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Examining Hyponormality in Toeplitz Operators: Non-Harmonic Symbols on Weighted Bergman Spaces

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Abstract: ***In this study, we explore the hyponormality of Toeplitz operators with non-harmonic symbols within the framework of weighted Bergman spaces. By examining necessary and sufficient conditions, we demonstrate that the Toeplitz operator is hyponormal when particular constants and functions meet the defined criteria. Further, we analyze scenarios in which Toeplitz operators become normal and hyponormal based on operator symbols and coefficients, providing both theoretical and practical insights.***

Keywords: Toeplitz operators, Hyponormality, non-harmonic symbols, Bergman spaces, weighted spaces.

1. Introduction

For $\varphi \ in \ L^{\infty}(\partial D)$, the Toeplitz operator T_{φ} is the operator on H^2 of the unit disk D defined by $T_{\varphi}u = P\varphi u$ where P is the orthogonal projection $L^2(\partial D)$ of on to H^2 .

An operator A is called hyponormal if its self- commutator $A^*A - AA^*$ is positive. On Bergman space $A^2(D)$ of analytic functions in $L^2(D)$ we defined, the Toeplitz operator T_{φ} by $T_{\varphi}f = P(\varphi f) \ for \ \varphi \in L^{\infty}(D)$, and $f \in A^2(D)$, and P is the orthogonal projection maps $L^2(D)$ onto $A^2(D)$.

Lastly, for $-1 < \alpha < \infty$, the weight Bergman space of all analytic functions in $L^2(D, dA_{\alpha})$, where $dA_{\alpha}(z) = (\alpha + 1)\,(1 - |z|^2)\, dA(z)$

Given abounded measurable function $\varphi \ in \ L^{\infty}(\partial D)$, the Toeplitz operator T_{φ} is the operator with the symbol $\varphi \ on \ A^2_{\alpha}(D)$ is defined by
$T_{\varphi}g := P(\varphi . g) \quad (g \in A^2_{\alpha})$

Where P is the orthogonal projection from $L^2(D, dA_{\alpha}) \ on \ to \ A^2_{\alpha}(D)$.

For $\varphi \ in \ L^{\infty}(\partial D)$, the Toeplitz operator T_{φ} is the operator on H^2 of the unit disk D defined by $T_{\varphi}u = P\varphi u$ where P is the orthogonal projection $L^2(\partial D)$ of on to H^2 . For an operator A is called hyponormal if its self- commutator $A^*A - AA^*$ is positive.

Aims
This paper aims to investigate the conditions under which Toeplitz operators with non-harmonic symbols exhibit hyponormality within weighted Bergman spaces.

Brown and Halmos began study algebraic properties of Toeplitz operators and in [4, p.98] , that T_{φ} is normal if and only if $\varphi = \alpha + \beta\rho$ where $\alpha \ and \ \beta$ are complex numbers and ρ is a real valued function in L^{∞} .

Properties of hyponormal Toeplitz operators have played an important role in work on Halmos's problem 5, [31], "is every subnormal Toeplitz operator either normal or analytic? "But characterization has been lacking.

In [5,12,13], and [15], the basic properties of the Bergman space and the Hardy space are well known.

The hyponormality of Toeplitz operators on the Hardy space has been developed in [9,8,19], and [30].

In [16], Cowen characterized the hyponormality of Toeplitz operator T_{φ} on H^2 by the properties of the symbols $\varphi \ in \ L^{\infty}(\pi)$ .Cowen's method is to reconstruct the operator-Theoretic problem of hyponormal Topelitz operator into the problem of finding a solution of equations of functionals. Recently, in [20,17], the authors characterized the hyponormality of Toeplitz operators on Bergman space with harmonic symbols.

The hyponormality of Toeplitz operators has been researched in [9,10,19]. In [29], the auther characterized the hyponormality T_{φ} on $H^2(\pi)$ of by using the properties of symbols $\varphi \ in \ L^{\infty}(\pi)$ as follows:

Cowen's theorem [8] for $\varphi \ in \ L^{\infty}(\pi)$, let
$\varepsilon(\varphi) := \{k \in H^{\infty} : \|k\|_{\infty} \leq 1 \ and \ \varphi - k\bar{\varphi} \in H^{\infty}(\pi)\}$.

Then is hyponormal if and only if $\varepsilon(\varphi)$ is nonempty.

The main idea of the proof of Cowen's Theorem is dilation theorem as in [34].

However, Cowen's Theorem cannot be utilized to $A^2_{\alpha}(D)$. So, for the weighted Bergman space, determining the Toeplitz operators Very difficult.

In fact, there seems to be very little study of hyponormal Toeplitz operators on$A^2_{\alpha}(D)$ in the literature. In [18, 16, 34, 35, 28], the Authors charctrized the hyponormality of Toeplitz operator T_{φ} in terms of the coefficient of the symbols φ under certain Assumptions on $A^2_{\alpha}(D)$. Moreover, since the hyponormal Toeplitz operators on$A_\alpha^2(D)$ is translation invariant, we can assume that the constant terms is zero.

This research contributes to the ongoing understanding of operator theory by characterizing the hyponormality conditions of Toeplitz operators, which has implications for further applications in functional and complex analysis.

2. Hyponormality of Toeplitz operators

Theorem (2.1): T_φ is normal if and only $\varphi = \alpha + \beta\rho$ where α and β are complex numbers and ρ is a real valued function in L^∞.

Definition (2.2): Let ψ is in L^∞ , the Hankel operator H_Ψ is operator on H^2 given by

$$H_\psi u = J(I-p)(\psi u) \ ,$$

Where J is unitary operator from H^2 onto $H^{2^\perp}$, $J\left(e^{-in\theta}\right) = e^{i(n-1)\theta}$.

Another way: Let v^*be a function define by $v^*\left(e^{i\theta}\right) = \overline{v(e^{-i\theta})}$, then H_Ψ is the operator on H^2 defined by

$$\langle 2uv, \overline{\psi}\rangle = \langle H_\psi u, v^*\rangle \text{ , for all } v \in H^\infty \qquad (1).$$

Remark (2.3): Necessary facts about Hankel operators include.

1) $H_{\psi_1} = H_{\psi_2}$ if and only if $(I-p)\psi_1 = (I-p)\psi_2$.
2) $\|H_\psi\| = inf\{\|\phi\|_\infty : (I-p)\psi = (I-p)\phi\}$
3) $H_\psi^* = H_{\psi^*}$.
4) Either H_ψ is one – to – one or ker $\left(H_\psi\right) = xH^2$ where $\chi$is an inner function. The closure of the range of H_Ψ is H^2 in the former case and $(\chi^* H^2)^\perp$in the latter.
5) $H_\psi U = U^* H_\psi$. Where U is unilateral (forward) shift on H^2.

Theorem (2.4): If ϕ is in $L^\infty(\partial D)$, where $\phi = f + \bar{g}$ for f and $g$in H^2 , then T_ϕ is hyponormal if and only if

$$g = cT_{\bar{h}}f$$

For some constant c and some function $h$in $H^\infty(\partial D)$ with $\|h\|_\infty \le 1$.

Proof: Let $\phi = f + \bar{g}$ where f and $g$are in H^2 . For every polynomial p in H^2 ,

$$\begin{aligned}
&\langle (T_\phi^* T_\phi - T_\phi T_\phi^*)(p), p\rangle = \langle T_\phi p, T_\phi p\rangle - \langle T_\phi^* p, T_\phi^* p\rangle \\
&= \langle fp + P\bar{g}p, fp + P\bar{g}p\rangle \\
&\qquad - \langle p\overline{f}p + gp, p\overline{f}p + gp\rangle \\
&= \langle \overline{f}p, \overline{f}P\rangle - \langle P\overline{f}P, P\overline{f}P\rangle - \langle \bar{g}p, \bar{g}p\rangle \\
&\qquad + \langle P\bar{g}p, P\bar{g}P\rangle \\
&= \langle \overline{f}P, (I-p)\overline{f}p\rangle - \langle \bar{g}P, (I-p)\bar{g}p\rangle \\
&= \langle (I-p)\overline{f}p, (I-p)\overline{fp}\rangle \\
&\qquad - \langle (1-p)\bar{g}p, (I-p)\bar{g}p\rangle \\
&= \left\|H_{\overline{f}}P\right\|^2 - \left\|H_{\overline{g}}P\right\|^2
\end{aligned}$$

Since the polynomial are dense in H^2 and since the Hankel and Toeplitz operators involved are bounded, then $T_\phi$is hyponormal if and only if for all u in H^2 ,

$$\left\|H_{\overline{g}}u\right\| \le \left\|H_{\overline{f}}u\right\| . \qquad (2)$$

Let k denote the closure of the range of $H_{\overline{f}}$, and let S denote the compression of U to K. Since k is invariant for U^* , the operator $S^*$is the restriction of $U^*$to k .

Suppose first T_ϕ is hyponormal. Define an operator A on the range of $H_{\overline{f}}$ by

$$A\left(H_{\overline{f}}u\right) = H_{\overline{g}}u.$$

If $H_{\overline{f}}u_1 = H_{\overline{f}}u_2$, so that $H_{\overline{f}}(u_1 - u_2) = 0$, then the inequality (2) implies that $H_{\overline{g}}(u_1 - u_2) = 0$ too and it follow that A is well defined. Moreover, inequality (2) implies $\|A\| \le 1$ so A has an extension to k, which will also be denoted A. With the same norm.

Now by the intertwining formula for Hankel operators and the fact that k is invariant for U^*, we have

$$H_{\overline{g}}U = AH_{\overline{f}}U = AU^* H_{\overline{f}} = AS^* H_{\overline{f}}$$

And also

$$H_{\overline{g}}U = U^* H_{\overline{g}} = U^* AH_{\overline{f}} = S^* AH_{\overline{f}} .$$

Since the range of $H_{\overline{f}}$ is dense in k , we find that $AS^* = S^*A$on k , or taking adjoints , that
$SA^* = A^*S$.

By the usual theory (2.2.8) of the unilateral shift $k = H^2$ there is function k in $H^\infty(\partial D)$ with $\|k\|_\infty = \|A^*\| = \|A\|$ such that $A^*$is the compression to k of T_k . Since $k$is invariant for $T_k^* = T_{\bar{k}}$this means that $A^*$is the restriction of $T_{\bar{k}}$to K and

$$H_{\overline{g}} = T_{\bar{k}} \rightleftarrows H_{\overline{f}} \qquad (3)$$

Conversely, if equation (3) holds for some k in $H^\infty(\partial D)$ with $\|k\|_\infty \le 1$, then clearly inequality (2) holds for all u_1, and T_ϕ is hyponormal. By using the formulation (1), equation (3) holds if and only for all H^∞ functions , v ,

$$\begin{aligned}
\langle zuv, g\rangle &= \langle H_{\overline{g}}u, v^*\rangle = \left\langle T_{\bar{k}} H_{\overline{f}}u, v^*\right\rangle \\
&= \left\langle H_{\overline{f}}u, kv^*\right\rangle = \langle zuk^* v \rightleftarrows f\rangle \\
&= \langle zuv, \overline{k^*}f\rangle = \langle zuv, T_{\overline{k^*}}f\rangle
\end{aligned}$$

Since the closed span of $\{zuv: u, v \in H^\infty\}$ is zH^2 this means that equation (3) holds if and only if

$$g = c + T_{\bar{h}}f$$

For $h = k^*$ (Note that $\|k\|_\infty = \|k^*\|_\infty$)

In the cases for which T_ϕ is normal, h is a constant of modulus 1 and in the cases for which T_ϕ is known to be subnormal but not normal h is a constant of modulus less than 1.

Remark (2.5): The function h that relate f and g is unique.

Proof: Suppose $h_1$and h_2 are in H^∞ and$c_1 + T_{\overline{h_1}}f = g = c_2 + T_{\overline{h_2}}f$. This is possible if and only if
$T_{\bar{z}}T_{\overline{h_1}}f = T_{\bar{z}}T_{\overline{h_2}}f$,

That is, if and only if

$$T_{\overline{zh_1 - zh_2}}f = 0$$

Thus, f must be in $(z\chi H^2)^\perp$ where $\chi$is the inner factor of $h_1 - h_2$. If f is not in any such subspace, the corresponding function h must be unique for every g.

On the other hand, if χ is an inner function such that f is in $(z\chi H^2)^\perp$ and$c_1 + T_{\overline{h_1}}f = g$, then for any h_3 in H^∞ and $h_2 = h_1 + z\chi h_3$, if follows that $g = c_2 + T_{\overline{h_2}}f$ for some constant c_2.

Definition (2.6): Let $H = \{v \in H^\infty : v = (0) = 0$ and $\|v\|_2 \le 1\}$. For f in H^2 and let G_f denote the set of g in H^2 such that for every u in H^2,

$$\sup_{v_0 \in H} |\langle uv_0, g\rangle| \le \sup_{v_0 \in H} |\langle uv_0, f\rangle|$$

If f is in H^∞ and u is in H^2 , then by (1),

$$\sup_{v_0 \in H} |\langle uv_0, f\rangle| = \left\|H_{\bar{f}}u\right\|$$

Theorem (2.7): If $f$and g are in H^2, then g is in G_f if and only if

$$g = c + T_{\bar{h}}f\ .$$

For some constant c and some function h in $H^\infty(\partial D)$ with $\|h\|_\infty \le 1$.

Proof: Let $\phi = f + \bar{g}$ where $f$and $g$are in H^2 . From [6], for every polynomial pin H^2 , $\langle (T_\phi^* T_\phi - T_\phi T_\phi^*)(p), p\rangle = \left\|H_{\bar{f}}P\right\|^2 - \left\|H_{\bar{g}}P\right\|^2$.

Since the polynomials are dense in H^2 and since the Hankel and Toeplitz operators involved are bounded, we see that T_ϕ is hypomormal if and only if for all u in H^2,
$\left\|H_{\bar{g}}u\right\| \le \left\|H_{\bar{f}}u\right\|$

Let $k$denote the closure of the range of $H_{\bar{f}}$and let S denote the compression of $U$to. Since K is invariant for U^*, the operator $S^*$is the restriction of U^* to K .

Suppose first that T_ϕ is hyponormal. Define an operator A on the range of $H_{\bar{f}}$ by

$$A\left(H_{\bar{f}}u\right) = H_{\bar{g}}u.$$

If $H_{\bar{f}}u_1 = H_{\bar{f}}u_2$, so that $H_{\bar{f}}(u_1 - u_2) = 0$, then the inequality (2) implies that $H_{\bar{g}}(u_1 - u_2) = 0$too and it follows that A is well defined. Morover (2) implies $\|A\| \le 1$ so$A$ has an extension to, which will also be denoted$A$, with the same norm.

Now by the intertwining formula for Hankel operators and the fact that$K$ is invariant for U^*, we have

$$H_g U = AH_{\bar{f}}U = AU^*H_{\bar{f}} = AS^*H_{\bar{f}}$$

And also
$H_{\bar{g}}U = U^*H_{\bar{g}} = U^*AH_{\bar{f}} = S^*AH_{\bar{f}}$.

Since the range of $H_{\bar{f}}$ is dense in, we find that $AS^* = S^*A$on K, or taking adjoints, that $SA^* = A^*S$.

By the usual theory of the unilateral shift if $= H^2$, there is a function k in $H^2(\partial D)$with $\|K\|_\infty = \|A^*\| = \|A\|$ such that $A^*$is the compression to K of T_k. Since K is invariant for $T_k^* = T_{\bar{k}}$. This means that A is the restriction of $T_{\bar{k}}$to K and
$H_{\bar{g}} = T_{\bar{k}}H_{\bar{f}}$,

is holds if and only if for all H^∞functions u, v,
$\langle zuv, g\rangle = \left\langle H_{\bar{g}}u, v^*\right\rangle = \left\langle T_{\bar{k}}H_{\bar{f}}u, V^*\right\rangle$
$= \left\langle H_f u, kv^*\right\rangle = \langle zuk^*v, f\rangle$
$= \left\langle zuv, \bar{k}^*f\right\rangle = \langle zuv, T_{\bar{k}}f\rangle$
by using relation (1) $\langle zuv, T_{\bar{k}^*}f\rangle = T_{\bar{k}}H_{\bar{f}} = H_{\bar{g}}$.

Now by using the definition there exist
$H = \{v \in H^\infty : v(0) = 0$ and $\|v\|_2 \le 1\}$.

If f is in $H^\infty$and u is in H^2 , then by relation (1)
$\sup_{v \in H} |\langle uv, f\rangle| = \left\|H_{\bar{f}}u\right\|$.

Thus (2) holds then g is in $G_f$definition such that for every u in H^2,
$\sup_{v \in H} |\langle uv, g\rangle| \le \sup_{v \in H} |\langle uv, f\rangle|$

Thus $T_\phi$is hyponormal , then from theorem (1)
$g = c + T_h f$, for $h = k^*(\|k\|_\infty = \|k^*\|_\infty)$.

Corollary (2.8): For $f$in H^2 , the following hold .
(i) f is in G_f.
(ii) If g is in G_f , then $g + \lambda$ is in G_f for all complex numbers λ.
(iii) $G_f$is balanced and convex, that is , if g_1 and $g_2$are in $G_f$and $|s_1| + |s_2| \le 1$, then $s_1 g_1 + s_2 g_2$ is also in G_f.
(iv) G_f is weakly closed.
(v)$T_{\overline{\chi}}G_f \subset G_f$ for every inner function χ.

Conversely, If $G$is a set that satisfies properties (i) to (v) , then $G \supset G_f$.

3. Toeplitz Operators with Non-Harmonic Symbols

Now consider the symbol φ of the form:
$\varphi(z) = a_{mn} z^m z^{-n}\ with\ m, n \in C$

Theorem (3.1): Let $\varphi(z) = a_{mn} z^m z^{-n}\ with\ m, n \in C$.
Then $T_\varphi\ on\ A^2(D)$ is hyponormal if and only if $m \ge n$.

Proof: If $m > n, then\ T_\varphi$ is hyponormal. Suppose that T_φ is hyponormal. By the definition of hyponormal Toeplitz operators , T_φ is hyponormal if and only if

$$\langle (T_\varphi^* T_\varphi - T_\varphi T_\varphi^*) \sum_{i=0}^{\infty} c_i z^i ,\quad \sum_{i=0}^{\infty} c_i z^i\rangle \ge 0$$

For all$c_i \in C$ we have:

$$\|T_\varphi \sum_{i=0}^{\infty} c_i z^i\ \|^2 - \|T_\varphi^* \sum_{i=0}^{\infty} c_i z^i\ \|^2$$

$$= \|T_{a m,n z^m z^{-n}} \sum_{i=0}^{\infty} c_i z^i \|^2 - \|T_{\underline{a} m,n z^{-m} z^n} \sum_{i=0}^{\infty} c_i z^i\ \|^2$$

$$= \|P\left(am,n^{z^m z^{-n}} \quad \sum_{i=0}^{\infty} c_i z^i\right)\|^2 - \|P\left(\underline{a}m,n^{z^{-m}z^n} \quad \sum_{i=0}^{\infty} c_i z^i\right)\|^2$$

$$= |a_{m,n}|^2 \sum_{i=max\{n-m,0\}}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 - |a_{m,n}|^2 \sum_{i=max\{m-n,0\}}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2 \geq 0.$$

Hence, T_φ is hyponormal if and only if

$$\sum_{i=max\{n-m,0\}}^{\infty} \frac{m+i-n+1}{(m+i+1)^2}|c_i|^2 \geq \sum_{i=max\{m-n,0\}}^{\infty} \frac{n+i-m+1}{(n+i+1)^2}|c_i|^2$$

For all$c_i \in C$ .Since 1 are arbitrary, we conclude that T_φ is hyponormal if and only if $m \geq n$.

Consider the hyponormality of Toeplitz operator with two terms non-harmonic symbols:

Theorem (3.2): Let $\varphi(z) = az^m z^{-n} + bz^n z^{-m}$. with nonnegative integers m, n , with $m \geq n$ and nonzerors $a, b \in C$. . Then T_φ on $A^2(D)$ is hyponormal if and only if $|a| \geq |b|$

Proof: In similar way to the proof of above theorem (3.1).

The following theorem is general characterization of hyponormal Toeplitz operators with the symbols of the form:

$$\varphi(z) = az^m z^{-n} + bz^s z^{-t}$$

$(m \geq n \geq 0, t \geq s \geq 0)$ *with some condions.*

Theorem (3.3): Let $\varphi(z) = az^m z^{-n} + bz^s z^{-t}$ with nonnegative integers m, n, s, t *with* $m \geq n, t \geq s, m \neq t, m-n = t-s$ and nonzeros $a, b \in C$. If T_φ on $A^2(D)$ is hyponormal then,

$$\{|a|^2 \geq \left\{\frac{(2m-n)^2}{(t+m-n)^2}, \wedge (m,n,t,s)\right\} |b|^2 \text{ if } t > m, |a|^2 \geq \left\{\frac{(m+1)^2}{(t+1)^2}, \wedge (m,n,t,s)\right\} |b|^2 \text{ if } t < m,$$

Where $\wedge (m,n,t,s) = max_{i \in [m-n,\infty)} \frac{\frac{(t+i-s+1)}{(t+i+1)^2} - \frac{(s+i-t+1)}{(s+i+1)^2}}{\frac{(m+i-n+1)}{(m+i+1)^2} - \frac{(n+i-m+1)}{(n+i+1)^2}}$.

Proof: Similarly, to the proof above.

4. Hyponormal Toeplitz operators with non-harmonic symbols on the weighted Bergman spaces

For $-1 < \alpha < \infty$, the weight Bergman space $A^2_\alpha(D)$ is defined by the space of all analytic functions in $L^2(D, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)\, dA(z)$$

If $f, g \in L^2(D, dA_\alpha)$, we have (the inner product)

$$\|f\| = \left(\int_D |f(z)|^2 dA_\alpha(z)\right)^{\frac{1}{2}}$$

The space $L^2(D, dA_\alpha)$ is a Hilbert space with the above inner product.

For any non-negative n *and* $z \in D$,let

$$e_n(z) = \sqrt{\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\,\Gamma(\alpha+2)}}\; z^n,$$

Where $\Gamma(.)$ is the usual Gamma function. Then $\{e_n\}$ is an orthogonal basis for the weight Bergman spaces.
If $f, g \in A^2_\alpha$ of the form:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

Then, the inner product of f *and* g is:

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\,\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} a_n \underline{b_n}$$

Let φ *in* $L^\infty(\partial D)$ be a bounded measureable function, the Toeplitz operator T_φ with symbol φ *on* $A^2_\alpha(D)$ is defined by:

$$T_\varphi g \coloneqq P(\varphi . g)\ (\ g \in A^2_\alpha(D))$$

Where P is the orthogonal projection from $L^2(D, dA_\alpha)$ *on* $A^2_\alpha(D)$.

Next, we present the Sufficient condition for hyponormal Toeplitz operators T_φ *on* $A^2_\alpha(D)$, where φ is of the form:

$$\varphi(z) = az^m z^{-n} + bz^s z^{-t}$$

under certain assumptions about the coefficients and degree of $\varphi(z)$

Theorem (4.1): For any $a, b \in C$ and nonnegative integers m, n, s, t, if $\varphi(z) = az^m z^{-n} + bz^s z^{-t}$ *with* $m-n = s-t \geq 0$

Then T_φ *on* $A^2_\alpha(D)$is hyponormal.

Example (4.2): If $\varphi(z) = i\, z^2 \underline{z} + (2+i)z^3 z^{-2}$, *then , by Theorm.* $(2,7), T_\varphi$ *on* $A^2_\alpha(D)$

Example (4.3): Let $\varphi(z) = r_1\, e^{i\theta_1} z^m z^{-n} + r_2\, e^{i\theta_2} z^s z^{-t}$
With $m-n = s-t \geq 0.$ *If* $|\theta_1 - \theta_2| < \frac{\pi}{2}$
Then T_φ *on* $A^2_\alpha(D)$ is hyponormal.

Next, we present the necessary condition for hyponormal Toeplitz operatorsT_φ with the symbol of the form:
$\varphi(z) = az^m z^{-n} + bz^s z^{-t}$ *and* $t \geq s$.

Theorem (4.4): Let $\varphi(z) = az^m z^{-n} + bz^s z^{-t}$ $with\ m \geq n, t \geq s\ , and\ a, b\ \in C$

If $m - n = t - s\ and\ m \neq t\ , T_\varphi\ on\ A_\alpha^2(D)\ is\ hyponormal\ then$

$$\{|a|^2 \geq max\{\frac{\wedge_\alpha\,(2t-s-1,s)}{\wedge_\alpha\,(2m-n-1,n)}, W\,(m,n\,,t\,,s)\}|b|^2 if\ t > m\ |a|^2\ \geq max\{\frac{\wedge_\alpha\,(t,s)}{\wedge_\alpha\,(m,n)}, W\,(m,n\,,t\,,s)\}|b|^2 if\ t < m$$

Where $W\,(m,n\,,t\,,s)\} = max_{i\in\{\,m-n\,,\infty)}\frac{\wedge_\alpha(t+i,s)-\wedge_\alpha(s+i,t)}{\wedge_\alpha(m+i,n)-\wedge_\alpha(n+i,m)}$.

Corollary (4.5): Let $\varphi(z) = a|z|^{2m} + b|z|^{2s}\ with\ a\ , b\ \in C$.

Then $T_\varphi\ on\ A_\alpha^2(D)$ is normal and hence hyponormal.

$\frac{1}{n+1}|a_n|^2\ <\ \infty.$

5. Conclusion

This paper establishes conditions under which Toeplitz operates with non-harmonic symbols are hyponormal in weighted Bergman spaces. By exploring both necessary and sufficient conditions, we provide insights that bridge theory with practical implications in operator analysis. Further studies could further examine the potential applications of these findings in advanced mathematical contexts.

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