Analyzing Order Statistics of Non-Identically Distributed Shifted Exponential Variables in Numerical Data

Zakeia A. Al-Saiary

Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah, Saudi Arabia
Email: zaalsear@uj.edu.sa

Abstract: ***In this study, we analyze the order statistics (OS) arising from independent, non-identically distributed (INID) Shifted Exponential (SH-E) random variables. We deduce the moments of the rth order statistic, provide graphical representations of the probability density function (PDF) and cumulative distribution function (CDF) and present numerical examples, including cases where n = 3. This research offers insights into the distributional properties of SH-E variables and their practical applications in numerical data analysis.***

Keywords: Shifted Exponential Distribution, Order Statistics, Non-identically Distributed Variables, Numerical Data, Statistical Analysis

1. Introduction

In reliability engineering and survival analysis, the two-parameter exponential distribution or (Shifted Exponential distribution) is widely used in modelling the lifetime data. It offers a robust extension to the classical exponential model. Its applications span finance and environmental science. In the last few years research focused on estimation improvements, model selection techniques, and exploring its applications in emerging fields. For Example**, [1]** proposed the PDF and CDF of two parameter exponential distribution to discuss its key properties related to order statistics. **[2]** used stress-strength reliability as a measure to compare the lifetimes of two systems and inferred it for the two-parameter exponential distribution using generalized order statistics first without constraint on the location and scale parameters, second when the scale parameters are equal. **[3]** presented an interval estimation for the two-parameter Exponential distribution based on upper record values. **[4]** proposed an estimation method of process performance index for the two-parameter exponential distribution with measurement errors to fill this gap. **[5]** used the linear exponential loss function method to derive parameter estimators from the exponential distribution of two parameters on type I censored data. **[6]** introduced the Max-EWMA chart, employing maximum likelihood estimators and exponentially weighted moving average (EWMA) statistics, to jointly monitor SH-E distribution parameters. The probability density function PDF of two- parameter SH-E distribution is given by

$$f(x) = \alpha e^{-\alpha(x-\nu)},\ x \geq \nu,\ \alpha,\nu > 0. \qquad (1)$$

And the cumulative distribution function CDF of two - parameter SH-E distribution is given by

$$F(x) = 1 - e^{-\alpha(x-\nu)},\ x \geq \nu,\ \alpha,\nu > 0 \qquad (2)$$

Where α and ν are scale and location parameters respectively.

On the other hand, the subject on non-identical order statistics is explained widely in the literature in **[7], [8], [9], [10], [11], [12]** and **[13]** for continuous distributions. Also, in **[14]** for discrete distributions. Recently, many authors interested with it for example**: [15]** considered new ordering among continuous distribution functions of independent, but not necessarily identically distributed random variables INID and introduced new order statistics defined as the observations from these ordered distribution functions. **[16]** presented methods of obtaining single moments of order statistics arising from possibly dependent and non-identically distributed discrete random variables. **[17]** provided new upper and lower bounds for the $Fi{:}n$ Furthermore, new approximations for $F1{:}n$ and $Fn{:}n$, as well as for other cases, are derived. Comparisons with respect to approximations suggested by [15] are also provided**. [18]** studied order statistics OS from independent non identically distributed INID samples for two large classes of statistical distributions: Exponentiated Distributions (ED) and Proportional Hazard Rate Models (PHRM). **[19]** derived the explicit expressions for moments of order statistics under the independent identically distributed (IID) case and independent not identically distributed INID case.

This research is motivated by the lack of prior studies on order statistics for independent, non-identically distributed variables following the Shifted Exponential distribution, as well as the recurrence relationships to the single moments of the rth order statistics from this distribution was not previously introduced, in which it is difficult to find a closed form.

In section 2, we obtained the probability density function PDF and the cumulative distribution function CDF of the rth OS arising from INID SH-E distribution and their graphical representation at n = 3. Computation of moments of the rth OS of INID random variables arising from SH-E distribution are found in section 3.

From [10], [12] and [13], the PDF& CDF of the rth INID OS are written as:

$$f_{r:n}(x) = \frac{1}{(r-1)!\ (n-r)!} \sum_{p} \prod_{a=1}^{r-1} F_{i_a}(x)\ f_{i_r}(x) \prod_{c=r+1}^{n} \{1- F_{i_c}(x)\} \quad (3)$$

Where $\sum_{p}$ denotes the summation over all $n!$ permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots n)$.

[10] write it in the form of permanent as:

$$f_{r:n}(x) = \frac{1}{(r-1)!\ (n-r)!}\ per \left[\underbrace{F(x)}_{r-1} \quad \underbrace{f(x)}_{1} \quad \{1 - \underbrace{F}_{n-r}(x)\} \right] \quad (4)$$

$$F_{r:n}(x) = \sum_{j=r}^{n} \sum_{p_j} \prod_{a=1}^{j} F_{i_a}(x) \prod_{c=j+1}^{n} [1 - F_{i_c}(x)] \quad (5)$$

Where $\sum_{p_j}$ is all permutations of $(i_1, i_2, \cdots, i_n)$ for $(1, \cdots, n)$ which satisfy $i_1 < i_2 < \cdots < i_j$ and $i_{j+1} < i_{j+2} < \cdots < i_n$.

Using permanent, equation (5) written as:

$$F_{r:n}(x) = \sum_{i=r}^{n} \frac{1}{i!\ (n-i)!}\ per \begin{bmatrix} F_1(x) & 1 - F_1(x) \\ \vdots & \\ \underbrace{F_n(x)}_{i} & 1 - \underbrace{F_n}_{n-i}(x) \end{bmatrix}, -\infty < x < \infty \quad (6)$$

2. Non-identical order statistics from SH-E Distribution:

For $X_1, X_2, \cdots, X_n$ be INID random variables from SH-E distribution with parameters $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ respectively:

$$F(x_i) = 1 - e^{-\alpha_i (x-\nu)}, x \geq \nu\ \alpha_i, \nu > 0; i = 1, 2, \ldots, n \quad (7)$$

$$f(x_i) = \alpha_i e^{-\alpha_i (x-\nu)}, x \geq \nu, \alpha_i, \nu > 0\ ; i = 1, 2, \ldots, n \quad (8)$$

and by substituting the equations (7) and (8) in equations (3) and (5) we obtain the CDF & PDF of the rth OS of INID SH-E variables as:

$$f_{r:n}(x) = \frac{1}{(r-1)!\ (n-r)!} \sum_p \prod_{a=1}^{r-1} [1 - e^{-\alpha_{i_a}(x-\nu)}] \times \alpha_{i_r} e^{-(x-\nu)\sum_{a=1}^{n-r+1} \alpha_{i_a}} \quad (9)$$

$$F_{r:n}(x)(x) = \sum_{j=r}^{n} \sum_{p_j} \prod_{c=1}^{j} [1 - e^{-\alpha_{i_c}(x-\nu)}] \times e^{-(x-\nu)\sum_{a=j+1}^{n} \alpha_{i_a}(x-\nu)} \quad (10)$$

2.1 Special cases of PDF from INID SH-E distribution:

If we substituting Eq. (7) and (8) in (3) or (4), we will obtain the PDF of the first, second & the last order statistic from INID SH-E distribution when n = 3 as:

$$f_{1:3}(x) = \sum_{i=1}^{3} \alpha_i\ e^{-(x-\nu)\sum_{i=1}^{3}\alpha_i}; x \geq \nu$$

$$f_{2:3}(x) = \sum_{1 \leq i_1 < i_2 \leq 3} e^{-(x-\nu)(\alpha_{i_1} + \alpha_{i_2})} (\alpha_{i_1} + \alpha_{i_2}) - 2 \sum_{i=1}^{3} \alpha_i\ e^{-(x-\nu)\sum_{i=1}^{3}\alpha_i}, x \geq \nu$$

$$f_{3:3}(x) = e^{-x\sum_{i=1}^{3}\alpha_i} \sum_{\substack{j=1 \\ j \neq i_1, i_2}}^{3} \alpha_j\ e^{\nu\alpha_j} \sum_{1 \leq i_1 < i_2 \leq 3} (e^{x\alpha_{i_1}} - e^{\nu\alpha_{i_1}})(e^{x\alpha_{i_2}} - e^{\nu\alpha_{i_2}}), x \geq \nu$$

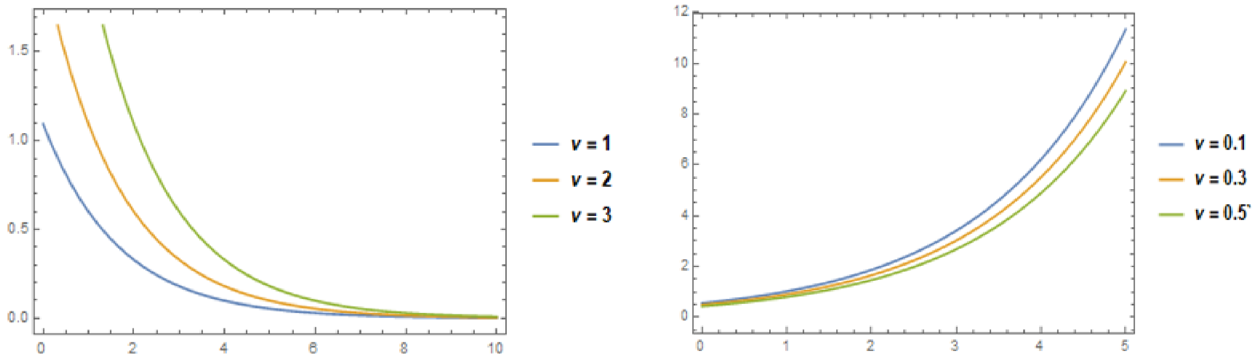

Figure 1: Graph of PDF's of the first INID SH-E OS at $\alpha_1 = 0.1,\ \alpha_2 = 0.2, \alpha_3 = 0.3$ and different values of ν

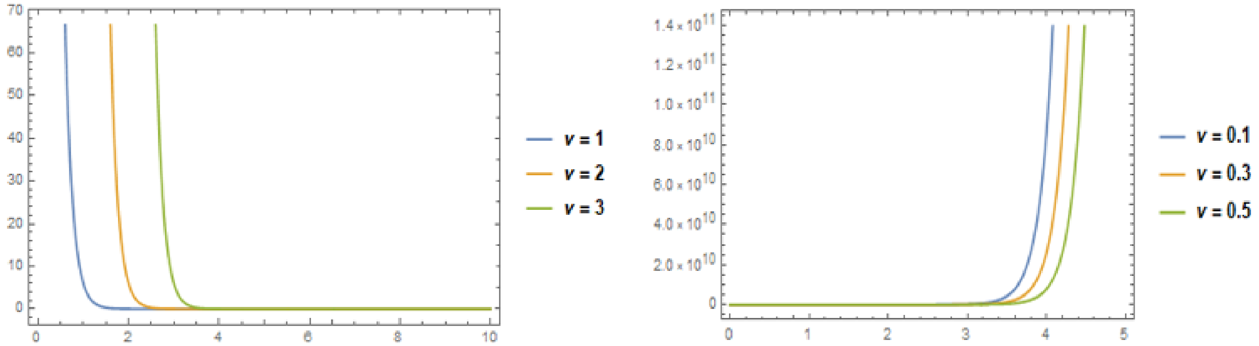

Figure 2: Graph of PDF's of the first INID SH-E OS at $\alpha_1 = 1,\ \alpha_2 = 2, \alpha_3 = 3$ and different values of ν

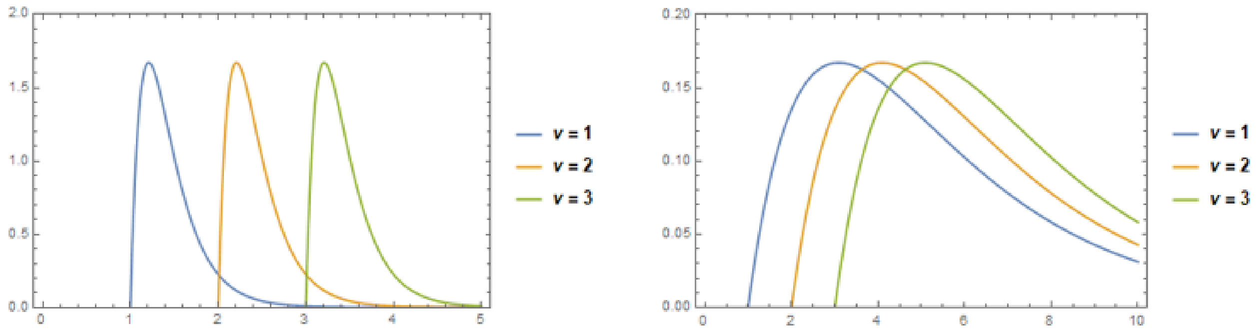

Figure 3: Graph of PDF's of the second INID SH-E OS at $\alpha_1 = 0.1,\ \alpha_2 = 0.2, \alpha_3 = 0.3$ and different values of ν

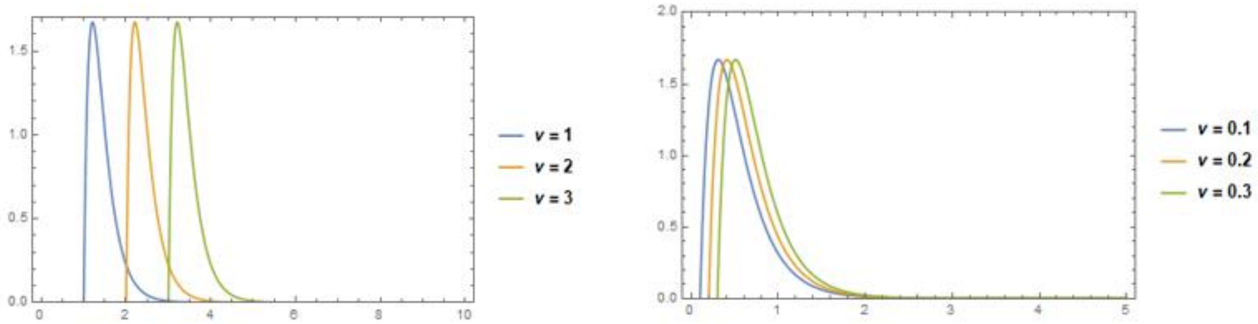

Figure 4: Graph of PDF's of the second INID SH-E OS at $\alpha_1 = 1,\ \alpha_2 = 2, \alpha_3 = 3$ and different values of ν

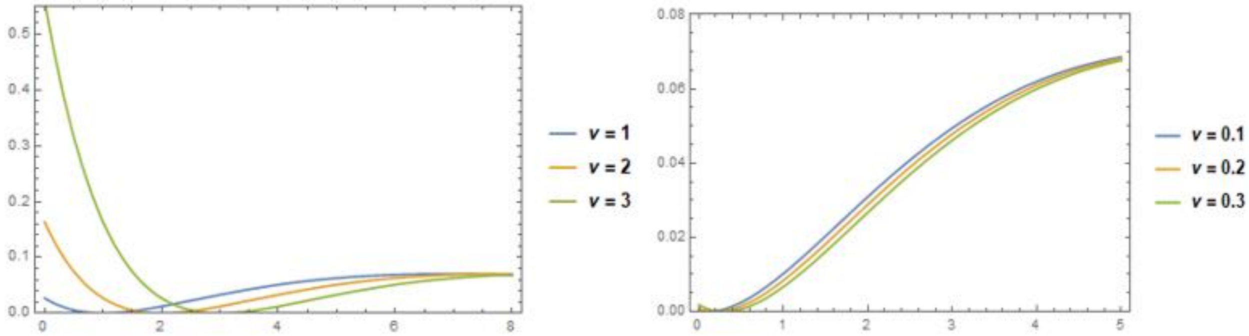

Figure 5: Graph of PDF's of the third INID SH-E OS at $\alpha_1 = 0.1,\ \alpha_2 = 0.2, \alpha_3 = 0.3$ and different values of ν

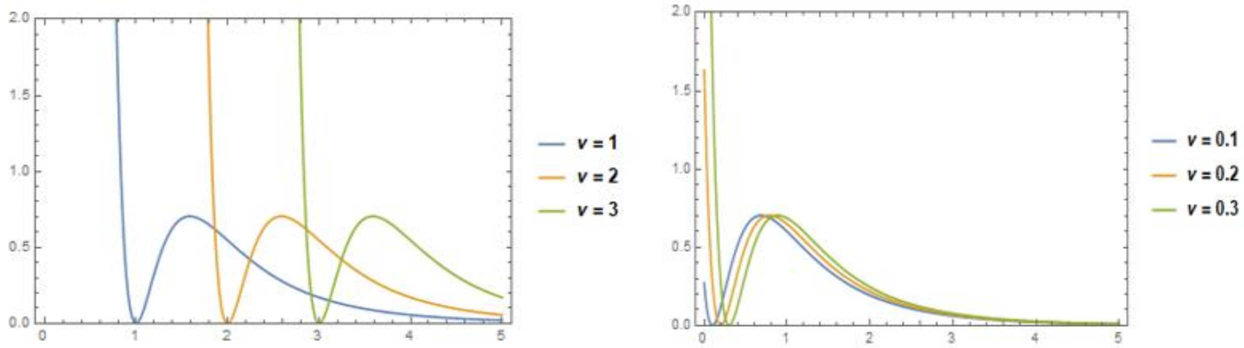

Figure 6: Graph of PDF's of the third INID SH-E OS at $\alpha_1 = 1,\ \alpha_2 = 2, \alpha_3 = 3$ and different values of ν

Figures (1&2) shows the curves of the pdf of the first INID SH-E Distribution for different values of the parameters ν. Figures (3&4) shows the curves of the pdf of the second INID INID SH-E Distribution for different values of the parameters ν. Figure (5&6) shows the curves of the pdf of the third INID SH-E Distribution for different values of ν.

2.2 Special cases of CDF of INID SH-E distribution:

If we are substituting equations (7) and (8) in (5) or (6), we will obtain the CDF of the first, second & the last order statistic from INID SH-E distribution when n = 3 as:

$$F_{1:3}(x) = 1 - e^{-(x-\nu)\sum_{i=1}^{3}\alpha_i}\ ;\ x \geq \nu$$

$$F_{2:3}(x) = 1 - \sum_{1 \leq i_1 < i_2 \leq 3}^{3} e^{-(x-\nu)(\alpha_{i_1}+\alpha_{i_2})} + 2e^{-(x-\nu)\sum_{i=1}^{3}\alpha_i}\ ;\ x \geq \nu$$

$$F_{3:3}(x) = \prod_{i=1}^{3}(1 - e^{-(x-\nu)\alpha_i});\quad x \geq \nu$$

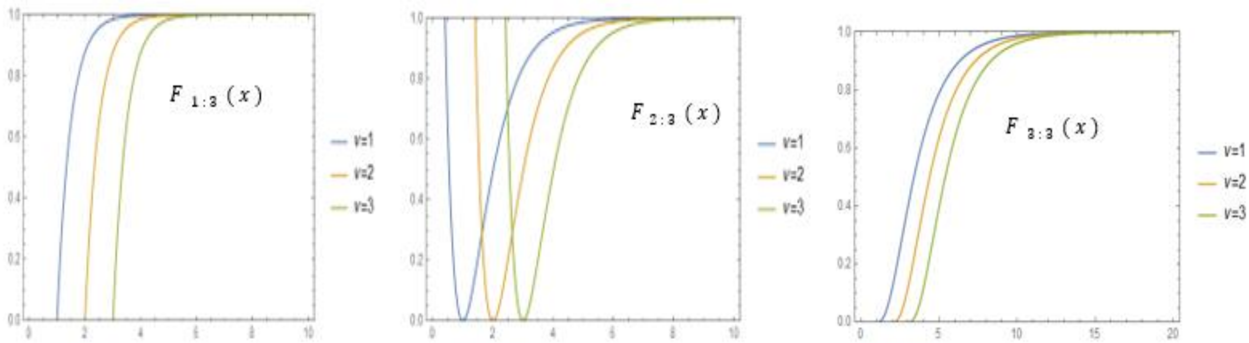

Figure 7: Graphs of CDF's of the first, second and third INID OS from SH-E distribution at $\alpha_1 = 0.5,\ \alpha_2 = 0.7, \alpha_3 = 0.9$ and different values of ν

3. The Recurrence Relation of the Single Moments of the rth OS arising from INID SH-E Random Variables

In this section, we will introduce a recurrence relation for the single moment of INID OS arising from Equation (7) and depending on the theorem in **[11].**

Theorem 3.1:

Let $X_1, X_2, \cdots, X_n$ be INID random variables from SH-E distribution, the k^th moment of all OS, $\mu_{r:n}^{(k)}$, for $1 \le r \le n$ and $k = 1, 2, \ldots$ is given by:

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} I_j(k) \qquad (11)$$

Where:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \sum_{s=0}^{k-1} k \binom{k-1}{s} \times \frac{\nu^{k-s-1} \Gamma(s+1)}{\left(\sum_{t=1}^{j} \alpha_{i_t}\right)^{s+1}} \qquad (12)$$

Proof:

In **[11]** Barakat & Abdelkader derived the single moments of INID OS and generalized this recurrence relation as equation (11)

Where:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_0^{\infty} x^{k-1} \prod_{t=1}^{j} G_{i_t}(x)\, dx, \quad j = 1, 2, \cdots, n \qquad (13)$$

And $G_{i_t}(x) = 1 - F_{i_t}(x)$, with $(i_1, i_2, \cdots, i_n)$ is a permutation of (1, 2, …, n) for which

$i_1 \le i_2 < \cdots < i_n$.

Now

Let $X_1, X_2, \cdots, X_n$ are INID OS from SH-E distribution, by substituting Equation (7) in equation (13), we obtain

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} k \times \int_{\nu}^{\infty} x^{k-1} \prod_{t=1}^{j} e^{-\alpha_{i_t}(x-\nu)}\, dx,$$

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} k \times \int_{\nu}^{\infty} x^{k-1} e^{-\sum_{t=1}^{j} \alpha_{i_t}(x-\nu)}\, dx, \qquad (14)$$

Using the transformation: $y = \sum_{t=1}^{j} \alpha_{i_t}(x - \nu)$ in equation (14), we obtain:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} k \times \int_0^{\infty} \left(\frac{y}{\sum_{t=1}^{j} \alpha_{i_t}} + \nu\right)^{k-1} e^{-y} \frac{dy}{\sum_{t=1}^{j} \alpha_{i_t}},$$

Using Binomial Expanding for the term: $\left(\frac{y}{\sum_{t=1}^{j} \alpha_{i_t}} + \nu\right)^{k-1}$ we deduce:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} k \sum_{s=0}^{k-1} \binom{k-1}{s} \times \frac{\nu^{k-s-1}}{\left(\sum_{t=1}^{j} \alpha_{i_t}\right)^{s+1}} \int_0^{\infty} y^s e^{-y}\, dy,$$

$$\therefore I_j(k) = k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \sum_{s=0}^{k-1} \binom{k-1}{s} \times \frac{\nu^{k-s-1} \Gamma(s+1)}{\left(\sum_{t=1}^{j} \alpha_{i_t}\right)^{s+1}}$$

Which gives the proof of the theorem.

Table 1: $I_j\,(k)$ using equation (12) when n = 3

j	$I_j\,(k)$
1	$I_1\,(k) = k \sum_{s=0}^{k-1} \binom{k-1}{s} \Gamma\,(s+1) \nu^{k-s-1} \left(\frac{1}{(\alpha_1)^{s+1}} + \frac{1}{(\alpha_2)^{s+1}} + \frac{1}{(\alpha_3)^{s+1}} \right)$
2	$I_2\,(k) = k \sum_{s=0}^{k-1} \binom{k-1}{s} \Gamma\,(s+1) \nu^{k-s-1} \left(\frac{1}{(\alpha_1+\alpha_2)^{s+1}} + \frac{1}{(\alpha_1+\alpha_3)^{s+1}} + \frac{1}{(\alpha_2+\alpha_3)^{s+1}} \right).$
3	$I_3\,(k) = k \sum_{s=0}^{k-1} \binom{k-1}{s} \Gamma\,(s+1) \nu^{k-s-1} \frac{1}{(\alpha_1+\alpha_2+\alpha_3)^{s+1}}.$

Corollary (3.1):

For the case of a sample of n (IID) random variables having closed CDF in equation (7) the $I_j\,(k)$ in theorem (3.1) simply reduces to

$$I_j\,(k) = k \binom{n}{j} \sum_{s=0}^{k-1} \binom{k-1}{s} \frac{\nu^{k-s-1}\Gamma\,(s+1)}{(j\,\alpha)^{s+1}} \qquad (15)$$

Table 2: $I_j\,(k)$ of IID case using equation (15) when n = 3

j	$I_j\,(k)$
1	$I_1\,(k) = n \sum_{s=0}^{k-1} k \binom{k-1}{s} \frac{\nu^{k-s-1}\Gamma\,(s+1)}{(\alpha)^{s+1}}$
2	$I_2\,(k) = \binom{n}{2} \sum_{s=0}^{k-1} k \binom{k-1}{s} \frac{\nu^{k-s-1}\Gamma\,(s+1)}{(2\,\alpha)^{s+1}}$
3	$I_3\,(k) = \binom{n}{3} \sum_{s=0}^{k-1} k \binom{k-1}{s} \frac{\nu^{k-s-1}\Gamma\,(s+1)}{(3\,\alpha)^{s+1}}$

Table 3: $\mu_{r:n}^{(k)}$ of INID OS using equation (12)

r \ n	1	2	3
1	$I_1\,(k)$	$I_2\,(k)$	$I_3\,(k)$
2		$I_1\,(k) - I_2\,(k)$	$I_2\,(k) - 2\,I_3\,(k)$
3			$I_1\,(k) - I_2\,(k) + I_3\,(k)$

Table 4: $E\,(x_{r:3})$, $E\,(x_{r:3}^2)$ and $V\,(x_{r:3})$ of INID SH-E distribution using equations (11&12) or Tables (1&3)

r	$E\,(x_{r:3})$	$E\,(x_{r:3}^2)$	$V\,(x_{r:3})$
1	$\frac{1}{\sum_{i=1}^{3}\alpha_i}$	$\frac{2\,\nu \sum_{i=1}^{3}\alpha_i + 2}{(\sum_{i=1}^{3}\alpha_i)^2}$	$\frac{2\,\nu \sum_{i=1}^{3}\alpha_i + 1}{(\sum_{i=1}^{3}\alpha_i)^2}$
2	$\frac{1}{\alpha_1+\alpha_2} + \frac{1}{\alpha_1+\alpha_3} + \frac{1}{\alpha_2+\alpha_3} - \frac{2}{\sum_{i=1}^{3}\alpha_i}$	$2\,\nu \left(\frac{1}{\alpha_1+\alpha_2} + \frac{1}{\alpha_1+\alpha_3} + \frac{1}{\alpha_2+\alpha_3} \right) + 2 \left(\frac{1}{(\alpha_1+\alpha_2)^2} + \frac{1}{(\alpha_1+\alpha_3)^2} + \frac{1}{(\alpha_2+\alpha_3)^2} \right) - \frac{4\,\nu \sum_{i=1}^{3}\alpha_i + 4}{(\sum_{i=1}^{3}\alpha_i)^2}$	$2\,\nu \left(\frac{1}{\alpha_1+\alpha_2} + \frac{1}{\alpha_1+\alpha_3} + \frac{1}{\alpha_2+\alpha_3} \right) + 2 \left(\frac{1}{(\alpha_1+\alpha_2)^2} + \frac{1}{(\alpha_1+\alpha_3)^2} + \frac{1}{(\alpha_2+\alpha_3)^2} \right) - \frac{4\,\nu \sum_{i=1}^{3}\alpha_i + 4}{(\sum_{i=1}^{3}\alpha_i)^2} - \left(\frac{1}{\alpha_1+\alpha_2} + \frac{1}{\alpha_1+\alpha_3} + \frac{1}{\alpha_2+\alpha_3} - \frac{2}{\sum_{i=1}^{3}\alpha_i} \right)^2$
3	$\sum_{i=1}^{3}\frac{1}{\alpha_i} - \frac{1}{\alpha_1+\alpha_2} - \frac{1}{\alpha_1+\alpha_3} - \frac{1}{\alpha_2+\alpha_3} + \frac{1}{\sum_{i=1}^{3}\alpha_i}$	$\sum_{i=1}^{3}\frac{2\,\nu}{\alpha_i} + \sum_{i=1}^{3}\frac{2}{(\alpha_i)^2} - \left[2\,\nu \left(\frac{1}{\alpha_1+\alpha_2} + \frac{1}{\alpha_1+\alpha_3} + \frac{1}{\alpha_2+\alpha_3} \right) + \left(\frac{2}{(\alpha_1+\alpha_2)^2} \frac{2}{(\alpha_1+\alpha_3)^2} \frac{2}{(\alpha_2+\alpha_3)^2} \right) \right] + \frac{2\,\nu \sum_{i=1}^{3}\alpha_i + 2}{(\sum_{i=1}^{3}\alpha_i)^2}$	$\sum_{i=1}^{3}\frac{2\,\nu}{\alpha_i} + \sum_{i=1}^{3}\frac{2}{(\alpha_i)^2} - \left[2\,\nu \left(\frac{1}{\alpha_1+\alpha_2} + \frac{1}{\alpha_1+\alpha_3} + \frac{1}{\alpha_2+\alpha_3} \right) + \left(\frac{2}{(\alpha_1+\alpha_2)^2} \frac{2}{(\alpha_1+\alpha_3)^2} \frac{2}{(\alpha_2+\alpha_3)^2} \right) \right] + \frac{2\,\nu \sum_{i=1}^{3}\alpha_i + 2}{(\sum_{i=1}^{3}\alpha_i)^2} - \left(\sum_{i=1}^{3}\frac{1}{\alpha_i} - \frac{1}{\alpha_1+\alpha_2} - \frac{1}{\alpha_1+\alpha_3} - \frac{1}{\alpha_2+\alpha_3} + \frac{1}{\sum_{i=1}^{3}\alpha_i} \right)^2$

4. Numerical applications:

Table 5: The values of $\mu_{2:2}^{(1)}$ arising from INID SH-E using equations (11&12)

α_2 \ α_1	0.5	1	1.5	2	2.5	3
1	2.33333	1.50000	1.266670	1.16667	1.11429	1.083330
1.5	2.16667	1.26667	1.000000	0.880952	0.816667	0.777778
2	2.10000	1.16667	0.880952	0.750000	0.677778	0.633333
2.5	2.06667	1.11429	0.816667	0.677778	0.600000	0.551515
3	2.04762	1.08333	0.777778	0.633333	0.551515	0.500000

Table 6: The values of $\mu_{r:3}^{(k)}(x)$ and $\boldsymbol{V}(\boldsymbol{x}_{r:3})$ arising from INID SH-E distribution for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3; \nu = 2$ using equations (11&12) or Table (3)

r	$E(x_{r:3})$	$E(x_{r:3}^2)$	$V(x_{r:3})$
1	0.166667	0.722222	0.69444
2	0.4500000	2.116110	1.913610
3	1.216670	6.627220	5.146940

5. Conclusion

The study of order statistics for independent but non-identically distributed random variables INID has gained renewed interest with the emergence of new distributions, particularly in the pursuit of calculating moments for these distributions. I encourage researchers to continue exploring methods for determining moments, especially within various distributions, using diverse approaches. Future studies could explore the application of these findings in real-world scenarios such as reliability engineering and predictive analytics.

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