A New Variant of Banach Space Valued Sequence Space Using Modulus Function

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Abstract: ***The main object of the paper is to introduce a new Banach space valued sequence space $ces_\theta(X, f, p)$ involving lacunary sequence and a modulus function. Various algebraic and topological properties of the space have been examined. Some inclusion relations between the space have been investigated and information on multipliers for $ces_\theta(X, f, p)$has been given. Our results generalize and unify the corresponding earlier results of Karakaya [3], Shiue [12], Sanhan and Suantai [11] and Bala [1].***

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Introduction

The notion of a modulus function was introduced in 1953 by Nakano[8]. We recall [7,8] that a modulus $f$is a function $f : [0, \infty) \rightarrow [0, \infty)$ such that

- i) $f(x) = 0$ if and only if $x = 0$,
- ii) $f(x + y) \leq f(x) + f(y)$for all $x \geq 0, y \geq 0$,
- iii) f is increasing,
- iv) f is continuous from the right at 0.

Because of (ii), $|f(x) - f(y)| \leq f(|x - y|)$ so that in view of (iv), f is continuous everywhere on $[0, \infty)$. A modulus may be unbounded (for example, $f(x) = x^p, 0 < p \leq 1$) or bounded (for example, $f(x) = \frac{x}{1+x}$).

It is easy to see that $f_1 + f_2$ is a modulus function when f_1 and f_2 are modulus functions, and that the function f^i (i is a $positive$ $integer$), the composition of a modulus function f with itself $i$times, is also a modulus function.

Ruckle [10] used the idea of a modulus function f to construct a class of FK spaces

$$L(f) = \{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

The space $L(f)$ is closely related to the space ℓ_1 which is an $L(f)$ space with $f(x) = x$ for all real $x \geq 0$.

By a lacunary sequence $\theta = (k_r); r = 0,1,2, \ldots,$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r / k_{r-1} will be denoted by q_r . The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [2] as follows:

$$N_\theta = \{x = (x_k) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} |x_k - l| = 0 \text{ for some } l\}.$$

There is a strong connection [2] between N_θ and the space w of strongly Cesàro summable sequences, which is defined by

$$w = \{x = (x_k) : \lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} |x_k - l| = 0 \text{ for some } l\}.$$

In the special case where $\theta = (2^r)$, we have $N_\theta = w$. Infact, for a lacunary sequence θ, $N_\theta = w$ if and only if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$[2, p. 511].

Let w, ℓ^0 denote the spaces of all scalar and real sequences, respectively. For $1 < p < \infty$, the Cesàro sequence space ces_p defined by

$$ces_p = \{x \in \ell^0 : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty\}$$

is a Banach space when equipped with the norm

$$\| x \| = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{1/p}$$

This space was first introduced by Shiue [12], which is useful in the theory of Matrix operator and others (see [4, 5]). Some geometric properties of the Cesàro sequence space ces_p were studied by many authors.

Sanhan and Suantai [11] introduced and studied a generalized Cesàro sequence space $ces(p)$, where $p = (p_n)$ is a bounded sequence of positive real numbers.

Quite recently, Karakaya[3], Ozturk and Basarir[9] introduced a new sequence space involving lacunary sequence and examined some geometric properties of this space equipped with Luxemburg norm.

Let $(X, \|.\|)$ be a Banach space over the complex field $\mathbb{C}$. Denote by $w(X)$ the space of all X-valued sequences. Let $p = (p_k)$ be a bounded sequence of positive real numbers.

We now introduce the Banach space valued sequence space $ces_\theta(X, \mathrm{f}, p)$ as follows.

$$ces_\theta(X, \mathrm{f}, p) = \left\{x \in w(X) : \sum_{r=1}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k \in I_r}\|\mathrm{x_k}\|\right)\right]^{p_r} < \infty\right\}.$$

Some well-known spaces are obtained by specializing X, f and p.

(i) If $X = \mathbb{C}$ and $f(x) = x$, then $ces_\theta(X, \mathrm{f}, p) = \ell(p, \theta)$ (Karakaya [3]).

(ii) If $X = \mathbb{C}$, $p_n = p (1 \le p < \infty)$ for all n, $f(x) = x$and $\theta = (2^r)$, then $ces_\theta(X, \mathrm{f}, p) = ces_p$ (Shiue [12]).

(iii) If $X = \mathbb{C}$, $f(x) = x$and $\theta = (2^r)$, then $ces_\theta(X, \mathrm{f}, p) = ces(p)$ (Sanhan and Suantai [11]).
(iv) If $X = \mathbb{C}$ and $\theta = (2^r)$, then $ces_\theta(X, \mathrm{f}, p) = ces(f, p)$ (Bala[1]).

The following inequalities (see, e.g., [6, p. 190]) are needed throughout the paper.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup_k p_k$, then for any complex a_k and b_k ,

(1) $$|a_k + b_k|^{p_k} \le C(|a_k|^{p_k} + |b_k|^{p_k}),$$

where $C = \max(1, 2^{H-1})$. Also for any complex λ,

(2) $$|\lambda|^{p_k} \le \max(1, |\lambda|^H).$$

Linear topological structure of $ces_\theta(X, \mathrm{f}, p)$ space

In this section we establish some algebraic and topological properties of the sequence space defined above. In order to discuss the properties of $ces_\theta(X, \mathrm{f}, p)$, we assume that (p_n) is bounded.

Theorem 2.1. $ces_\theta(X, \mathrm{f}, p)$ is a linear space over the complex field $\mathbb{C}$.

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.2. $ces_\theta(X, \mathrm{f}, p)$ is a topological linear space, paranormed by

$$g(x) = \left(\sum_{r=1}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k \in I_r}\|x_k\|\right)\right]^{p_r}\right)^{\frac{1}{M}} \qquad (2.1)$$

where $H = \sup p_r < \infty$ and $M = \max(1, H)$.

The proof follows by using standard techniques and the fact that every paranormed space is a topological linear space [13, p. 37].

Corollary 2.3. If p is a constant sequence, then $ces_\theta(X, \mathrm{f}, p)$ is a normed space for $p \geq 1$ and a p-normed space for $p < 1$.

Theorem 2.4. $ces_\theta(X, \mathrm{f}, p)$ is a Fréchet space paranormed by (2.1).

Proof. In view of Theorem 2.2 it suffices to prove the completeness of $ces_\theta(X, \mathrm{f}, p)$. Let $\left(x^{(i)}\right)$ be a Cauchy sequence in $ces_\theta(X, \mathrm{f}, p)$. Then $g\left(x^{(i)} - x^{(j)}\right) \to 0$ as $i, j \to \infty$, that is

$$\left(\sum_{r=1}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)} - x_k^{(j)}\right\|\right)\right]^{p_r}\right)^{\frac{1}{M}} \to 0, as \quad i, j \ \infty, \qquad (2.2)$$

which implies that for each fixed k, $\left\|x_k^{(i)} - x_k^{(j)}\right\| \to 0$ as $i, j \to \infty$ and so $\left(x_k^{(i)}\right)$ is a Cauchy sequence in X for each fixed k. Since X is complete, there exists a sequence $x = (x_k)$ such that $x_k \in X$ for each $k \in \mathbb{N}$ and $\lim_{i\to\infty} x_k^{(i)} = x_k$ for each k. Now from (2.2), we have for $\epsilon > 0$, there exists a natural number K such that

$$\sum_{r=1}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)} - x_k^{(j)}\right\|\right)\right]^{p_r} < \epsilon^M \qquad for\ i, j > K. \qquad (2.3)$$

Since for any fixed natural number r_0, we have from (2.3),

$$\sum_{r=1}^{r_0}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)} - x_k^{(j)}\right\|\right)\right]^{p_r} < \epsilon^M \qquad for\ i, j > K,$$

by taking $j \to \infty$ in the above expression we obtain

$$\sum_{r=1}^{r_0}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)} - x_k\right\|\right)\right]^{p_r} < \epsilon^M \qquad for\ i > K.$$

Since r_0 is arbitrary, by taking $r_0 \to \infty$, we obtain

$$\sum_{r=1}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)} - x_k\right\|\right)\right]^{p_r} < \epsilon^M \qquad for\ i > K,$$

that is, $g\left(x^{(i)} - x\right) < \epsilon$ for $i > K$.
To show that $x \in ces_\theta(X, \mathrm{f}, p)$, let $i > K$ and fix r_0. Since $p_r/M \leq 1$ and $M \geq 1$, using Minkowski's inequality, we have

$$\left(\sum_{r=1}^{r_0}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)\right]^{p_r}\right)^{\frac{1}{M}}$$

$$= \left(\sum_{r=1}^{r_0}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k - x_k^{(i)} + x_k^{(i)}\right\|\right)\right]^{p_r}\right)^{\frac{1}{M}}$$

$$\leq \left(\sum_{r=1}^{r_0}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k - x_k^{(i)}\right\|\right)\right]^{p_r}\right)^{\frac{1}{M}} + \left(\sum_{r=1}^{r_0}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\left\|x_k^{(i)}\right\|\right)\right]^{p_r}\right)^{\frac{1}{M}}$$

$$< \varepsilon + g\left(x^{(i)}\right),$$

from which it follows that $x \in ces_\theta(X, \mathrm{f}, p)$ and the space is complete.

Corollary 2.5. If p is a constant sequence and $p \geq 1$, then $ces_\theta(X, \mathrm{f}, p)$ is a Banach space.

Inclusion between $ces_\theta(X, \mathrm{f}, p)$ spaces

We now investigate some inclusion relations between $ces_\theta(X, \mathrm{f}, p)$ spaces.

Theorem 3.1. If $p = (p_r)$ and $q = (q_r)$ are bounded sequences of positive real numbers with $0 < p_r \leq q_r < \infty$ for each r, then $ces_\theta(X, \mathrm{f}, p) \subseteq ces_\theta(X, \mathrm{f}, q)$.

Proof. Let $x \in ces_\theta(X, \mathrm{f}, p)$. Then $\sum_{r=1}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)\right]^{p_r} < \infty$ .This implies that

$\mathrm{f}\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right) \leq 1$ for sufficiently large values of r, say $r \geq r_0$ for some fixed $r_0 \in \mathbb{N}$. Since $p_r \leq q_r$, we have

$$\sum_{r\geq r_0}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)\right]^{q_r} \leq \sum_{r\geq r_0}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)\right]^{p_r} < \infty.$$

This shows that $x \in ces_\theta(X, \mathrm{f}, q)$ and completes the proof.

Theorem 3.2. If (t_r) and (u_r) are bounded sequences of positive real numbers with

$0 < t_r, u_r < \infty$ and if $p_r = \min(t_r, u_r)$, $q_r = \max(t_r, u_r)$, then

$ces_\theta(X, \mathrm{f}, p) = ces_\theta(X, \mathrm{f}, t) \cap ces_\theta(X, \mathrm{f}, u)$ and $ces_\theta(X, \mathrm{f}, q) = G$ where G is the subspace of $w(X)$ generated by $ces_\theta(X, \mathrm{f}, t) \cup ces_\theta(X, \mathrm{f}, u)$.

Proof. It follows from Theorem 3.1 that $ces_\theta(X, \mathrm{f}, p) \subseteq ces_\theta(X, \mathrm{f}, t) \cap ces_\theta(X, \mathrm{f}, u)$ and that

$G \subseteq ces_\theta(X, \mathrm{f}, q)$. For any complex λ, $|\lambda|^{p_r} \leq \max(|\lambda|^{t_r}, |\lambda|^{u_r})$; thus

$ces_\theta(X, \mathrm{f}, t) \cap ces_\theta(X, \mathrm{f}, u) \subseteq ces_\theta(X, \mathrm{f}, p)$. Let $A = \{r: t_r \geq u_r\}$ and $B = \{r: t_r < u_r\}$. If

$x \in ces_\theta(X, \mathrm{f}, q)$, we write

$$y_r = x_r (r \in A) \quad \text{and} \quad y_r = 0 (r \in B); and$$
$$z_r = 0 (r \in A) \quad \text{and} \quad z_r = x_r (r \in B).$$

$$\sum_{r=1}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\|y_k\|\right)\right]^{t_r} = \sum_{r\in A} + \sum_{r\in B} = \sum_{r\in A}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)\right]^{q_r} < \infty,$$

and so $y \in ces_\theta(X, \mathrm{f}, t) \subseteq G$. Similarly, $z \in ces_\theta(X, \mathrm{f}, u) \subseteq G$. Thus, $x = y + z \in G$. We have proved that $ces_\theta(X, \mathrm{f}, q) \subseteq G$, which completes the proof.

Corollary 3.3. The three conditions $ces_\theta(X, \mathrm{f}, t) \subseteq ces_\theta(X, \mathrm{f}, u)$, $ces_\theta(X, \mathrm{f}, p) = ces_\theta(X, f, t)$ and $ces_\theta(X, \mathrm{f}, q) = ces_\theta(X, \mathrm{f}, u)$ are equivalent.

Corollary 3.4. $ces_\theta(X, \mathrm{f}, t) = ces_\theta(X, \mathrm{f}, u)$ if and only if $ces_\theta(X, f, p) = ces_\theta(X, \mathrm{f}, q)$.

The Space of Multipliers of $ces_\theta(X, \mathrm{f}, p)$

For any set $E \subset w(X)$ the space of multipliers of E, denoted by S(E), is given by

$S(E) = \{a = (a_k) \in \mathrm{w(X)} : ax = (a_k x_k) \in \mathrm{E} \text{ for all } \mathrm{x} = (x_k) \in \mathrm{E}\}$.

Theorem 4.1. For Banach algebra X, $l_\infty(X) \subseteq S[ces_\theta(X, \mathrm{f}, p)]$, where

$$l_\infty(X) = \{a = (a_k) \in \mathrm{w(X)} : sup_k\|a_k\| < \infty\}.$$

Proof. Let $a = (a_k) \in l_\infty(X)$, $T = sup_k\|a_k\|$ and $x = (x_k) \in ces_\theta(X, \mathrm{f}, p)$. Since X is Banach algebra, by definition of modulus function (ii), (iii) and inequality (1.2), we have

$$\sum_{r=1}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\|a_k x_k\|\right)\right]^{p_r} \leq (1+[T])^H \sum_{r=1}^{\infty}\left[f\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k\|\right)\right]^{p_r}$$

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