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A New Common Fixed-Point Theorem for Weakly C-Contractive Mappings

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Abstract: ***The study of common fixed-point theorems is one of the most important and active areas of research in mathematics. It has important applications across various disciplines including mathematical physics, computer science and economics. In this paper we have established a new common fixed-point theorem for weakly C-contractive mappings in complete metric space, which is a generalization of a result in the existing literature.***

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1. Introduction

The fixed-point technique is the most important and powerful tool used for solving the nonlinear operator equations. In fixed-point theory we find the points which remain invariant under the action of a mapping. From many decades, the study of fixed-point theory is one of the most active areas of research work in analysis and topology. The importance of the study of nonlinear problems was recognized firstly by the French Mathematician Henry Poincare. He predicted that 'the study of nonlinearity will be the core part of the future Mathematics'. Since then, many techniques and methods have been developed for studying the nonlinear equations.

The first fixed-point theorem was proved in 1911 by Luitzen Brouwer [2]. In 1922, the Polish Mathematician Stefan Banach [1] established the famous contraction mapping theorem which is also known as the Banach contraction mapping principle. This theorem guarantees the existence (and uniqueness) of fixed-points of certain selfmaps on metric spaces. A constructive method for finding those fixed-points is also provided in the same result. Many mathematicians generalized the Banach contraction theorem in different spaces. In 1930, Schauder [20] generalized the Brouwer's fixed-point theorem to Banach spaces. This result was further generalized to multivalued functions by S. Kakutani [8]. In 1935, Tychonoff [22] proved the fixed-point theorem for the case when K is a compact convex subset of a locally convex space. Robert Cauty [3] proved the full result, in 2001, without the assumption of local convexity. In 1998, Jungck and Rhoades [7] introduced the concept of weakly compatible pair of mappings. In past few decades, many authors obtained common fixed-point theorems for several classes of mappings on different metric spaces such as complete metric spaces, compact metric spaces, partially ordered metric spaces and many more see [3, 5, 7, 9-14, 17, 21, 23].

In the present paper, we establish a common fixed-point theorem for weakly C-contractive mappings satisfying the weakly compatibility condition without using the continuity. So, it is the generalization of the Choudhury's fixed-point theorem [5].

2. Preliminaries

Definition 2.1:[11] Let X be a nonempty set, then a point x in X is said to be a fixed-point of mapping $\phi: X \longrightarrow X$ if $\phi x = x$.
Definition 2.2:[9] Let (X, ϱ) be a metric space, then a selfmap ϕ on X is called a contraction if there exists $\lambda \in [0,1)$ such that
$\varrho(\phi x, \phi y) \leq \lambda\, \varrho(x, y), \forall\, x, y \in X$.

Banach contraction mapping theorem [1] is one of the most important and central results of functional analysis, it states that:

Theorem 2.1:[1] Every contraction mapping on a complete metric space possesses a unique fixed-point, that is, if (X, ϱ) is a complete metric space and a mapping $\phi: X \longrightarrow X$ is such that

$$\varrho(\phi x, \phi y) \leq \lambda\, \varrho(x, y), \forall\, x, y \in X \quad .... (2.1)$$

where $0 \leq \lambda < 1$, then ϕ possesses a unique fixed-point.

Relation (2.1) implies that the mapping ϕ is continuous. Many researchers worked on the problem of finding the contractive conditions that will imply the existence of fixed-point in a complete metric space but will not imply the continuity. Kannan [15, 16] answered this problem by extending the Banach contraction principle as follows:

Theorem 2.2:[15] Let (X, ϱ) be a complete metric space. If a mapping $\phi: X \longrightarrow X$ satisfies the condition

$$\varrho(\phi x, \phi y) \leq \lambda\, [\, \varrho(x, \phi x) + \varrho(y, \phi y)\,] \quad .... (2.2)$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{2}\right)$, then ϕ has a unique fixed-point.

Kannan [15] has shown that the mapping ϕ need not be continuous. Clearly, the conditions in Banach and Kannan theorems are independent.

Definition 2.3:[15] A mapping is said to be a Kannan type mapping (or K-mapping) if it satisfies (2.2).

Many authors established various fixed-point results for Kannan type mappings and their generalizations see [4, 6, 9-14, 17, 21, 23, 24].

Reich [18] generalized the Banach contraction principle by unifying conditions (2.1) and (2.2) for the mappings $\phi: X \longrightarrow X$ by assuming the following condition:

for all $x, y \in X$, $\varrho(\phi x, \phi y) \leq \lambda\, \varrho(x, y) + \eta\, \varrho(x, \phi x) + \mu\, \varrho(y, \phi y)$ .... (2.3)

where λ, η and μ are nonnegative constants satisfying $\lambda + \eta + \mu < 1$.

Reich proved that the Banach and Kannan conclusions about the existence and uniqueness of fixed-point hold also for the contractions satisfying (2.3).

Gerald Jungck and B.E. Rhoades [7] introduced the concept of weakly compatible maps as follows:

Definition 2.4:[7] Two selfmaps ϕ and ψ defined on a metric space (X, ϱ) are said to be weakly compatible if for all $x \in X$,

$$\phi x = \psi x \implies \phi\psi x = \psi\phi x.$$

In 1972, Chatterjea [4] introduced the concept of C-contraction as follows:

Definition 2.5:[4] A mapping $\phi: X \longrightarrow X$, where (X, ϱ) is a metric space, is said to be a C- contraction (or Chatterjea type contraction) if there exists a constant $\lambda \in \left(0, \frac{1}{2}\right)$ such that for all $x, y \in X$,

$$\varrho(\phi x, \phi y) \leq \lambda\, [\, \varrho(x, \phi y) + \varrho(y, \phi x)\,] \quad .... (2.4)$$

He has proved the existence and uniqueness of fixed-point of the mapping ϕ satisfying (2.4) in the following theorem:

Theorem 2.3:[4] Every C- contraction defined on a complete metric space (X, ϱ) possesses a unique fixed-point.

Continuity of C- contraction is not necessary for proving theorem 2.3. Rhoades [19] analyzed various definitions of contraction mappings and established that conditions (2.1), (2.2) and (2.4) are independent of one another.

Kirk [17] introduced the asymptotic contraction and generalized the Banach contraction mapping theorem. In metric spaces weakly contractive mapping is defined as follows:

Definition 2.6: [13] Let (X, ϱ) be a complete metric space, then a mapping $\phi: X \longrightarrow X$ is said to be weakly contractive if for all $x, y \in X$,

$$\varrho(\phi x, \phi y) \leq \varrho(x, y) - \psi(\varrho(x, y)) \quad .... (2.5)$$

where $\psi: [0, \infty) \longrightarrow [0, \infty)$ is continuous, nondecreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim_{x \to \infty} \psi(x) = \infty$.

If we put $\psi(x) = \lambda x$, where $\lambda \in (0, 1)$, then the weak contraction (2.5) becomes the Banach contraction (2.1).

Choudhury [5] introduced the concept of weakly C-contractive mapping as follows:

Definition 2.7:[5] A mapping $\phi: X \longrightarrow X$, where (X, ϱ) is a metric space, is called weakly C- contraction if for all $x, y \in X$,

$$\varrho(\phi x, \phi y) \leq \frac{1}{2}[\, \varrho(x, \phi y) + \varrho(y, \phi x)] - \psi\big(\varrho(x, \phi y), \varrho(y, \phi x)\big) \quad .... (2.6)$$

where $\psi: [0, \infty)^2 \longrightarrow [0, \infty)$ is continuous map satisfying $\phi(x, y) = 0$ if and only if $x = y = 0$.

In (2.6) if we put $\psi(x, y) = \lambda\,(x + y)$, where $\lambda \in \left(0, \frac{1}{2}\right)$, then it reduces to (2.4) and thus weak C- contraction is generalization of C- contraction.

Following result was developed by Choudhari [5]:

Theorem 2.4:[5] Every weakly C- contraction defined on a complete metric space (X, ϱ) possesses a unique fixed-point.

3. Main Result

We have generalized the fixed-point theorem 2.4 as follows:

Theorem 3.1: Let A be a nonempty closed subset of a complete metric space (X, ϱ) and $\sigma: [0, \infty)^2 \longrightarrow [0, \infty)$ be a continuous map satisfying

$$\sigma(x, y) = 0 \text{ if and only if } x = y = 0. \quad .... (3.1)$$

Let ϕ and ψ be two selfmaps defined on A satisfying the condition:

for each pair $(x, y) \in X \times X$

$$\varrho(\phi x, \psi y) \leq \frac{1}{2}[\, \varrho(\varphi x, \psi y) + \varrho(\varphi y, \phi x)] - \sigma\big(\varrho(\varphi x, \psi y), \varrho(\varphi y, \phi x)\big) \quad .... (3.2)$$

where $\varphi: A \longrightarrow X$ satisfies

(a) $\phi A \subset \varphi A$, $\psi A \subset \varphi A$ and
(b) the pairs (ϕ, φ) and (ψ, φ) are weakly compatible.

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Moreover let $\varphi(A)$ be a closed subset of X. Then mappings ϕ , φ and ψ possesses a unique common fixed-point.

Proof: Let x_0 be an arbitrary point in A. Then by the assumption (a) there exists two sequences $\{x_n\}$ and $\{y_n\}$ satisfying

$\phi x_{2n} = \varphi x_{2n+1} = y_{2n}, \psi x_{2n+1} = \varphi x_{2n+2} = y_{2n+1},$ for all $n \in N$.

At first we shall prove that $\lim_{n\to\infty} \varrho(y_n, y_{n+1}) = 0$.

Consider $n = 2k$, then we have

$$\varrho(y_{2k}, y_{2k+1}) = \varrho(\phi x_{2k}, \psi x_{2k+1})$$
$$\leq \frac{1}{2}[\varrho(\varphi x_{2k}, \psi x_{2k+1}) + \varrho(\varphi x_{2k+1}, \phi x_{2k})]$$
$$- \sigma(\varrho(\varphi x_{2k}, \psi x_{2k+1}), \varrho(\varphi x_{2k+1}, \phi x_{2k}))$$
$$= \frac{1}{2}[\varrho(y_{2k-1}, y_{2k+1}) + \varrho(y_{2k}, y_{2k})] -$$
$$\sigma(\varrho(y_{2k-1}, y_{2k+1}), \varrho(y_{2k}, y_{2k}))$$
$$\Rightarrow \varrho(y_{2k}, y_{2k+1}) \leq \frac{1}{2}\varrho(y_{2k-1}, y_{2k+1}) -$$
$$\sigma(\varrho(y_{2k-1}, y_{2k+1}), 0) \quad .... (3.3)$$
$$\leq \frac{1}{2}[\varrho(y_{2k-1}, y_{2k+1})] \quad .... (3.4)$$
$$\leq \frac{1}{2}[\varrho(y_{2k-1}, y_{2k}) + \varrho(y_{2k}, y_{2k+1})] \quad .... (3.5)$$
$$\Rightarrow \varrho(y_{2k}, y_{2k+1}) - \frac{1}{2}\varrho(y_{2k}, y_{2k+1}) \leq \frac{1}{2}\varrho(y_{2k-1}, y_{2k})$$

Hence, we get $\varrho(y_{2k}, y_{2k+1}) \leq \varrho(y_{2k-1}, y_{2k})$.

Similarly, for $n = 2k + 1$, we get

$$\varrho(y_{2k+1}, y_{2k+2}) \leq \varrho(y_{2k}, y_{2k+1})$$

this shows that $\{\varrho(y_n, y_{n+1})\}$ is a monotonically decreasing sequence of nonnegative real numbers and therefore it converges, say to l

i.e. let $\lim_{n\to\infty} \varrho(y_n, y_{n+1}) = l$. .... (3.6)

From (3.4) and (3.5), we have

$$\varrho(y_n, y_{n+1}) \leq \frac{1}{2}\varrho(y_{n-1}, y_{n+1})$$
$$\leq \frac{1}{2}[\varrho(y_{n-1}, y_n) + \varrho(y_n, y_{n+1})].$$

From above inequality, letting $n \to \infty$ and using (3.6), we get

$$l \leq \lim_{n\to\infty} \frac{1}{2}\varrho(y_{n-1}, y_{n+1}) \leq l.$$

Thus $\lim_{n\to\infty} \varrho(y_{n-1}, y_{n+1}) = 2l$.

Now from (3.3), by letting $k \to \infty$ and by using the continuity of mapping σ, we get

$$l \leq \frac{1}{2} 2l - \sigma(2l, 0) \Rightarrow \sigma(2l, 0) = 0.$$

which by using (3.1) implies that $l = 0$,

thus $\lim_{n\to\infty} \varrho(y_n, y_{n+1}) = 0$. .... (3.7)

Further we will prove that the sequence $\{y_n\}$ is Cauchy. But as sequence $\{\varrho(y_n, y_{n+1})\}$ is monotonically decreasing, therefore it is sufficient to show that the subsequence $\{y_{2n}\}$ is Cauchy.

If possible, let us suppose that the subsequence $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists some $\varepsilon > 0$ for which the subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of the sequence $\{y_{2n}\}$ can be obtained such that $n(k)$ is the least index satisfying

$$\varrho(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon, \quad .... (3.8)$$

with $k < m(k) < n(k)$.

Then obviously we have

$$\varrho(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon \quad .... (3.9)$$

Again, by the triangle inequality, we obtain

$$\varrho(y_{2m(k)}, y_{2n(k)}) \leq \varrho(y_{2m(k)}, y_{2n(k)-2}) +$$
$$\varrho(y_{2n(k)-2}, y_{2n(k)-1}) + \varrho(y_{2n(k)-1}, y_{2n(k)})$$
$$\leq \varepsilon + \varrho(y_{2n(k)-2}, y_{2n(k)-1}) + \varrho(y_{2n(k)-1}, y_{2n(k)})$$

From above inequality, letting $k \to \infty$ and using (3.7) and (3.8), we get

$$\varepsilon \leq \lim_{k\to\infty} \varrho(y_{2m(k)}, y_{2n(k)}) \leq \varepsilon$$
$$\Rightarrow \lim_{k\to\infty} \varrho(y_{2m(k)}, y_{2n(k)}) = \varepsilon \quad .... (3.10)$$

We also have

$$[\varrho(y_{2m(k)}, y_{2n(k)+1}) - \varrho(y_{2m(k)}, y_{2n(k)})] \leq$$
$$\varrho(y_{2n(k)}, y_{2n(k)+1}) \quad .... (3.11)$$

and

$$[\varrho(y_{2n(k)}, y_{2m(k)-2}) - \varrho(y_{2n(k)}, y_{2m(k)-1})] \leq$$
$$\varrho(y_{2m(k)-2}, y_{2m(k)-1}) \quad .... (3.12)$$

Now from (3.7), (3.10), (3.11) and (3.12) we get

$$\lim_{k\to\infty} \varrho(y_{2m(k)-1}, y_{2n(k)}) = \lim_{k\to\infty} \varrho(y_{2m(k)-2}, y_{2n(k)}) =$$
$$\varepsilon, \quad .... (3.13)$$

and from (3.2) we get

$$\varrho(y_{2m(k)-1}, y_{2n(k)}) = \varrho(\phi x_{2n(k)}, \psi x_{2m(k)-1})$$
$$\leq \frac{1}{2}[\varrho(\varphi x_{2n(k)}, \psi x_{2m(k)-1}) + \varrho(\varphi x_{2m(k)-1}, \phi x_{2n(k)})]$$
$$- \sigma(\varrho(\varphi x_{2n(k)}, \psi x_{2m(k)-1}), \varrho(\varphi x_{2m(k)-1}, \phi x_{2n(k)}))$$

Now as σ is continuous, therefore letting $k \to \infty$ in above inequality and using (3.13), we obtain

$$\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) - \sigma(\varepsilon, \varepsilon) \Rightarrow \sigma(\varepsilon, \varepsilon) = 0,$$

and therefore, by using (3.1) we obtain $\varepsilon = 0$, which is a contradiction. Hence $\{y_n\}$ must be a Cauchy sequence.

Finally, we prove that the mappings ϕ, φ and ψ have a common fixed-point. Now as $\{y_n\}$ is a Cauchy sequence in the complete metric space (X, ϱ), so there exists $z \in X$ satisfying $\lim_{n\to\infty} y_n = z$. Again as A is closed and $\{y_n\} \subset A$, therefore z is in A. Further by our supposition, $\varphi(A)$ is closed, therefore there exists $t \in A$ such that $z = \varphi t$. Now, for all $n \in N$, we have

$\varrho(\phi t, y_{2n+1}) = \varrho(\phi t, \psi x_{2n+1})$
$\leq \frac{1}{2}\,[\,\varrho(\varphi t, \psi x_{2n+1}) + \varrho(\varphi x_{2n+1}, \phi t)\,] -$
$\sigma\left(\varrho(\varphi t, \psi x_{2n+1}), \varrho(\varphi x_{2n+1}, \phi t)\right)$
$= \frac{1}{2}\,[\,\varrho(z, y_{2n+1}) + \varrho(y_{2n}, \phi t)\,] -$
$\sigma\left(\varrho(\varphi t, \psi x_{2n+1}), \varrho(\varphi x_{2n+1}, \phi t)\right).$
Letting $n \longrightarrow \infty$, we get
$\varrho(\phi t, z) \leq \frac{1}{2}\,[\,\varrho(z, z) + \varrho(z, \phi t)\,] -$
$\sigma\left(\varrho(\varphi t, z), \varrho(z, \phi t)\right)$
therefore
$\sigma\left(0, \varrho(z, \phi t)\right) \leq -\frac{1}{2}\left(\varrho(\phi t, z)\right) \leq 0$
$\Longrightarrow\ \varrho(z, \phi t) = 0 \ \Longrightarrow\ \phi t = z.$
Similarly, we get
$\psi t = z.$
Therefore, we get
$\phi t = \psi t = \varphi t = z.$
Again, as the pairs (ϕ, φ) and (ψ, φ) are weakly compatible, therefore we have
$\phi z = \psi z = \varphi z.$
Further we have
$\varrho(\phi z, y_{2n+1}) = \varrho(\phi z, \psi x_{2n+1})$
$\leq \frac{1}{2}[\,\varrho(\varphi z, \psi x_{2n+1}) + \varrho(\varphi x_{2n+1}, \phi z)] -$
$\sigma\left(\varrho(\varphi z, \psi x_{2n+1}), \varrho(\varphi x_{2n+1}, \phi z)\right)$
$= \frac{1}{2}\,[\,\varrho(\varphi z, y_{2n+1}) + \varrho(y_{2n}, \phi z)\,] -$
$\sigma\left(\varrho(\varphi z, y_{2n+1}), \varrho(y_{2n}, \phi z)\right).$
Letting $n \longrightarrow \infty$, as $\phi z = \psi z = \varphi z$, we get
$\varrho(\phi z, z) = \frac{1}{2}\,[\,\varrho(\phi z, z) + \varrho(z, \phi z)\,] -$
$\sigma\left(\varrho(\phi z, z), \varrho(z, \phi z)\right)$
$\Longrightarrow\ \sigma\left(\varrho(\phi z, z), \varrho(z, \phi z)\right) = 0$
$\Longrightarrow\ \varrho(\phi z, z) = 0 \ \Longrightarrow\ \phi z = z$
and therefore from $\phi z = \psi z = \varphi z$,
we obtain
$\phi z = \psi z = \varphi z = z.$

This shows that the mappings ϕ, φ and ψ have common fixed-point.

Further, if z_1, z_2 are two common fixed-points of these mappings, then from the inequality (3.2) we have
$\varrho(z_1, z_2) = \varrho(\phi z_1, \psi z_2)$
$\leq \frac{1}{2}[\,\varrho(\varphi z_1, \psi z_2) + \varrho(\varphi z_2, \phi z_1)] -$
$\sigma\left(\varrho(\varphi z_1, \psi z_2), \varrho(\varphi z_2, \phi z_1)\right)$
$= \varrho(z_1, z_2) - \ \sigma\left(\varrho(z_1, z_2), \varrho(z_1, z_2)\right)$
$\Longrightarrow\ \sigma\left(\varrho(z_1, z_2), \varrho(z_1, z_2)\right) = 0$
which by (3.1) implies that $z_1 = z_2$.

Hence the common fixed-point is unique.

4. Conclusion

In this paper we have established a common fixed-point theorem for weakly C-1contraction mappings by using C-1contraction, weakly C-1contraction and weakly compatibility condition and without using the continuity. We have generalized the fixed-point theorem developed by B. S. Choudhury in complete metric spaces.

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