International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Exploring Properties and Isomorphisms of Automorphism Groups in Regular Covering Spaces

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Abstract: ***The aim of this article is mainly focus on studying the properties of the automorphism group of a regular covering space. Regular covering spaces are those in which the isotropy subgroup corresponds to any point of the space is a normal subgroup of the fundamental group of the given topological space. By considering homomorphisms and autmorphisms of the covering spaces of a topological space, the details about all possible covering spaces of a given space can be obtained. The detailed study of First Poincare group is obtained from [1], [4] [6]& [10] . The theory of covering spaces and its properties were included in [3], [7]. For discussing about regular covering spaces, it is essential to define the fundamental group action on fibre set and we show that this action is transitive. Here we define an automorphism for a covering space and show that the set of all automorphisms forms a group under composition of transformations. Then discussion is about some properties of the automorphism group for a regular covering space. We also define an automorphism group for fibre set, by taking the restriction of the defined automorphism of regular covering space to fibre set. Also we show that the two automorphism groups so defined are isomorphic and we analyse the results.***

Keywords: Regular Covering space, Fundamental group, Homogenius space, Transitive action, Homotopy, Auto morph ism, fibre set, group action

1. Introduction

Topology is the study of surfaces. It deals with continuity of functions for a topological space and closeness of points. French Mathematician Henri Poincare (1854-1912) initiated the study of Algebraic Topology. Some geometric problems including paths and surfaces motivate the study of Algebraic Topology. It gives description about the topological structure of a space by relating it with an algebraic system. The comprehensive study of Algebraic Topology is obtained from [11], [12].

Regular covering spaces are a special class of covering spaces which is the characteristic of the stabilizer of the concerned topological spaces and such space characterises the automorphism group. In this article we discuss about the automorphism for a regular covering space. Then we derive a group of automorphisms for a regular covering space. We also define an automorphism group for fibre set. The discussion of isomorphism between these two automorphism groups is the key note for this article.

2. Preliminaries

In this section, we shall mention about the basic preliminary results needed for the further work of this article. For the definition and Theorems of Group action, the reader can use [8] & [9]. The reader can refer [2], [4], [5] & [6] to get basic results of Algebraic topology

2.1 Preliminaries from Abstract Algebra

In this section, we shall include the basic results from Abstract Algebra. For that we use [1], [9] & [10] for references

2.1.1 Definition [9]

Let G be a group and let E be a nonempty set. Let there be a mapping from EX G → E which maps (x, g) → x. g for x in E and g in G satisfying the conditions x.1 = x and x. (g1. g2) = (x.g1). g2 for x in E and g1, g2 in G. Here 1 is identity element of G.

Then E is called **right G-set or right G-space.**

Similarly we can define a left G-set.

2.1.2 Definition [9]

Let E be a G-set and x in E. The stabilizer of x, denoted by G_x is the set of all elements of G which leaves x fixed.

More concretely, $G_x = \{g \in G / x.\ g = x\}$

Then G_x will be a subgroup of G called **isotropy subgroup** of G associated with x This subgroup is also called **stabilizer** of x.

2.1.3 Definition [10]

A group G acts **transitively** on a set E, if for x, y in E, there corresponds g in G with y = x .g. When the action of G on E is transitive, then E is called a **transitive G- space** or a homogenius G- space.

2.1.4 Lemma [9]

If H is a normal subgroup of G, then N[H] = G, where N[H] denotes the normaliser of the subgroup H.

2.1.5 Theorem [9]

Let **E** be a homogenius **G-set**. If H is the stabilizer of x_0 in E**,** then the automorphism group A (E) of E is isomorphic to N [H] / H

2.2 Preliminaries from Algebraic Topology

Here we include the results from Algebraic Topology that are required for further work. We use [2] , [3], [6], & [17] for references.

2.2.1 Definition [3]
A **path** in a space X is a contimuous function from I =[0,1] in to X.

Let : I → X be a path in X. Then $\alpha(0)$ is called initial point and $\alpha(1)$ is called terminal point of the path α

2.2.2 Definition [6]
A **loop** in a topological space X is a closed path in X. If α : I → X is a loop in X, then $\alpha(0) = \alpha(1)$. The common end point is called **base point** of the loop.

2.2.3 Definition [6]
Let α and β be two loops in X having base point x0.

Then α **is equivalent to** β (**α is homotopic to β**) , if there exists a continuous map H: I X I → X such that
H (t, 0) = α (t)
H (t, 1) = β (t)
H (0, s) = H (1, s) = x0 for t, s in [0,1]

2.2.4 Definition [3]
If α and β are two loops in X with same base point x0, then the **loop product** $\alpha * \beta$ is defined as

$$\alpha * \beta \ (t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Then $\alpha*\beta$ is a loop in X based at x0

Now we shall give definition of Fundamental group of a topological space. A clear idea about this group can be obtained from [2], [5], [7] & [11].

2.2.5 Definition [2]
The relation *Homotopy of loops* is an equivalence relation on the set of all loops in the space X with same base point. The corresponding equivalence classes are called homotopy classes.

Let π (X, x) be the collection of all homotopy classes of loops in X based at x which is a group under the operation o, called **Fundamental group of X at x or First Poincare group of X at x .**

2.2.6 Definition [3]
A **covering space** for a topological space X is a pair ($\tilde{X}$, p) consisting of a spaxe X and a continuous map p: $\tilde{X}$ → X with the property that for each point x in X, there exists a path connected open neighbourhood U in X such that each path component of p^{-1} (U) is mapped homeomorphically on to U by p. The neighbourhood U is called admissible neighbourhood and the mapping p is called **covering projection.**

2.2.7 Definition [6]
Let ($\tilde{X}$, p) be a covering space of X and let f: I → X be a path in X. then a pat g: I → $\tilde{X}$ in $\tilde{X}$ such that pg=f is called **covering path (lifting path)** of f.

2.2.8 Lemma [6]
For the covering space ($\tilde{X}$, p) of X, let $\tilde{x}_0 \in \tilde{X}$ and $p\tilde{x}_0 = x_0$. Then for any path f : I → X with initial point x0, there exists a unique path g: I → $\tilde{X}$ with initial point $\tilde{x}_0$ such that p g = f

Proof:-
Proof is in [6]

2.2.9 Lemma [3]

Let Y be a connected space and ($\tilde{X}$, p), a covering space of X. If f0, f1: Y → X are continuous maps such that pf0 = pf1 . then the set of all points at which f0 and f1 agree is either null set or full set Y

Proof:
Proof is in [3]

2.2.10 Defintion [2]
For any space X, if ($\tilde{X}$, p) is a covering space of X. Then the set $p^{-1}(x)$ consist of all $\tilde{x}$ in $\tilde{X}$ such that p $\tilde{x} = x$ is called **fibre set.**

2.2.11 Theorem [2]:
The fibre set is a right π (X, x) space

Proof
Let [β] ∈ π (X, x) and $\tilde{x} \in p^{-1}(x)$

Then there is a unique path $\tilde{\beta}$ in $\tilde{X}$, with initial point $\tilde{x}$ and such that $p\tilde{\beta} = \beta$

Now for [β] in π (X, x) and for $\tilde{x}$ in $p^{-1}(x)$, define $\tilde{x}.[\beta] = \tilde{\beta}$ (1)

Thus the mapping from p^{-1} (x) X π (X, x)→ π (X, x) defines an action of π (X, x) on the set $p^{-1}(x)$.

Thus $p^{-1}(x)$ is a right π (X, x) space.

Hence the proof

2.2 .12 Theorem [2]
Let ($\tilde{X}$, p) be a covering space of the space X and x ∈ X. Then the set $p^{-1}(x)$ is a transitive right π (X , x) space.

Proof

Let $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x)$

Then $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$, and $\tilde{X}$ is path connected.

Thus there exists a path $\tilde{\beta}$ with $\tilde{\beta}$ (0) = $\tilde{x}_0$ and $\tilde{\beta}$ (1) = $\tilde{x}_1$

Paper ID: SR23824230900 DOI: 10.21275/SR23824230900

Put $\beta = p\tilde{\beta}$

Then $[\beta] \in \pi$ (X, x) and $\tilde{x}_0 . [\beta] = \tilde{\beta}(1) = \tilde{x}_1$

Hence the Theorem

2.2.13 Theorem

Let $(\tilde{X}, p)$ be a covering space of B and $\tilde{x}_0 \in \tilde{X}$ with $p\tilde{x}_0$ = x0. Then the map p*: $\pi(\tilde{X}, \tilde{x}_0) \rightarrow \pi$ (X, x0) defined by p*: $[\beta] = [p\beta]$ is a monomorphism.

Proof

By using the concept of equivalence of paths, the reader can easily prove the theorem.

Remark:

For any point $\tilde{x} \in p^{-1}$ (x), the isotropy subgroup corresponding to $\tilde{x}$ is the subgroup p*$\pi(\tilde{X}, \tilde{x})$ of π(X, x).

3. Main Results and Discussions

In this section, we shall include the main results of our study

3.1 Definition

A covering space $(\tilde{X}, p)$ is called a **regular covering space** of a space X, if there exists a point x∈X such that for some $\tilde{x} \in p^{-1}$ (x), the subgroup p* $(\tilde{X}, \tilde{x})$ is a normal subgroup of π (X, x).

3.2 Definition

Let $(\tilde{X}_1, p_1)$ **and** $(\tilde{X}_2, p_2)$ be two covering spaces of a space X.

A homomorphism **of** $(\tilde{X}_1$, **p1)** → $(\tilde{X}_2$, **p2) is a continuous map** $\emptyset$: $\tilde{X}_1 \rightarrow \tilde{X}_2$

Such that p2 $\emptyset$= p1

3.3 Definition

Let $(\tilde{Y}_1, p_1)$ and $(\tilde{Y}_2, p_2)$ be two covering spaces of a space Y. A homomorphism $\emptyset$: $\tilde{Y}_1 \rightarrow \tilde{Y}_2$ is an **isomorphism**, if there exists a homomorphism φ: $\tilde{Y}_2 \rightarrow \tilde{Y}_1$ Such that both the compositions $\emptyset\varphi$ and $\varphi\ \emptyset$ are identity maps on respective sets.

3.4 Definition

An **automorphism** of a covering space is an isomorphism on to itself. Automorphisms are usually called **covering transformations.**

Remark

The set of all automorphisms of a covering space $(\tilde{X}, p)$ of X, denoted by A $(\tilde{X}, p)$, is a group under composition of maps called **automorphism group** of $(\tilde{X}, p)$

3.5 Theorem:

Let $(\tilde{X}_1$, p1) and $(\tilde{X}_2$, p2) be two covering spaces of a space X and let $\emptyset_0$ and $\emptyset_1$ be homomorphisms of $(\tilde{X}_1$ then $\emptyset_0 = \emptyset_1$, p_1) to $(\tilde{X}_2$, p2). If $\emptyset_0$ and $\emptyset_1$ agree at some $\tilde{x} \in \tilde{X}_1$, then

Proof

Let $(\tilde{X}_1$, p1) and $(\tilde{X}_2, p_2)$ be two covering spaces of a space X.

Now $\emptyset_0$ and $\emptyset_1$ are homomorphisms from $\tilde{X}_1$ in to $\tilde{X}_2$.

So $\emptyset_0$ and $\emptyset_1$ are two continuous maps from $(\tilde{X}_1$ P2$\emptyset_0$ = p1 and p2 $\emptyset_1$ = p1

Then $P_2\emptyset_0 = p_2\ \emptyset_1$

Also by hypothesis, $\emptyset_0(\tilde{x}) = \emptyset_1(\tilde{x})$ Then it follows that $\emptyset_0 = \emptyset_1$.

Hence the proof.

3.6 Corollary

If $\emptyset \in$ A $(\tilde{X}, p)$ and $\emptyset$ is not the identity map, then $\emptyset$ has no fixed points.

Proof:-

Let $\emptyset \in$ A $(\tilde{X}, p)$ and $\emptyset$ is not the identity map I. Suppose for some $\tilde{x} \in \tilde{X}$, $\emptyset(\tilde{x}) = \tilde{x}$

That is $\emptyset(\tilde{x}) = I(\tilde{x})$

Since $\emptyset$ and I are two homomorphisms from $(\tilde{X}$,p) to $(\tilde{X}$,p) such that $\emptyset(\tilde{x}) = I(\tilde{x})$ for some $\tilde{x} \in \tilde{X}$

Then by theorem 2.5, it follows that $\emptyset$ = I, which is a contradiction.

Thus $\emptyset$ has no fixed points.

3.7 Theorem

Let $(\tilde{X}, p)$ be a covering space of X and let $\emptyset \in$ A $(\tilde{X}, p)$. Then for $\tilde{x} \in p^{-1}$ (x) and $[\beta] \in \pi$(X, x), we have $\emptyset(\tilde{x} . [\beta]) = \emptyset(\tilde{x}). [\beta]$

That is, each element $\emptyset$ induces an automorphism on the fibre set

Proof:

Let $\emptyset \in$ A $(\tilde{X}, p)$ and $[\beta] \in \pi$(X, x)

Then β is a path in X such that β (0) = x and p $(\tilde{x})$ = x.

Then there is a unique lifting $\tilde{\beta}$ in $\tilde{X}$ such that $\tilde{\beta}(0) = \tilde{x}$ and p $\tilde{\beta} = \beta$

Paper ID: SR23824230900 DOI: 10.21275/SR23824230900

Then the action is given by $\tilde{x} . \beta = \tilde{\beta}$ (1).

For $\emptyset \in A(\tilde{X}, p)$, we have $\emptyset$ is an automorphism from $\tilde{X}$ to $\tilde{X}$ ie $\emptyset$ is a continuous map from $\tilde{X}$ to $\tilde{X}$ such that $p\emptyset = p$.

Now $\emptyset\tilde{\beta}$ is a path in $\tilde{X}$ with initial point $\emptyset \tilde{\beta}(0) = \emptyset((0)) = \emptyset(\tilde{x})$ and $p(\emptyset \tilde{\beta}) = p \emptyset(\tilde{\beta}) = \beta$

The terminal point of $\emptyset \tilde{\beta}$ is $\emptyset \tilde{\beta}(1) = \emptyset(\tilde{\beta}(1))$

$$= \emptyset(\tilde{x} .[\beta])$$

So $\emptyset \tilde{\beta}$ is a lift of β with initial point $\emptyset(\tilde{x})$ and terminal point $\emptyset(\tilde{x}. [\beta])$

So we have, $\emptyset(\tilde{x}). [\beta] = \emptyset\tilde{\beta}(1)$
$= \emptyset(\tilde{x} . [\beta])$

Also $p(\emptyset(\tilde{x})) = p\, \emptyset(\tilde{x}) = p(\tilde{x}) = x$.

Hence $\emptyset(\tilde{x}) \in p^{-1}(x)$

Then the restriction of $\emptyset$ to $p^{-1}(x)$ induces an automorphism of the set $p^{-1}(x)$.

Hence the proof.

Remark:
The set of automorphisms of the set $p^{-1}(x)$ denoted by $A(p^{-1}(x))$ is a group under composition of maps

3.8 Theorem

Let $(\tilde{X}, p)$ be a covering space of X . Then the group of automorphisms $A(\tilde{X}, p)$ is isomorphic to the group $A(p^{-1}(x))$

Proof:
Let $(\tilde{X}, p)$ be a covering space of X and $A(\tilde{X}, p)$ denote the automorphism group. For $\emptyset \in A(\tilde{X}, p)$, let $\varphi = \emptyset / p^{-1}(x)$

Then φ is an automorphism of $p^{-1}(x)$ to $p^{-1}(x)$

Let $f: A(\tilde{X}, p) \to A(p^{-1}(x))$ by $f(\emptyset) = \varphi$

Now we shall show that f is an isomorphism.

By definition, $f(\emptyset_1 \circ \emptyset_2) = \emptyset_1 \circ \emptyset_2 / p^{-1}(x)$ For $\tilde{x} \in p^{-1}(x)$, we have
$\emptyset_1 \circ \emptyset_2 (e) = \emptyset_1(\emptyset_2(\tilde{x}))$
$= \emptyset_1(\varphi_2(\tilde{x}))$
$= \varphi_1(\varphi_2(\tilde{x}))$
$= \varphi_1 \circ \varphi_2 (\tilde{x})$

Since $\tilde{x} \in p^{-1}(x)$ is arbitrary, $\emptyset_1 \circ \emptyset_2 / p^{-1}(x) = \varphi_1 \circ \varphi_2$ i.e. $f(\emptyset_1 \circ \emptyset_2) = f(\emptyset_1)$ of $(\emptyset_2)$ Thus f is a homomorphism.

Now to prove that f is an injective mapping,

Let $\varphi_1 = \varphi_2$

i.e. $\varphi_1(\tilde{x}) = \varphi_2(\tilde{x})$, for $\tilde{x} \in p^{-1}(x)$

Then $\emptyset_1(\tilde{x}) = \emptyset_2(\tilde{x})$, for $\tilde{x} \in p^{-1}(x)$

Thus $\emptyset_1 = \emptyset_2$

So f is an injective mapping.

Now every φ is the restriction to $p^{-1}(x)$ of some $\emptyset \in A(\tilde{X}, p)$ and then $\varphi = f(\emptyset)$

Hence f is a surjective mapping.

Thus $A(\tilde{X}, p) \cong A(p^{-1}(x))$ Hence the proof.

Corollary
For a covering space $(\tilde{X}, p)$ of X , the group $A(\tilde{X}, p)$ is isomorphic to $N[p^*\pi(\tilde{X}, \tilde{x})] / p^*\pi(\tilde{X}, \tilde{x})$, for $x \in X$ and $\tilde{x} \in p^{-1}(x)$

Proof
We have $p^{-1}(x)$ is a homogenius right $\pi(X, x)$ space.

The stabilizer corresponding to $\tilde{x} \in p^{-1}(x)$ is the subgroup $p^* \pi(\tilde{X}, \tilde{x})$.

Then $N[p^*\pi(\tilde{X}, \tilde{x})$ denotes the normaliser of $p^* \pi(\tilde{X}, \tilde{x})$ in $\pi(X, x)$.

Since $p^{-1}(x)$ is a homogenius space, we have

$A(p^{-1}(x))$ is isomorphic to the quotient group $N[p^*\pi(\tilde{X}, \tilde{x})] / p^*\pi(\tilde{X}, \tilde{x})$.

But by Theorem 2.8, it follows that $A(\tilde{X}, p) \cong A(p^{-1}(x))$

Thus, $A(\tilde{X}, p)$ is isomorphic to the quotient group $N[p^*\pi(\tilde{X}, \tilde{x})] / p^*\pi(\tilde{X}, \tilde{x})$

Hence the theorem.

3.9 Theorem

For a regular covering space $(\tilde{X}, p)$ of X , the automorphism group $A(\tilde{X}, p)$) and the quotient group $\pi(X, x) / p^*\pi(\tilde{X}, \tilde{x})$ for any $x \in X$ and $\tilde{x} \in p^{-1}(x)$ are isomorphic.

Proof:

Let $(\tilde{X}, p)$ be a normal covering space of X.

Then by definition of noemal covering space , $p^* \pi(\tilde{X}, \tilde{x})$ is a normal subgroup of $\pi(X, x)$.

Then we have $N[p^*\pi(\tilde{X}, \tilde{x})] = \pi(X, x)$.

By Corollary 2.9, we have

A ($\tilde{X}$, p) ≅ N[p*π ($\tilde{X}$, $\tilde{x}$)] / p*π ($\tilde{X}$, $\tilde{x}$)

Thus A ($\tilde{X}$, p) is isomorphic to π (X , x)/ p*π ($\tilde{X}$, $\tilde{x}$)

Hence the theorem.

4. Conclusion

In this article we have derived two automorphism groups namely automorphism group of covering space and automorphism group of fibre set. Then we have shown that the two automorphism groups so defined are isomorphic and we have analysed the properties of the automorphism group of a regular covering space. We have shown that a regular covering space characterises the isotropy subgroup of the underlying topological spaces . This study has many advantages. It motivates the researchers to do collaborative research with Algebraic Topology. By using results from Abstract Algebra, researchers can correlate this study to other branches of Mathematics. The defined isomorphism between automorphism groups is a stepping stone for outstanding research in this area. The study will motivate researchers for future work on Algebraic Topology.

Acknowledgement
I am very grateful to all my teachers who taught me Algebraic Topology and gave me valuable suggestions for doing this work.

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