

Rogers-Ramanujan Type Identities Modulo 9, 11, 18 and 22

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Abstract: In this paper, some identities of Rogers-Ramanujan Type related to modulo 9, 11, 18 and 22 is derived with the incorporation of generalized Bailey pairs and some standard results established by Andrew V. Sills [1] using some q – difference relations.

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1. Introduction

We begin by recalling the famous Rogers-Ramanujan Identities:

The Rogers-Ramanujan Identities:

For $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{n=0}^{\infty} \frac{1}{1 - q^{5n+2}}, \text{ where } n \not\equiv 0, 2, 3 \pmod{5}$$

and,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{n=0}^{\infty} \frac{1}{1 - q^{5n+3}}, \text{ where } n \not\equiv 0, 1, 4 \pmod{5}$$

where $(q; q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$, for $n \geq 1$

which are known as the celebrated original Rogers-Ramanujan Identity. These two identities have motivated extensive research over the past hundred years. These identities are due to L.J. Rogers [5] and were rediscovered independently by S. Ramanujan [7] and I. Schur [4]. In 1940's W.N. Bailey undertook a careful study of Rogers work and greatly simplified into two papers [8] and [9]. In these papers, Bailey proved some more generalized formula that helps to find more identities of Rogers-Ramanujan Type.

Definitions 1.1:

For $|q| < 1$, the q -shifted factorial is defined by

$$(a; q)_0 = 1$$

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \text{ for } n \geq 1$$

$$\text{and } (a; q)_{\infty} = \prod_{k=1}^{\infty} (1 - aq^k).$$

$$\text{It follows that } (a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}$$

The multiple q -shifted factorial is defined by

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$$

$$(a_1, a_2, \dots, a_m; q)_{\infty} = (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_m; q)_{\infty}.$$

Jacobi's Triple Product Identity: (see [3] 2.2.10 and 2.2.11)

$$(zq^{\frac{1}{2}}, z^{-1}q^{\frac{1}{2}}, q; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n^2/2} \quad (1.1)$$

And its corollary

$$\begin{aligned} \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)\frac{n(n+1)}{2} - in} \\ = \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)\frac{n(n+1)}{2} - in} (1 - q^{(2n+1)i}) \\ = \prod_{n=0}^{\infty} (1 - q^{(2k+1)(n+1)}) (1 - q^{(2k+1)n+i}) (1 - q^{(2k+1)(n+1)-i}) \quad (1.2) \end{aligned}$$

Definition 1.2: A pair of sequences $(\alpha_n(a, q), \beta_n(a, q))$ is called a Bailey pair if for $n \geq 0$,

$$\beta_n(a, q) = \sum_{r=0}^n \frac{\alpha_r(a, q)}{(q, q)_{n-r} (aq, q)_{n+r}} \quad (1.3)$$

In [8] and [9], Bailey proved the following result known as "Bailey Lemma".

Bailey's Lemma: If $(\alpha_r(a, q), \beta_j(a, q))$ form a Bailey pair, then

$$\begin{aligned} \frac{1}{(\frac{aq}{\rho_1}; q)_n (\frac{aq}{\rho_2}; q)_n} \sum_{j \geq 0} \frac{(\rho_1; q)_j (\rho_2; q)_j (\frac{aq}{\rho_1 \rho_2}; q)_{n-j}}{(q; q)_{n-j}} (\frac{aq}{\rho_1 \rho_2})^j \beta_j(a; q) \\ = \sum_{r=0}^n \frac{(\rho_1; q)_r (\rho_2; q)_r}{(\frac{aq}{\rho_1}; q)_r (\frac{aq}{\rho_2}; q)_r (q; q)_{n-r} (aq; q)_{n+r}} (\frac{aq}{\rho_1 \rho_2})^r \alpha_r(a; q) \quad (1.4) \end{aligned}$$

Some important consequences of this lemma are the following corollary: ([see [1], Eqn. (9), (10)])

Corollary 1.1: If $(\alpha_m(a, q), \beta_j(a, q))$ form a Bailey pair, then

$$\sum_{j \geq 0} a^j q^{j^2} \beta_j(a, q) = \frac{1}{(aq; q)_{\infty}} \sum_{m=0}^{\infty} a^m q^{m^2} \alpha_m(a, q) \quad (1.5)$$

$$\begin{aligned} \sum_{j \geq 0} a^j q^{j^2} (-q; q^2)_j \beta_j(a, q^2) = \\ \frac{(-aq; q^2)_{\infty}}{(aq^2; q^2)_{\infty}} \sum_{m=0}^{\infty} \frac{a^m q^{m^2} (-q; q^2)_m}{(-aq; q^2)_m} \alpha_m(a, q^2) \quad (1.6) \end{aligned}$$

In [8] and [9], Bailey considered several Bailey pairs which are special cases of a more general Bailey pair involving additional parameters d and k .

Parameterized Bailey pair:

Let $\lambda = -\frac{3}{2}d^2 + dk + \frac{1}{2}d$, $h = \lfloor \frac{2\lambda}{d} \rfloor$, and $t = d + h + 2$.

Let $\alpha_{d,k,m}(a, q) =$

$$\begin{cases} \frac{(-1)^r a^{(k-d)r} q^{(dk-d^2+\frac{d}{2})r^2 - \frac{d}{2}r} (aq^{2d}, q^{2d})_r (a, q^d)_r}{(a, q^{2d})_r (q^d, q^d)_r} \\ 0 \quad \text{if } m = dr, \text{ and otherwise} \end{cases}$$

and

$$\beta_{d,k,m}(a, q) =$$

$$\begin{cases} \lim_{r \rightarrow 0} \frac{W_t(a; \gamma_1, \gamma_2, \dots, \gamma_h, \mu_1, \mu_2, \dots, \mu_d; q^d, \tau^h a^{k-d} q^{nd})}{(a, aq; q)_n} & \text{if } \lambda \geq 0 \\ \lim_{r \rightarrow 0} \frac{W_{t+1}(a; \delta_1, \delta_2, \dots, \delta_h, \mu_1, \mu_2, \dots, \mu_d; q^d, \frac{a^{k-d} q^{nd}}{\tau^h})}{(a, aq; q)_n} & \text{if } \lambda < 0 \end{cases}$$

where $\gamma_j = \frac{q^{\lambda/h}}{\tau}$, $\mu_j = q^{d-j-n}$, $\delta_j = \tau a q^{d-\lambda/h}$.

$${}_{s+1}W_s(a_1; a_4, a_5, \dots, a_{s+1}; q; z) = {}_{s+1}\phi_s$$

$$\left[\begin{matrix} a_1, qa_1^{1/2}, -qa_1^{1/2}, a_4, \dots, a_{s+1}; q, z \\ a_1^{1/2}, -a_1^{1/2}, \frac{qa_1}{a_4}, \dots, \frac{qa_1}{a_{s+1}} \end{matrix} \right],$$

and,

$${}_{s+1}\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_{s+1}; q, z \\ b_1, b_2, \dots, b_s \end{matrix} \right] = \sum_{r=0}^{\infty} \frac{(a_1, a_2, \dots, a_{s+1}; q)_r}{(q, b_1, b_2, \dots, b_s; q)_r} \cdot z^r.$$

Then $\alpha_{d,k,m}(a, q)$ and $\beta_{d,k,m}(a, q)$ form a Bailey pair.

Bailey considered the special cases $\alpha_{d,k,m}(a, q)$ for $(d, k) = (1, 2), (2, 2), (2, 3)$ and $(3, 4)$ in [8]. Each of these four (d, k) sets is particularly nice, as the resulting expression for $\alpha_{d,k,m}(a, q)$ is summable by Jackson's theorem ([2], 238, eqn(II - 20)). Thus, $\beta_{d,k,m}(a, q)$ reduces to a finite product, and upon substituting it in (1.5) the left hand side of the resulting $a - RRT$ identity will be a single-fold sum.

Definition 1.3: For $k \geq 1$, and $1 \leq i \leq k$,

$$Q_{d,k,i}(a) = Q_{d,k,i}(a, q) = \frac{1}{(aq; q)_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{kn} q^{(dk+\frac{d}{2})n^2 + (k-i+\frac{1}{2})dn} (1-a^i q^{(2n+1)di}) (aq^d, q^d)_n}{(q^d, q^d)_n} \quad (1.7)$$

In [1], Andrew V. Sills has derived the following results with incorporation of the parameterized Bailey pairs and some q -difference equations as noted in [1].

Theorem 1.1: The following q -difference equation is valid: (See [1], eqn.(19) and (20))

$$Q_{d,k,1}(a, q) = \frac{1}{(aq; q)_{d-1}} Q_{d,k,k}(aq^d, q) \quad (1.8) \quad \text{and for } 2 \leq i \leq k,$$

$$Q_{d,k,i}(a, q) = Q_{d,k,i-1}(a, q) + \frac{a^{i-1} q^{(i-1)d}}{(aq; q)_{d-1}} Q_{d,k,k-1}(aq^d, q) \quad (1.9)$$

Theorem 1.2: For $i = 1, 2, 3, 4$ (see [1], Theorem 3.16, p. 19)

$$F_{3,4,i}(a, q) = Q_{3,4,i}(a, q) \quad (1.10)$$

where,

$$F_{3,4,1}(a, q) = \sum_{n=0}^{\infty} \frac{a^n q^{n^2+3n} (aq^3; q^3)_n}{(aq; q)_{2n+2} (q; q)_n}$$

$$F_{3,4,2}(a, q) = \sum_{n=0}^{\infty} \frac{a^n q^{n^2+2n} (aq^3; q^3)_n}{(aq; q)_{2n+2} (q; q)_n}$$

$$F_{3,4,3}(a, q) = \sum_{n=0}^{\infty} \frac{a^n q^{n^2+n} (aq^3; q^3)_n}{(aq; q)_{2n+1} (q; q)_n}$$

$$F_{3,4,4}(a, q) = \sum_{n=0}^{\infty} \frac{a^n q^{n^2} (a; q^3)_n}{(a; q)_{2n} (q; q)_n}$$

Theorem 1.3: For $i = 1, 2, 3, 4, 5$ (see [1], Theorem 3.19, p. 20)

$$F_{3,5,i}(a, q) = Q_{3,5,i}(a, q) \quad (1.11)$$

where $F_{3,5,1}(a, q) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r} q^{n^2+3r^2+3n+3r} (aq^3; q^3)_{n-r}}{(aq; q)_{2n+2} (q; q)_{n-3r} (q^3; q^3)_r}$

$$F_{3,5,2}(a, q) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r} q^{n^2+3r^2+3n+3r} (aq^3; q^3)_{n-r} (1+aq^{3r+3})}{(aq; q)_{2n+2} (q; q)_{n-3r} (q^3; q^3)_r}$$

$$F_{3,5,3}(a, q) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{n^2+3r^2-3} (a; q^3)_{n-r} (q^{3r} + aq^{6r+3} - 1)}{(a; q)_{2n} (q; q)_{n-3r} (q^3; q^3)_r}$$

$$F_{3,5,4}(a, q) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{n^2+3r^2+3r} (a; q^3)_{n-r}}{(a; q)_{2n} (q; q)_{n-3r} (q^3; q^3)_r}$$

$$F_{3,5,5}(a, q) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{n^2+3r^2} (a; q^3)_{n-r}}{(a; q)_{2n} (q; q)_{n-3r} (q^3; q^3)_r}$$

2. We derive some transformations from (1.10) and (1.11) which will be used in obtaining Identities related to Rogers-Ramanujan Type:

Setting $i = 1, 2, 3, 4$ successively in (1.10) and then replacing q by $q^{1/3}$ and $q^{2/3}$ respectively for each this particular value of i , we obtain the following eight transformations:

$$\sum_{n=0}^{\infty} \frac{a^n q^{(n^2+3n)/3} (aq; q)_n}{(aq^{1/3}, q^{1/3})_{2n+2} (q^{1/3}, q^{1/3})_n} = \frac{1}{(aq^{1/3}, q^{1/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{4n} q^{(9n^2+7n)/2} (1-aq^{2n+1}) (aq; q)_n}{(q; q)_n} \quad (2.1)$$

$$\sum_{n=0}^{\infty} \frac{a^n q^{(2n^2+6n)/3} (aq^2; q^2)_n}{(aq^{2/3}, q^{2/3})_{2n+2} (q^{2/3}, q^{2/3})_n} = \frac{1}{(aq^{2/3}, q^{2/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{4n} q^{(9n^2+7n)/2} (1-aq^{4n+2}) (aq^2; q^2)_n}{(q^2; q^2)_n} \quad (2.2)$$

$$\sum_{n=0}^{\infty} \frac{a^n q^{(n^2+2n)/3} (aq; q)_n}{(aq^{1/3}, q^{1/3})_{2n+2} (q^{1/3}, q^{1/3})_n} = \frac{1}{(aq^{1/3}, q^{1/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{4n} q^{(9n^2+5n)/2} (1-a^2 q^{4n+2}) (aq; q)_n}{(q; q)_n} \quad (2.3)$$

$$\sum_{n=0}^{\infty} \frac{a^n q^{(2n^2+4n)/3} (aq; q)_n}{(aq^{2/3}, q^{2/3})_{2n+2} (q^{2/3}, q^{2/3})_n} = \frac{1}{(aq^{2/3}, q^{2/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{4n} q^{(9n^2+5n)/2} (1-a^2 q^{8n+4}) (aq^2; q^2)_n}{(q^2; q^2)_n} \quad (2.4)$$

$$\sum_{n=0}^{\infty} \frac{a^n q^{(n^2+n)/3} (aq; q)_n}{(aq^{1/3}, q^{1/3})_{2n+1} (q^{1/3}, q^{1/3})_n} = \frac{1}{(aq^{1/3}, q^{1/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{4n} q^{(9n^2+3n)/2} (1-a^3 q^{(6n+3)}) (aq; q)_n}{(q; q)_n} \quad (2.5)$$

$$\sum_{n=0}^{\infty} \frac{a^n q^{(2n^2+2n)/3} (aq^2; q^2)_n}{(aq^{2/3}; q^{2/3})_{2n+1} (q^{2/3}; q^{2/3})_n} = \frac{1}{(aq^{2/3}; q^{2/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{4n} q^{(9n^2+3n)} (1-a^3 q^{(12n+6)}) (aq^2; q^2)_n}{(q^2; q^2)_n} \quad (2.6)$$

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/3} (a; q)_n}{(aq^{1/3}; q^{1/3})_{2n} (q^{1/3}; q^{1/3})_n} = \frac{1}{(aq^{1/3}; q^{1/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{4n} q^{(9n^2+n)/2} (1-a^4 q^{(8n+4)}) (aq; q)_n}{(q; q)_n} \quad (2.7)$$

$$\sum_{n=0}^{\infty} \frac{a^n q^{2n^2/3} (a; q^2)_n}{(a; q^2/3)_{2n} (q^2/3; q^2/3)_n} = \frac{1}{(aq^{2/3}; q^{2/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{4n} q^{(9n^2+n)} (1-a^4 q^{(16n+8)}) (aq^2; q^2)_n}{(q^2; q^2)_n} \quad (2.8)$$

In similar way, setting $i = 1, 2, 3, 4, 5$ successively in (1.11) and then replacing q by $q^{1/3}$ and $q^{2/3}$ respectively for each particular value of i , we obtain the following ten transformations:

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r} q^{(n^2+3r^2+3n+3r)/3} (aq; q)_{n-r}}{(aq^{1/3}; q^{1/3})_{2n+2} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(aq^{1/3}; q^{1/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+9n)/2} (1-aq^{2n+1}) (aq; q)_n}{(q; q)_n} \quad (2.9)$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r} q^{(2n^2+6r^2+6n+6r)/3} (aq^2; q^2)_{n-r}}{(aq^{2/3}; q^{2/3})_{2n+2} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r} = \frac{1}{(aq^{2/3}; q^{2/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+9n)} (1-aq^{4n+2}) (aq^2; q^2)_n}{(q^2; q^2)_n} \quad (2.10)$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r} q^{(n^2+3r^2+3n+3r)/3} (aq; q)_{n-r} (1+aq^{r+1})}{(aq^{1/3}; q^{1/3})_{2n+2} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(aq^{1/3}; q^{1/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+7n)/2} (1-a^2 q^{(4n+2)}) (aq; q)_n}{(q; q)_n} \quad (2.11)$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r} q^{(2n^2+6r^2+6n+6r)/3} (aq^2; q^2)_{n-r} (1+aq^{2r+2})}{(aq^{2/3}; q^{2/3})_{2n+2} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r} = \frac{1}{(aq^{2/3}; q^{2/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+7n)} (1-a^2 q^{(8n+4)}) (aq^2; q^2)_n}{(q^2; q^2)_n} \quad (2.12)$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{(n^2+3r^2-3)/3} (a; q)_{n-r} (q^r + aq^{2r+1} - 1)}{(a; q^{1/3})_{2n} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(aq^{1/3}; q^{1/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+5n)/2} (1-a^3 q^{(6n+3)}) (aq; q)_n}{(q; q)_n} \quad (2.13)$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{(2n^2+6r^2-6)/3} (a; q^2)_{n-r} (q^{2r} + aq^{4r+2} - 1)}{(a; q^{2/3})_{2n} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r} = \frac{1}{(aq^{2/3}; q^{2/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+5n)} (1-a^3 q^{(12n+6)}) (aq^2; q^2)_n}{(q^2; q^2)_n} \quad (2.14)$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{(n^2+3r^2+3r)/3} (a; q^{1/3})_{n-r}}{(a; q^{1/3})_{2n} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(aq^{1/3}; q^{1/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+3n)/2} (1-a^4 q^{(8n+4)}) (aq; q)_n}{(q; q)_n} \quad (2.15)$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{(2n^2+6r^2+6r)/3} (a; q^2)_{n-r}}{(a; q^2/3)_{2n} (q^2/3; q^2/3)_{n-3r} (q^2; q^2)_r} = \frac{1}{(aq^{2/3}; q^{2/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+3n)} (1-a^4 q^{(16n+8)}) (aq^2; q^2)_n}{(q^2; q^2)_n} \quad (2.16)$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{(n^2+3r^2)/3} (a; q)_{n-r}}{(a; q^{1/3})_{2n} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(aq^{1/3}; q^{1/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+n)/2} (1-a^5 q^{(10n+5)}) (aq; q)_n}{(q; q)_n} \quad (2.17)$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{(2n^2+6r^2)/3} (a; q^2)_{n-r}}{(a; q^2/3)_{2n} (q^2/3; q^2/3)_{n-3r} (q^2; q^2)_r} = \frac{1}{(aq^{2/3}; q^{2/3})_{\infty}} \sum_{n \geq 0} \frac{(-1)^n a^{5n} q^{(11n^2+n)} (1-a^5 q^{(20n+10)}) (aq^2; q^2)_n}{(q^2; q^2)_n} \quad (2.18)$$

3. Main Results

3.1 Rogers-Ramanujan Type Identities Modulo 9:

Setting $a = 1, q$ successively in the transformations (2.1), (2.3) and (2.5) respectively and then using (1.1), the following identities of Rogers-Ramanujan Type is found:

$$\begin{aligned} \frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/3} (q; q)_n}{(q^{1/3}; q^{1/3})_{2n+2} (q^{1/3}; q^{1/3})_n} &= \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{\frac{9n^2+7n}{2}} (1 - q^{2n+1}) \\ &= \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{9n^2+7n}{2}} \\ &= \prod_{n=0}^{\infty} \frac{1}{1 - q^n}, \end{aligned}$$

where $n \not\equiv 0, 1, 8 \pmod{9}$ (3.1.1)

$$\begin{aligned} \frac{(q; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(n^2+6n+9)/3} (q; q)_{n+1}}{(q; q^{1/3})_{2n+3} (q^{1/3}; q^{1/3})_n} &= \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{9n^2+n}{2}} + q^3 \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{9n^2+15n}{2}} \right) \\ &= \prod_{n=0}^{\infty} \frac{1}{1 - q^n} + q^3 \prod_{n=0}^{\infty} \frac{1}{1 - q^n} \quad (3.1.2) \end{aligned}$$

where $n \not\equiv 0, 4, 5 \pmod{9}$ and $n \not\equiv 0, 3, 6 \pmod{9}$

$$\begin{aligned} \frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(n^2+2n)/3} (q; q)_n}{(q^{1/3}; q^{1/3})_{2n+2} (q^{1/3}; q^{1/3})_n} &= \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{9n^2+5n}{2}} \\ &= \prod_{n=0}^{\infty} \frac{1}{1 - q^n}, \end{aligned}$$

where $n \not\equiv 0, 2, 7 \pmod{9}$ (3.1.3)

$$\frac{(q; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(n^2+5n)/3} (q; q)_{n+1}}{(q; q^{1/3})_{2n+3} (q^{1/3}; q^{1/3})_n} = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(9n^2+3n)/2} + q^2 \sum_{n=-\infty}^{\infty} (-1)^n q^{(9n^2+13n)/2} \right) \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^2 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.1.4)$$

where $n \not\equiv 0, 3, 6 \pmod{9}$ and $n \not\equiv 0, 2, 7 \pmod{9}$

$$\frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/3} (q; q)_n}{(q^{1/3}; q^{1/3})_{2n+1} (q^{1/3}; q^{1/3})_n} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(9n^2+3n)/2} \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0, 3, 6 \pmod{9} \quad (3.1.5)$$

$$\frac{(q; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(n^2+4n+3)/3} (q^2; q)_n}{(q; q^{1/3})_{2n+2} (q^{1/3}; q^{1/3})_n} = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(9n^2+5n)/2} + q \sum_{n=-\infty}^{\infty} (-1)^n q^{(9n^2+11n)/2} \right) \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.1.6)$$

where $n \not\equiv 0, 2, 7 \pmod{9}$ and $n \not\equiv 0, 1, 8 \pmod{9}$

Also, setting $a = q$ in the transformation (2.7), we find

$$\frac{(q; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2} (q; q)_n}{(q^{4/3}; q^{1/3})_{2n} (q^{1/3}; q^{1/3})_n} = \prod_{n=0}^{\infty} \frac{1}{1-q^n} + \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.1.7)$$

where $n \not\equiv 0 \pmod{9}$ and $n \not\equiv 0, 1, 8 \pmod{9}$

3.2 Rogers-Ramanujan Type Identities Modulo 11:

Setting $a = 1, q$ successively in the transformations (2.9), (2.11), (2.13), (2.15) and (2.17) respectively, we find the following identities of Rogers-Ramanujan Type:

$$\frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(n^2+3r^2+3n+3r)/3} (q; q)_{n-r}}{(q^{1/3}; q^{1/3})_{2n+2} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 1, 10 \pmod{11}$ (3.2.1)

$$\frac{(q; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(n^2+3r^2+6n+6r+12)/3} (q; q)_{n-r+1}}{(q; q^{1/3})_{2n+3} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} \\ = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+n}{2}} + q^4 \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+19n}{2}} \right) \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^4 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.2.2)$$

where $n \not\equiv 0, 5, 6 \pmod{11}$ and $n \not\equiv 0, 4, 7 \pmod{11}$

$$\frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(n^2+3r^2+3n+3r)/3} (q; q)_{n-r} (1+q^{r+1})}{(q^{1/3}; q^{1/3})_{2n+2} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+7n}{2}} \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0, 2, 9 \pmod{11} \quad (3.2.3)$$

$$\frac{(q; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(n^2+3r^2+6n+6r+9)/3} (q^2; q)_{n-r} (1+q^{r+2})}{(q^{4/3}; q^{1/3})_{2n+2} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} \\ = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+3n}{2}} + q^3 \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+17n}{2}} \right) \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^3 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.2.4)$$

where $n \not\equiv 0, 4, 7 \pmod{11}$ and $n \not\equiv 0, 3, 8 \pmod{11}$

$$\frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(n^2+3r^2-3)/3} (q; q)_{n-r-1} (q^r + a q^{2r+1} - 1)}{(q^{1/3}; q^{1/3})_{2n-1} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+5n}{2}} \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0, 3, 8 \pmod{11} \quad (3.2.5)$$

$$\frac{(q; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(n^2+3r^2+3n+3r)/3} (q; q)_{n-r} (q^r + q^{2r+2} - 1)}{(q; q^{1/3})_{2n} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} \\ = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+5n}{2}} + q^2 \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+15n}{2}} \right) \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^2 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.2.6)$$

where $n \not\equiv 0, 3, 8 \pmod{11}$ and $n \not\equiv 0, 2, 9 \pmod{11}$

$$\frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(n^2+3r^2+3r)/3} (q^{1/3}; q^{1/3})_{n-r-1}}{(q^{1/3}; q^{1/3})_{2n-1} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+3n}{2}} \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0, 4, 7 \pmod{11} \quad (3.2.7)$$

$$\frac{(q; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(n^2+3r^2+3n+6r)/3} (q; q^{1/3})_{n-r}}{(q; q^{1/3})_{2n} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+7n}{2}} + q \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{11n^2+13n}{2}} \right) \\ = \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.2.8)$$

where $n \not\equiv 0, 2, 9 \pmod{11}$ and $n \not\equiv 0, 1, 10 \pmod{11}$

$$\frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(n^2+3r^2)/3} (q; q)_{n-r-1}}{(q^{1/3}; q^{1/3})_{2n-1} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(11n^2+n)/2}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0, 5, 6 \pmod{11} \quad (3.2.9)$$

$$\frac{(q; q^{1/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{(n^2+3r^2+3n+3r-3)/3} (q; q)_{n-r}}{(q; q^{1/3})_{2n} (q^{1/3}; q^{1/3})_{n-3r} (q; q)_r} = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(11n^2+9n)/2} \right)$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0, 1, 10 \pmod{11} \quad (3.2.10)$$

3.3 Rogers-Ramanujan Type Identities Modulo 18:

Setting $a = 1, q^2$ successively in the transformations (2.2), (2.4), (2.6) and (2.8) respectively, we find the following identities of Rogers-Ramanujan Type:

$$\frac{(q^{2/3}; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(2n^2+6n)/3} (q^2; q^2)_n}{(q^{2/3}; q^{2/3})_{2n+2} (q^{2/3}; q^{2/3})_n} = \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 2, 16 \pmod{18}$ (3.3.1)

$$\frac{(q^2; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(2n^2+12n+18)/3} (q^2; q^2)_{n+1}}{(q^2; q^{2/3})_{2n+3} (q^{2/3}; q^{2/3})_n} = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2-n} + q^6 \sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2+15n} \right)$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^6 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.3.2)$$

where $n \not\equiv 0, 8, 10 \pmod{18}$ and $n \not\equiv 0, 6, 12 \pmod{18}$

$$\frac{(q^{2/3}; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(2n^2+4n)/3} (q; q)_n}{(q^{2/3}; q^{2/3})_{2n+2} (q^{2/3}; q^{2/3})_n} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(9n^2+5n)}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 4, 14 \pmod{18}$ (3.3.3)

$$\frac{(q^2; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(2n^2+10n+12)/3} (q^2; q^2)_{n+1}}{(q^2; q^{2/3})_{2n+3} (q^{2/3}; q^{2/3})_n} = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(9n^2+13n)} + q^4 \sum_{n=-\infty}^{\infty} (-1)^n q^{(9n^2+3n)} \right)$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^4 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.3.4)$$

where $n \not\equiv 0, 4, 14 \pmod{18}$ and $n \not\equiv 0, 6, 12 \pmod{18}$

$$\frac{(q^{2/3}; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(2n^2+2n)/3} (q^2; q^2)_n}{(q^{2/3}; q^{2/3})_{2n+1} (q^{2/3}; q^{2/3})_n} = \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 6, 12 \pmod{18}$ (3.3.5)

$$\frac{(q^2; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(2n^2+8n+6)/3} (q^2; q^2)_{n+1}}{(q^2; q^{2/3})_{2n+2} (q^{2/3}; q^{2/3})_n} = \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^2 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.3.6)$$

where $n \not\equiv 0, 4, 14 \pmod{18}$ and $n \not\equiv 0, 2, 16 \pmod{18}$

$$\frac{(q^{2/3}; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{2n^2/3} (aq^2; q^2)_{n-1}}{(aq^{2/3}; q^{2/3})_{2n-1} (q^{2/3}; q^{2/3})_n} = \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 8, 10 \pmod{18}$ (3.3.7)

$$\frac{(q^2; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{(2n^2+6n)/3} (q^2; q^2)_n}{(q^2; q^{2/3})_{2n} (q^{2/3}; q^{2/3})_n} = \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(9n^2+9n)} (1 - q^{(16n+16)}) (1 - q^{2n+2}) \right)$$

$$= 1 + \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 2, 16 \pmod{18}$ (3.3.8)

3.4 Rogers-Ramanujan Type Identities Modulo 22:

Setting $a = 1, q^2$ successively in the transformations (2.10), (2.12), (2.14), (2.16) and (2.18) respectively, we find the following identities of Rogers-Ramanujan Type:

$$\frac{(q^{2/3}; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(2n^2+6r^2+6n+6r)/3} (q^2; q^2)_{n-r}}{(q^{2/3}; q^{2/3})_{2n+2} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(11n^2+9n)}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 2, 20 \pmod{22}$ (3.4.1)

$$\frac{(q^2; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(2n^2+6r^2+12n+12r+24)/3} (q^2; q^2)_{n-r+1}}{(q^2; q^{2/3})_{2n+3} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r} = \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^8 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.4.2)$$

where $n \not\equiv 0, 10, 12 \pmod{22}$ and $n \not\equiv 0, 8, 14 \pmod{22}$

$$\frac{(q^{2/3}; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(2n^2+6r^2+6n+6r)/3} (q^2; q^2)_{n-r} (1+q^{2r+2})}{(q^{2/3}; q^{2/3})_{2n+2} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(11n^2+7n)}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 4, 18 \pmod{22}$ (3.4.3)

$$\frac{(q^2; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(2n^2+6r^2+12n+12r+18)/3} (q^2; q^2)_{n-r+1} (1+q^{2r+4})}{(q^2; q^{2/3})_{2n+3} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r}$$

$$= \frac{1}{(q; q)_{\infty}} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(11n^2+3n)} + q^6 \sum_{n=-\infty}^{\infty} (-1)^n q^{(11n^2+17n)} \right)$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^6 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.4.4)$$

where $n \not\equiv 0, 8, 14 \pmod{22}$ and $n \not\equiv 0, 6, 16 \pmod{22}$

$$\frac{(q^{2/3}; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(2n^2+6r^2-6)/3} (q^2; q^2)_{n-r-1} (q^{2r} + q^{4r+2} - 1)}{(q^{2/3}; q^{2/3})_{2n-1} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 6, 16 \pmod{22}$ (3.4.5)

$$\frac{(q^2; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(2n^2+6r^2+6n+6r)/3} (q^2; q^2)_{n-r-1} (q^{2r} + q^{4r+4} - 1)}{(q^{2/3}; q^{2/3})_{2n-1} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^4 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.4.6)$$

where $n \not\equiv 0, 6, 16 \pmod{22}$ and $n \not\equiv 0, 4, 18 \pmod{22}$

$$\frac{(q^{2/3}; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(2n^2+6r^2+6r)/3} (q^2; q^2)_{n-r-1}}{(q^{2/3}; q^{2/3})_{2n-1} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r} = \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 8, 14 \pmod{22}$ (3.4.7)

$$\frac{(q^2; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(2n^2+6r^2+6n+12r)/3} (q^2; q^2)_{n-r}}{(q^2; q^{2/3})_{2n} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n} + q^2 \prod_{n=0}^{\infty} \frac{1}{1-q^n} \quad (3.4.8)$$

where $n \not\equiv 0, 4, 18 \pmod{22}$ and $n \not\equiv 0, 2, 20 \pmod{22}$

$$\frac{(q^{2/3}; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{q^{(2n^2+6r^2)/3} (q^2; q^2)_{n-r-1}}{(q^{2/3}; q^{2/3})_{2n-1} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r}$$

$$= \prod_{n=0}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0, 10, 12 \pmod{22} \quad (3.4.9) \text{ and}$$

$$\frac{(q^2; q^{2/3})_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^{n+r-1} q^{(2n^2+6r^2+6n+6r-6)/3} (q^2; q^2)_{n-r}}{(q^2; q^{2/3})_{2n} (q^{2/3}; q^{2/3})_{n-3r} (q^2; q^2)_r} = \prod_{n=0}^{\infty} \frac{1}{1-q^n},$$

where $n \not\equiv 0, 2, 20 \pmod{22}$ (3.4.10)

4. Conclusion

This paper was motivated by the work of Andrew V. Sills method employed in [1] where some analytical aspect has been considered to derive some more identities of Rogers-Ramanujan Type of modulo 9, 11, 18, and 22. Some other identities may also be found by more inspection on the values of the parameter a . Also there is a scope of obtaining more identities by incorporating some particular identities from the Slater's famous list of 130 identities of Rogers-Ramanujan type.

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