We get then by Proposition 2.3

$$\rho_q\left(\frac{v}{c_0}\right) = \int_\Omega \frac{|\nabla v|^{q(x)}}{c_0^{q(x)}} dx + \int_{\partial\Omega} \frac{|v|^{q(x)}}{c_0^{q(x)}} d\sigma \leq 1. \tag{3.5}$$

We know that $0 < c_0^{q(x)} \leq \max(c_0^{q^-}, c_0^{q^+}) := M$, which implies that $\frac{1}{c_0^{q(x)}} \geq \frac{1}{M}$. Thus

$$\rho_q\left(\frac{v}{c_0}\right) \geq \frac{\rho_q(v)}{M}. \tag{3.6}$$

Combining (3.5) and (3.6), we find

$$\rho_q(v) \leq M.$$

Let $\bar{c} = \min\limits_{v\in F_k\cap S} \int_{\partial\Omega} |v|^{\alpha(x)} d\sigma > 0$ and since $1 < q^- < p^-$ and $0 < t \leq 1$, then we get

$$\begin{aligned}
\sup_{u\in H_k(t)} \Phi_i(u) \leq & \frac{t^{q^-}}{q^-} + \frac{t^{q^-} M}{q^-} - \frac{\lambda \bar{c} t^{\alpha^+}}{\alpha^+} \\
\leq & t^{q^-} \left(\frac{1+M}{q^-} - \frac{\lambda \bar{c}}{\alpha^+ t^{q^- - \alpha^+}} \right).
\end{aligned}$$

At this point, since $\alpha^+ < q^-$, we can find a positive constant λ_0 such that for every $\lambda > \lambda_0$, there exists a sufficiently small $t_k \in (0, 1]$ satisfying

$$\frac{1+M}{q^-} - \frac{\lambda \bar{c}}{\alpha^+ t_k^{q^- - \alpha^+}} < 0.$$

Then, we have

$$\sup_{u\in H_k(t_k)} \Phi_i(u) < 0.$$

This completes the proof. □

Proof of Theorem 3.5. By applying Lemmas 3.6, 3.7 and Theorem 2.5, it follows that the problem (1.1) possesses a sequence of weak solutions (u_k) for which $\Phi_i(u_k) < 0$ and $\lim\limits_{k\to+\infty} u_k = 0$. □

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Example 2. *Let Ω be the unit disk in $\mathbb{R}^2$, i.e., $\Omega = \{x \in \mathbb{R}^2 : \|x\| < 1\}$. Its boundary is $\partial\Omega = \{x \in \mathbb{R}^2 : \|x\| = 1\}$.*
We need continuous functions $\alpha(x)$, $q(x)$, and $p(x)$ on $\overline{\Omega}$ such that $\alpha(x) < q(x) < p(x)$ for all $x \in \overline{\Omega}$. For simplicity, let:

$$\alpha(x) = 2 + \frac{1}{2}(\|x\| - 1), \quad q(x) = 3 + \frac{1}{4}(\|x\| - 1), \quad p(x) = 4 + \frac{1}{8}(\|x\| - 1)$$

This ensures $\alpha(x) < q(x) < p(x)$ for all $x \in \overline{\Omega}$.
We need to define $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ that satisfies (H1'), (H2), and (H3).
Let's define:

$$f(x,t) = \begin{cases} d|t|^{s(x)-2}t & \text{if } |t| \le 1 \\ d|t|^{\mu-2}t & \text{if } |t| > 1 \end{cases}$$

where $s(x) = 2.5 + \frac{1}{2}(\|x\| - 1)$, which is a function in $C_+(\overline{\Omega})$ with $s^+ < p^- = 3.875$, $d > 0$ and $\mu = 5 > p^+ = 4$.
Now, we check the hypotheses:
For $|t| \le 1$,

$$|f(x,t)| = d|t|^{s(x)-1} \le d|t|^{s(x)-1}$$

Since $s(x) < p(x)$ for all $x \in \overline{\Omega}$, f satisfies (H1').
For $|t| > 1$,

$$F(x,t) = \int_0^t f(x,s)\,ds = \int_0^t d|s|^{\mu-2}s\,ds = \frac{d}{\mu}|t|^{\mu}$$

Then,

$$0 < \mu F(x,t) = d|t|^{\mu} = f(x,t)t$$

ensuring $0 < \mu F(x,t) \le f(x,t)t$ for $|t| > l = 1$.
For $t \in \mathbb{R}$,

$$f(x,-t) = -f(x,t)$$

remains true due to the symmetry in the definition. Finally, by Theorem 3.5, there exists $\lambda_0 > 0$, such that for any $\lambda > \lambda_0$ there exists a sequence (u_k) of nontrivial weak solutions for the problem (1.1) in this case.

4. The sublinear case

In case (ii) of the problem (1.1), the associated energy functional Φ_{ii}, which is defined on $W^{1,p(x)}(\Omega)$, is given by:

$$\begin{aligned} \Phi_{ii}(u) = &\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx + \int_{\partial\Omega} \left(\frac{|u|^{p(x)}}{p(x)} + \frac{|u|^{q(x)}}{q(x)} \right) d\sigma \\ &- \lambda \int_{\partial\Omega} F(x,u)\,d\sigma. \end{aligned} \tag{4.1}$$

And we have

$$\begin{aligned} \langle \Phi'_{ii}(u), v \rangle = &\int_\Omega \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2} \right) \nabla u \nabla v dx + \int_{\partial\Omega} \left(|u|^{p(x)-2} + |u|^{q(x)-2} \right) uv d\sigma \\ &- \lambda \int_{\partial\Omega} f(x,u) v d\sigma, \quad \text{for any } u, v \in W^{1,p(x)}(\Omega). \end{aligned} \tag{4.2}$$

As in the third section, the weak solutions of problem (1.1) given by (4.2) are exactly the critical points of Φ_{ii} defined by (4.1).

4.1. **The existence results.**

Theorem 4.1. *Assume that $f : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a function satisfying the conditions (H0), (H4) and (H5). Then, there exists a constant $\lambda_1 > 0$, such that for every $\lambda > \lambda_1$, the problem (1.1) in the case (ii) has at least one nontrivial weak solution.*

To prove our Theorem 4.1, we will apply the Weierstrass type theorem 2.6. We start with the following two Lemmas.

Lemma 4.2. *For any $\lambda > 0$, the functional Φ_{ii} is (sequentially) weakly lower semicontinuous.*

Proof. The same approach used in the proof of Lemma 5 in [5], by replacing the term $\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx$ by $\int_{\partial\Omega} \frac{|u|^{p(x)}}{p(x)}d\sigma$, shows that the functional

$$u \mapsto \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx + \int_{\partial\Omega} \left(\frac{|u|^{p(x)}}{p(x)} + \frac{|u|^{q(x)}}{q(x)} \right) d\sigma$$

defined on $W^{1,p(x)}(\Omega)$ is weakly lower semicontinuous. At the same time, hypotheses $(H5)$ implies the existence of a positive constant k' such that

$$|f(x,t)| \leq k'(1 + |t|^{p(x)-1}) \quad \text{for all } t \in \mathbb{R} \text{ and a.e } x \in \partial\Omega. \tag{4.3}$$

Hence, since $W^{1,p(x)}(\Omega)$ is compactly embedded to $L^{p(x)}(\partial\Omega)$, standard arguments infer that Φ_{ii} is weakly lower semicontinuous for every $\lambda > 0$ and the proof of the lemma is complete. □

Lemma 4.3. *For any $\lambda > 0$, the functional Φ_{ii} given by (4.1) is coercive and satisfies the Palais-Smale condition.*

Proof. We first show that Φ_{ii} is coercive. To do this, fix $\epsilon > 0$. Using condition $(H5)$, we can find $\delta > 0$ such that

$$|f(x,t)| \leq \epsilon |t|^{p(x)-1} \quad \text{for all } |t| \geq \delta \text{ and a.e. } x \in \partial\Omega.$$

By integrating this inequality, we obtain

$$|F(x,t)| \leq \epsilon \frac{|t|^{p(x)}}{p^-} + \max_{|s|\leq\delta} |f(x,s)||t| \quad \text{for all } t \in \mathbb{R}. \tag{4.4}$$

Applying Proposition 2.3, we find that for any $u \in W^{1,p(x)}(\Omega)$ with $\|u\| > 1$,

$$\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx + \int_{\partial\Omega} \left(\frac{|u|^{p(x)}}{p(x)} + \frac{|u|^{q(x)}}{q(x)} \right) d\sigma \geq \frac{1}{p^+}\|u\|^{p^-}. \tag{4.5}$$

From inequalities (4.4) and (4.5), we deduce that for any $u \in W^{1,p(x)}(\Omega)$ with $\|u\| > 1$,

$$\Phi_{ii}(u) \geq \left(\frac{1}{p^+} - \frac{\lambda\epsilon}{p^-} \right) \|u\|^{p^-} - \lambda \int_{\partial\Omega} \max_{|s|\leq\delta} |f(x,s)||u|\, d\sigma.$$

Choose ϵ small enough so that $\frac{1}{p^+} - \frac{\lambda\epsilon}{p^-} > 0$. Moreover, due to the continuity of f and to the continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^1(\partial\Omega)$, there exists $k" > 0$ such that

$$\Phi_{ii}(u) \geq \left(\frac{1}{p^+} - \frac{\lambda\epsilon}{p^-} \right) \|u\|^{p^-} - \lambda k" \|u\| \quad \text{for all } u \in W^{1,p(x)}(\Omega) \text{ with } \|u\| > 1.$$

As $p^- > 1$, we have $\Phi_{ii}(u) \to +\infty$ as $\|u\| \to +\infty$, which implies that Φ_{ii} is coercive. We will now show that Φ_{ii} satisfies the Palais-Smale condition. To do this, let $c \geq 0$ and $(u_n)_n \subset W^{1,p(x)}(\Omega)$ be a sequence such that $|\Phi_{ii}(u_n)| < c$ and $\Phi'_{ii}(u_n) \to 0$ as $n \to +\infty$. Since Φ_{ii} is coercive, it is straightforward to see that $(u_n)_n$ is bounded. To show that $(u_n)_n$ has a subsequence converging to a critical point of Φ_{ii}, we follow the same approach as in the proof of Lemma 3.3, using (4.3) in place of $(H1)$ and Φ_{ii} instead of Φ_i. Thus, Φ_{ii} satisfies the Palais-Smale condition, completing the proof of the lemma. □

We are now able to show Theorem 4.1.

Proof of Theorem 4.1. Applying Theorem 2.6 along with Lemmas 4.2 and 4.3, we conclude that there exists a global minimizer u_1 for Φ_{ii} in $W^{1,p(x)}(\Omega)$. This minimizer u_1 also provides a weak solution to the problem (1.1) in case (ii). The next question is whether u_1 is nontrivial. To address this, we need to examine the conditions that ensure the nontriviality of u_1. It is evident that, for every $\lambda > 0$, the following inequality holds:

$$\Phi_{ii}(u_1) \leq \Phi_{ii}(u) \quad \text{for all } u \in W^{1,p(x)}(\Omega). \tag{4.6}$$

Using (H4), we can choose $\lambda_1 > 0$ as follows

$$\lambda_1 = \left[\sup_{u \in W^{1,p(x)}(\Omega), u \neq 0} \frac{\lambda \int_{\partial\Omega} F(x,u) d\sigma}{\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx + \int_{\partial\Omega} \left(\frac{|u|^{p(x)}}{p(x)} + \frac{|u|^{q(x)}}{q(x)} \right) d\sigma} \right]^{-1}.$$

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Assumption $(H4)$ ensures that for every $\lambda > \lambda_1$, there exists a function $\omega \in W^{1,p(x)}(\Omega)$ such that

$$\begin{aligned}\Phi_{ii}(\omega) = &\int_\Omega \left(\frac{|\nabla\omega|^{p(x)}}{p(x)} + \frac{|\nabla\omega|^{q(x)}}{q(x)}\right) dx + \int_{\partial\Omega} \left(\frac{|\omega|^{p(x)}}{p(x)} + \frac{|\omega|^{q(x)}}{q(x)}\right) d\sigma \\ &- \lambda \int_{\partial\Omega} F(x,\omega) d\sigma < 0.\end{aligned} \tag{4.7}$$

We conclude from (4.6) and (4.7) that

$$\Phi_{ii}(u_1) \le \Phi_{ii}(\omega) \quad \text{for all } \lambda > \lambda_1. \tag{4.8}$$

Since we have $\Phi_{ii}(0) = 0$, (4.8) gives $u_1 \neq 0$. Thus, for $\lambda > \lambda_1$, the problem (1.1) in the case (ii) admits a nontrivial weak solution, which completes the proof of Theorem 4.1. □

Example 3. *Let $\Omega = (0,1)^2$ and $x = (x_1, x_2)$. Define:*

$$p(x) = 3 + x_1 x_2$$

$$q(x) = 2 + \frac{x_1 x_2}{2}$$

It is clear that $q(x) < p(x)$ for all $x \in \overline{\Omega} = [0,1]^2$.
Define $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,u) = \frac{u}{1+|u|}$$

In this case, we have the problem

$$\begin{cases} \triangle_{p(x)} u + \triangle_{q(x)} u = 0 & \text{in } \Omega, \\ (|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2}) \frac{\partial u}{\partial \nu} = \lambda \frac{u}{1+|u|} - |u|^{p(x)-2}u - |u|^{q(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

The function f satisfies the following hypotheses:
Hypothesis (H0): f is continuous in both variables x and t.
Hypothesis (H4): Let $t_0 = 1$. Then for $x \in \partial\Omega$:

$$F(x,t_0) = \int_0^{t_0} f(x,s)\, ds = \int_0^1 \frac{s}{1+|s|}\, ds > 0$$

Hypothesis (H5): For all $x \in \partial\Omega$:

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p(x)-1}} = \lim_{|t|\to+\infty} \frac{\frac{t}{1+|t|}}{|t|^{3+x_1x_2-1}} = \lim_{|t|\to+\infty} \frac{1}{|t|^{2+x_1x_2}(1+|t|)} = 0$$

since $2 + x_1x_2 \ge 2$ and $1 + |t|$ grows linearly with t. Therefore, by Theorem 4.1, there exists a constant $\lambda_1 > 0$, such that for every $\lambda > \lambda_1$, the problem (4.9) has at least one nontrivial weak solution.

4.2. **The multiplicity results.**

Theorem 4.4. *Assume that $f : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a function satisfying the conditions (H0), (H4)-(H6). Then, the problem (1.1) has at least two nontrivial weak solutions for every $\lambda > \lambda_1$ in the case (ii), where λ_1 is the one found in Theorem 4.1.*

The proof of this theorem is basically relies on Theorems 4.1 and 2.4.
Theorem 4.1 assures the existence of a nontrivial weak solution u_1 to problem (1.1) in the case (ii), for all $\lambda > \lambda_1$. In order to find a second weak solution for all $\lambda > \lambda_1$, we turn to Theorem 2.4. By Lemma 3.6, we know that Φ_{ii} satisfies the Palais-Smale condition for all $\lambda > 0$. Since $\Phi_{ii}(0) = 0$ and u_1 is a nontrivial function with $\Phi_{ii}(u_1) < 0$, as shown by equation (4.8), we now need to establish that Φ_{ii} has a mountain pass geometry for all $\lambda > \lambda_1$. Specifically, we need to find two positive constants, ρ and $b < \|u_1\|$, such that $\Phi_{ii}(u) \ge \rho$ for all u with $\|u\| = b$. To do this, choose $\epsilon > 0$ and $s \in C(\overline{\Omega}, \mathbb{R})$ such that $p^+ < s^- < s^+ < p^\partial$. According to conditions $(H5)$ and $(H6)$, there exist $\delta_1 \ge 1$ and $\delta_2 > 0$ such that

$$|f(x,t)| \le \epsilon |t|^{s(x)-1} \quad \text{for } |t| > \delta_1 \text{ and for a.e. } x \in \partial\Omega,$$

$$|f(x,t)| \le \epsilon |t|^{p(x)-1} \quad \text{for } |t| < \delta_2 \text{ and for a.e. } x \in \partial\Omega.$$

It follows that there exists a constant $k_2 > 0$ chosen sufficiently big to have the following inequality

$$|F(x,t)| \le \epsilon |t|^{p(x)} + k_2 |t|^{s(x)} \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \partial\Omega.$$

This leads to

$$\begin{aligned}\Phi_{ii}(u) &\geq \left(\frac{1}{p^+} - \lambda\epsilon\right)\rho_p(u) - \lambda k_2 \int_{\partial\Omega} |u|^{s(x)} d\sigma \\ &\geq \left(\frac{1}{p^+} - \lambda\epsilon\right)\rho_p(u) - \lambda k_2 c\|u\|^{s-},\end{aligned}$$

for all $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = b < 1$ and c is a positive constant due to the continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\partial\Omega)$. Select ϵ to be sufficiently small so that $\frac{1}{p^+} - \lambda\epsilon > 0$. By Proposition 2.3, for any $\lambda > \lambda_1$ and $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = b < 1$, it follows that

$$\Phi_{ii}(u) \geq \left(\frac{1}{p^+} - \lambda\epsilon\right)\|u\|^{p^+} - \lambda k_2 c\|u\|^{s-}.$$

Since $p^+ < s^-$, for sufficiently small b, there exists a constant $\rho > 0$ such that for all $\lambda > \lambda_1 > 0$ and $u \in W^{1,p(x)}(\Omega)$ with $\|u\| = b < \min\{1, \|u_1\|\}$, we have $\Phi_{ii}(u) \geq \rho > 0$. Consequently, we can now apply Theorem 2.4 to find a second critical point $u_2 \in W^{1,p(x)}(\Omega)$ such that

$$\Phi_{ii}(u_2) = \inf_{\gamma\in P} \sup_{u\in\gamma} \Phi_{ii}(u) \geq \rho > 0.$$

This gives us the required result for problem (1.1), which is a second nontrivial weak solution.

Example 4. *Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^2$. Define $p(x)$ and $q(x)$ as functions depending on the radial distance from the origin:*

$$p(x) = 3 + \sin(\pi\|x\|)$$

$$q(x) = 2 + \sin(\pi\|x\|)$$

Here, $\|x\|$ denotes the Euclidean norm of x. Clearly, for all $x \in \overline{B(0,1)}$, we have $q(x) < p(x)$ since $2 + \sin(\|x\|) < 3 + \sin(\|x\|)$.
Define $f : \partial B(0,1) \times \mathbb{R} \to \mathbb{R}$ by:

$$f(x,u) = \frac{\sin(u)}{1 + |u|^{p(x)}}$$

The problem (1.1) becomes

$$\begin{cases} \triangle_{p(x)}u + \triangle_{q(x)}u = 0 & \text{in } \Omega, \\ (|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2})\frac{\partial u}{\partial \nu} = \lambda \frac{\sin(u)}{1+|u|^{p(x)}} - |u|^{p(x)-2}u - |u|^{q(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{4.10}$$

Now, let's verify the hypotheses for the function f:
- (H0) : $f(x,t) = \frac{\sin(t)}{1+|t|^{p(x)}}$ is continuous in x and t since the sine function and the denominator are continuous functions.
- (H4) : Take $t_0 = 1$. Then, since $\partial B(0,1)$ is the boundary of the unit ball, it is the circle of radius 1 centered at the origin. On this boundary, $\|x\| = 1$ for all $x \in \partial B(0,1)$, and thus $p(x) = 3$.

$$F(x,t_0) = \int_0^1 \frac{\sin(1)}{1 + 1^3} ds = \int_0^1 \frac{\sin(1)}{2} ds > 0$$

Hence, $F(x,t_0) > 0$ almost everywhere on $\partial B(0,1)$.

- (H5) : For all $x \in \partial B(0,1)$,

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p(x)-1}} = \lim_{|t|\to+\infty} \frac{\sin(t)}{|t|^2(1+|t|^3)} = 0$$

- (H6) : For all $x \in \partial B(0,1)$,

$$\lim_{|t|\to 0} \frac{f(x,t)}{|t|^{p(x)-1}} = \lim_{|t|\to 0} \frac{\sin(t)}{|t|^2(1+|t|^3)} = 0$$

By Theorem 4.4, the problem (4.10) has at least two nontrivial weak solutions for every $\lambda > \lambda_1$, where λ_1 is the one found in Theorem 4.1.

5. Conclusion and perspectives

In this paper, we studied a nonlinear elliptic problem driven by the $(p(x), q(x))$-Laplace operator in a bounded domain. Using critical point theory, we proved existence and multiplicity theorems for this problem under Steklov boundary conditions. Specifically, the use of several mountain pass theorems and a traditional Weierstrass-type theorem proved to be effective. The methods and results presented in this work are different from those concerning the (p, q)-Laplacian.

There are two kinds of difficulties encountered: firstly, selecting specific conditions that enable the utilization of nonlinear analysis theorems; and secondly, effectively applying these theorems.

The findings from this study open several exciting avenues for future research, particularly in the area of control theory. We propose the following research directions to explore the applications of $(p(x), q(x))$-Laplacian problems in control systems, drawing on recent advances in the field:

Our results on $(p(x), q(x))$-Laplacian problems could be extended to investigate the controllability of discrete-time semilinear systems. The framework for controllability developed in [21] can be adapted to analyze how $(p(x), q(x))$-Laplacian operators influence the approximate controllability of discrete-time fractional evolution equations. Future research could focus on applying these concepts to develop new methods for ensuring controllability in systems governed by $(p(x), q(x))$-Laplacian equations.

Another promising direction is to explore approximate controllability in semilinear fractional control systems. The methods described in [32] for analyzing semilinear delay control systems of fractional order could be adapted to investigate the controllability of systems described by $(p(x), q(x))$-Laplacian differential equations. This research could lead to the formulation of new criteria for achieving approximate controllability in such systems.

The theory developed for complete controllability of semilinear stochastic systems with delay in [31] offers a foundation for extending the $(p(x), q(x))$-Laplacian framework to stochastic control systems. Future research could explore how the results for complete controllability of stochastic systems can be applied to $(p(x), q(x))$-Laplacian problems, potentially leading to new control strategies and methods for managing stochastic dynamics in nonlinear systems.

By pursuing these future research directions, we aim to bridge the gap between theoretical advances in $(p(x), q(x))$-Laplacian problems and practical applications in control theory. These efforts could lead to the development of innovative control strategies and a deeper understanding of how nonlinear differential equations can be managed in both deterministic and stochastic frameworks.

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