Exploring Properties and Applications of the F-Structure Equation $F^{4k} + F^k = 0$ in Differential Manifolds

Rajiv Kumar¹, Lakhan Singh²

¹Digamber Jain College, Baraut (Baghpat); UP, India 250611
Email: rajiv73kr[at]gmail.com
Orcid id: https://orcid.org/0000-0001-7601-3279

²Digamber Jain College, Baraut (Baghpat); UP, India 250611
Email: lakhansingh.077[at]gmail.com

Abstract: In this paper, we have studied various properties of the F-structure equation $F^{4k} + F^k = 0$, $k$ being a positive integer. Nijenhuis tensors, metric F-structure, kernel, tangent and normal vectors have also been discussed.

Keywords: DM, PO, ACS, NT, Ker, TVS, NVS

Notations Through the paper we use the following abbreviations in the place of standard technical terms.
DM- Differential manifold
PO- Projection operator
ACS- Almost complex structure
NT- Nijenhuis tensor
Ker- Kernel
TVS- Tangent vectors
NVS- Normal Vectors

1. Introduction

Various authors and researchers have studied differentiable manifolds, real and complex manifolds, and the F-structure equations from time to time. After the reviewed literature mentioned in the references [1], [2], [3], [4] ………….,[14] we find that currently, this field is alive for academicians and researchers. So, we posed a sequel of [6], [7],[8], and [9]. Let $M^n$ be a differentiable manifold of class $C^c$ and $F$ be a $(1,1)$ tensor of class $C^c$ defined on $M^n$ by-

$$F^{4k} + F^k = 0$$

(1.1)

We define the operators $l$ and $m$ on $M^n$, satisfying-

$$l = -F^{3k}, m = I + F^{3k}, I$$

denotes identity operator (1.2)

From (1.1) and (1.2) we have,

$$l + m = I, l^2 = l, m^2 = m, lm = ml = 0, F^k l = lF^k = F^k, F^k m = mF^k = 0$$

(1.3)

Theorem (1.1): Let the $(1,1)$ tensor $\alpha$ and $\beta$ satisfy-

$$\alpha = m + F^k, \beta = m - F^{2k}$$

then, (1.4)

$$\alpha^3 = m - l, \alpha^b = I = \alpha\beta$$

(1.5)

Proof: Using (1.2), (1.3) and (1.4) we get the results.

Theorem (1.2): Let the $(1,1)$ tensor $p$ and $q$ satisfy-

$$p = m + F^{3k}, q = m - F^k$$

then, (1.6)
\( p^2 = q^3 = I \) \( (1.7) \)

**Proof:** Using (1.2), (1.3) and (1.6) we get (1.7).

**Theorem (1.3):** Let \( k \) be even and \( \text{rank}(F) = n \) then,
\[ l = I, m = 0 \] \( (1.8) \)

and \( F^{3k/2} \) acts as an almost complex structure.

**Proof:** From the fact
\[ \text{rank}(F) + \text{nulity}(F) = \dim M^n = n \] \( (1.9) \)

We have
\[ \text{rank}(F) = 0 \implies \ker F = \{0\} \]

Thus \( FX = 0 \implies X = 0 \).

Let \( FX_1 = FX_2 \Rightarrow F(X_1 - X_2) = 0 \implies X_1 = X_2 \) or \( F \) is 1-1, moreover \( M^n \) being finite dimensional \( F \) is onto also. Thus \( F \) and hence \( F^k \) is invertible.

Operating \( F^{-k} \) on \( F^k I = I \) \( F^{-k} \) and on \( F^k m = m F^k = 0 \) we get \( I = I, m = 0 \). Operating \( F^{-k} \) on (1.1) we have \( F^{3k} + I = 0 \Rightarrow F^{3k/2} \) acts as an almost complex structure.

**2. NT**

Let \( N, l, m \) denote the Nijenhuis tensors corresponding to the operators \( F, l \) and \( m \) respectively. Then,

\[ N_F(X, Y) = [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY] \] \( (2.1) \)

\[ N_l(X, Y) = [lX, lY] + l^2[X, Y] - l[lX, Y] - l[X, lY] \] \( (2.2) \)

\[ N_m(X, Y) = [mX, mY] + m^2[X, Y] - m[mX, Y] - m[X, mY] \] \( (2.3) \)

**Theorem (2.1):** Let \( F, l, m \) satisfy (1.1), (1.2) and (1.3) then,

\[ N_F(mX, mY) = F^{2k}[mX, mY] \] \( (2.4) \)

\[ F^k N_F(mX, mY) = -l[mX, mY] \] \( (2.5) \)

\[ F^k N_F(mX, mY) + N_l(mX, mY) = 0 \] \( (2.6) \)

\[ N_m(lX, mY) = 0 \] \( (2.7) \)

**Proof:** Using (1.2), (1.3), (2.1), (2.2) and (2.3) we get the results.

**3. M – Structure**

Let \( lX(X, Y) = g(FX, Y) \) is skew symmetric then,

\[ g(FX, Y) = -g(X, FY) \] \( (3.1) \)

**Theorem (3.1):** Let \( F \) satisfies (1.1) then,

\[ g(F^k X, F^{2k} Y) = (-1)^{k+1}[g(X, Y) - l m(X, Y)] \] \( (3.2) \)

where,

\[ l m(X, Y) = g(X, mY) \] \( (3.3) \)
**Proof:** Using (1.2), (1.3), (3.1) and (3.3) we get-
\[ g(F^k X, F^{2k} Y) = (-1)^k g(X, F^{3k} Y) \]
\[ = (-1)^k g(X, -lY) \]
\[ = (-1)^{k+1} g(X, lY) \]
\[ = (-1)^{k+1} g(X, (I - m)Y) \]
\[ = (-1)^{k+1} [g(X, Y) - g(X, mY)] \]
\[ = (-1)^{k+1} [g(X, Y) - m(X, Y)] \quad (3.4) \]

**4. Ker, tangent and normal vectors**

We define-
\[ \text{ker } F = \langle X : FX = 0 \rangle \quad (4.1) \]
\[ \text{tan } F = \langle X : FX \parallel X \rangle = \langle X : FX = \lambda X \rangle \quad (4.2) \]
\[ \text{Nor } F = \langle X : g(X, FY) = 0, \forall Y \rangle \quad (4.3) \]

**Theorem (4.1):** Let \( F \) satisfies (1.1) then,
\[ \text{ker } F^k = \text{ker } F^{4k} \quad (4.4) \]
\[ \text{tan } F^k = \text{tan } F^{4k} \quad (4.5) \]
\[ \text{Nor } F^k = \text{Nor } F^{4k} \quad (4.6) \]

**Proof:** Using (1.1), (4.1), (4.2) and (4.3) we get the results. We proved only (4.6).

Let \( X \in \text{Nor } F^k \Rightarrow g(X, F^k Y) = 0 \)
\[ \Rightarrow g(X, -F^{4k} Y) = 0 \]
\[ \Rightarrow g(X, F^{4k} Y) = 0 \]
\[ \Rightarrow X \in \text{Nor } F^{4k} \]
Thus,
\[ \text{Nor } F^k \subseteq \text{Nor } F^{4k} \quad (4.7) \]
Again let \( X \in \text{Nor } F^{4k} \Rightarrow g(X, F^{4k} Y) = 0 \)
\[ \Rightarrow g(X, F^k Y) = 0 \]
\[ \Rightarrow X \in \text{Nor } F^k \]
\[ \text{Nor } F^{4k} \subseteq \text{Nor } F^k \quad (4.8) \]
From (4.7) and (4.8) we get (4.6).

**Acknowledgement**

The authors thank reviewers and editors for their beautiful, pinpoint comments, helpful exchanges, and suggestions regarding the title.
References