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Some Generalized Fixed Point and Common Fixed Point Theorems in S-Metric Spaces

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Abstract: ***This article delves into the realm of S-metric spaces, an extension of metric spaces, exploring their fundamental properties and applications in fixed point theory. Beginning with a historical overview of key fixed point theorems and their limitations, the paper introduces the concept of S-metric spaces and provides definitions for essential terms. The main focus lies on establishing several fixed point theorems and common fixed points in complete S-metric spaces. The theorems presented here provide not only the existence of fixed points but also uniqueness, contributing to a deeper understanding of these spaces and their applications. The article combines theoretical explanations with rigorous proofs, making it a valuable resource for researchers and practitioners in functional analysis and topology.***

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MSC: 54H25, 47H10

1. Introduction and Preliminaries

French mathematician Frechet [8] pioneered the concept of metric space in 1906, while the definition given by Husdroff in 1914 was commonly used. One of the earliest successes in algebraic topology was the Brouwer [2] fixed point theorem, which serves as the basis for more general fixed point theorems that are extremely important to functional analysis, but this theorem can't provide the uniqueness of the fixed point. Later, Banach [1] in 1922, established the fixed point theorem, which is also known as the Banach contraction principle. This principle provides the existence and uniqueness of a self-mapping on a metric space. The idea of metric spaces has been widely generalized in the literature by many authors (refer [3], [5], [7], [10], [13], [14], [21], [22] and so on).These fixed point theorems were strongly followed by other authors in generalized metric spaces as well. The idea of D-metric space was introduced by Dhage[6] while Mustafa and Sims [11] established D* metric space which is the modification of D metric space. In 2005, Mustafa and Sims [12] also introduced G-metric space. Consequently, S- metric space was established by Sedghi *et al.* [15], who also proved fixed point theorems in this space. More research on S-metric space can be found in ([16], [17], [19], [20]). In this paper, we established some fixed point and common fixed point theorems in S-metric spaces which generalize, extends and improves some fixed point results in existing literature.

We began by reviewing some fundamental definitions that comes about for S- metric spaces which will be required within the sequel.

Definition 1.1 [15]: "Let $\mathcal{X}$ be a non-empty set. An S-metric on $\mathcal{X}$is a mapping $\mathcal{S}: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ which satisfies the following condition:
$(\mathcal{S}_1)$ $\mathcal{S}(u,v,w) = 0$ if and only if $u = v = w = 0$;
$(\mathcal{S}_2)$ $\mathcal{S}(u,v,w) \leq \mathcal{S}(u,u,a) + \mathcal{S}(v,v,a) + \mathcal{S}(w,w,a)$, for all $u,v,w,a \,\epsilon\, \mathcal{X}$.
The pair $(\mathcal{X},\mathcal{S})$ is called an S-metric space."

Example 1.2 [15]: "Let $\mathcal{X} = \mathbb{R}$. Then $\mathcal{S}(u,v,w)$is an S-metric on $\mathbb{R}$ given by $\mathcal{S}(u,v,w) = |u-w| + |v-w|$,which is known as usual S-metric space on $\mathcal{X}$."

Lemma 1.3 [15]: "If $(\mathcal{X},\mathcal{S})$ is an S-metric space on a non-empty set $\mathcal{X}$, then $(\mathcal{X},\mathcal{S})$ satisfy the symmetric condition, that is$\mathcal{S}(u,u,v) = \mathcal{S}(v,v,u)$, for all $u,v \,\epsilon\, \mathcal{X}$."

Definition 1.4[15] "Let $(\mathcal{X},\mathcal{S})$ be an S-metric space. For $r > 0$ and $u \,\epsilon\, \mathcal{X}$ we define the open ball $B_s(u,r)$ and closed ball and $B_s[u,r]$ with a center u and radius r as follows:

$$B_s(u,r) = \{v \,\epsilon\, \mathcal{X}: \mathcal{S}(v,v,u) < r\}$$

$B_s[u,r] = \{v \,\epsilon\, \mathcal{X}: \mathcal{S}(v,v,u) \leq r\}$."

Definition 1.5 [17]: "A sequence $\{u_n\}$ in $(\mathcal{X},\mathcal{S})$ is said to be convergent to some point $u \,\epsilon\, \mathcal{X}$, if $\mathcal{S}(u_n, u_n, u) \to 0$ as $n \to \infty$."

Definition 1.6[17]: "A sequence $\{u_n\}$ in $(\mathcal{X},\mathcal{S})$ is said to be Cauchy sequence if $\mathcal{S}(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$."

Definition 1.7 [17]: "An S -metric space $(\mathcal{X},\mathcal{S})$ is said to be complete if every Cauchy sequence in X is convergent in $\mathcal{X}$."

Lemma 1.8 [17]: "Let $(X,\mathcal{S})$ be an S-metric space. If $u_n \to u$ *and* $v_n \to v$ then $\mathcal{S}(u_n, u_n, v_n) \to \mathcal{S}(u,u,v)$."

Lemma 1.9 [15]: "Let $(\mathcal{X},\mathcal{S})$ be an S-metric space and $\{u_n\}$ is a convergent sequence in $\mathcal{X}$. Then $\lim_{n\to\infty} u_n$ is unique."

Lemma 1.10 [15]: "If $\{u_n\}$ is a sequence of elements from S-metric space $(\mathcal{X},\mathcal{S})$ satisfying the following property $\mathcal{S}(u_n, u_n, u_{n+1}) \leq k\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n)$, for each $k \in [0, 1)$ where $n \in \mathbb{N}$, then $\{u_n\}$ is a Cauchy sequence."

2. Main Results

In this section, certain fixed point and common fixed point theorems in S-metric spaces are proved. The following is our first main result:

Theorem 2.1: Let $(\mathcal{X},\mathcal{S})$ be a complete S-metric space. A mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ is such that

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) \leq p\,\mathcal{S}(u,u,v) + q\,[\mathcal{S}(u,u,\mathcal{T}u)+\mathcal{S}(v,v,\mathcal{T}v)] + r\,[\mathcal{S}(u,u,\mathcal{T}v)+\mathcal{S}(v,v,\mathcal{T}u)] + s\left[\frac{\mathcal{S}(u,u,v)\mathcal{S}(u,u,\mathcal{T}v)}{2\mathcal{S}(u,u,v)+\mathcal{S}(v,v,\mathcal{T}v)}\right] + t\left[\frac{\mathcal{S}(u,u,\mathcal{T}v)\mathcal{S}(v,v,\mathcal{T}v)}{2\mathcal{S}(u,u,v)+\mathcal{S}(v,v,\mathcal{T}v)}\right]. \quad (2.1)$$

For all $u,v\in\mathcal{X}$ with $u\neq v$ and $p,q,r,s,t\in[0,1)$ such that $p+2q+3r+s+t<1$ and $p+2r<1$. Then, $\mathcal{T}$ contains a fixed point which is also unique in $\mathcal{X}$.

Proof: Let a sequence be $\{u_n\}$ in $\mathcal{X}$ defined as for $u_0\in\mathcal{X}$, $\mathcal{T}u_n = u_{n+1}$ for all $n = 0,1,2,\ldots$. From (2.1) we have

$$\begin{aligned}
\mathcal{S}(u_n,u_n,u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n)\\
&\leq p\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + q\,[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1}) + \mathcal{S}(u_n,u_n,\mathcal{T}u_n)]\\
&\quad + r\,[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)+\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})]\\
&\quad + s\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)}{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}\right]\\
&\quad + t\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}\right]\\
&\leq p\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + q\,[\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_n,u_n,u_{n+1})]\\
&\quad + r\,[\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})+\mathcal{S}(u_n,u_n,u_n)]\\
&\quad + s\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})}{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})}\right]\\
&\quad + t\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})\mathcal{S}(u_n,u_n,u_{n+1})}{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})}\right]\\
&\leq p\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + q\,[\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_n,u_n,u_{n+1})]\\
&\quad + r\,[2\,\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})]\\
&\quad + s\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})\}}{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})}\right]\\
&\quad + t\left[\frac{\{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})\}\mathcal{S}(u_n,u_n,u_{n+1})}{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})}\right]\\
&\leq p\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + q\,[\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_n,u_n,u_{n+1})]\\
&\quad + r\,[2\,\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})]\\
&\quad + s\,\mathcal{S}(u_{n-1},u_{n-1},u_n)\\
&\quad + t\,\mathcal{S}(u_{n-1},u_{n-1},u_n).
\end{aligned}$$

$$(1-q-r-t)\mathcal{S}(u_n,u_n,u_{n+1}) \leq (p+q+2r+s)\mathcal{S}(u_{n-1},u_{n-1},u_n),$$

which implies that

$$\mathcal{S}(u_n,u_n,u_{n+1}) \leq \left(\frac{p+q+2r+s}{1-q-r-t}\right)\mathcal{S}(u_{n-1},u_{n-1},u_n)$$

$$\mathcal{S}(u_n,u_n,u_{n+1}) \leq k\,\mathcal{S}(u_{n-1},u_{n-1},u_n),$$

where $k = \left(\frac{p+q+2r+s}{1-q-r-t}\right) < 1$, Since $p+2q+3r+s+t<1$.

Therefore, we have

$$\mathcal{S}(u_n,u_n,u_{n+1}) \leq k\,\mathcal{S}(u_{n-1},u_{n-1},u_n) \leq k^2\mathcal{S}(u_{n-2},u_{n-2},u_{n-1}) \leq \cdots$$

Continuing this process up to n iterates. We have

$$\mathcal{S}(u_n,u_n,u_{n+1}) \leq k^n\mathcal{S}(u_0,u_0,u_1).$$

As $0\leq k\leq 1$, so for $n\to\infty, k^n\to 0$ and hence $\mathcal{S}(u_n,u_n,u_{n+1})\to 0$.

Thus, $\{u_n\}$ is a Cauchy sequence in a complete S-metric space. Therefore, there exists a point $w\in\mathcal{X}$ such that $u_n\to w$.

Next, we show that $\mathcal{T}$ has a fixed point. From (2.1) we have

$$\begin{aligned}
\mathcal{S}(u_n,u_n,\mathcal{T}w) &= \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}w)\\
&\leq p\,\mathcal{S}(u_{n-1},u_{n-1},w) + q\,[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1}) + \mathcal{S}(w,w,\mathcal{T}w)]\\
&\quad + r\,[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w)+\mathcal{S}(w,w,\mathcal{T}u_{n-1})]\\
&\quad + s\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},w)\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w)}{2\mathcal{S}(u_{n-1},u_{n-1},w)+\mathcal{S}(w,w,\mathcal{T}w)}\right] + t\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w)\mathcal{S}(w,w,\mathcal{T}w)}{2\mathcal{S}(u_{n-1},u_{n-1},w)+\mathcal{S}(w,w,\mathcal{T}w)}\right]. \quad (2.2)
\end{aligned}$$

Taking lim as $n\to\infty$ in (2.2), we have

$$\begin{aligned}
\mathcal{S}(w,w,\mathcal{T}w) &\leq p\,\mathcal{S}(w,w,w) + q\,[\mathcal{S}(w,w,\mathcal{T}w) + \mathcal{S}(w,w,\mathcal{T}w)]\\
&\quad + r\,[\mathcal{S}(w,w,\mathcal{T}w)+\mathcal{S}(w,w,\mathcal{T}w)]\\
&\quad + s\left[\frac{\mathcal{S}(w,w,w)\mathcal{S}(w,w,\mathcal{T}w)}{2\mathcal{S}(w,w,w)+\mathcal{S}(w,w,\mathcal{T}w)}\right] + t\left[\frac{\mathcal{S}(w,w,\mathcal{T}w)\mathcal{S}(w,w,\mathcal{T}w)}{2\mathcal{S}(w,w,w)+\mathcal{S}(w,w,\mathcal{T}w)}\right]
\end{aligned}$$

$$\mathcal{S}(w,w,\mathcal{T}w) \leq (2q+2r+t)\mathcal{S}(w,w,\mathcal{T}w),$$

which implies that

$\mathcal{S}(w,w,\mathcal{T}w) = 0$.

Hence, we get $\mathcal{T}w = w$, therefore $\mathcal{T}$ has a fixed point.

Now we claim that the fixed point is unique.

If possible, consider w^* be another fixed point of $\mathcal{T}$.

Using (2.1), we have

$$\begin{aligned}
\mathcal{S}(w,w,w^*) &= \mathcal{S}(\mathcal{T}w,\mathcal{T}w,\mathcal{T}w^*)\\
&\leq p\,\mathcal{S}(w,w,w^*) + q\,[\mathcal{S}(w,w,\mathcal{T}w)+\mathcal{S}(w^*,w^*,\mathcal{T}w^*)]\\
&\quad + r\,[\mathcal{S}(w,w,\mathcal{T}w^*)+\mathcal{S}(w^*,w^*,\mathcal{T}w)]\\
&\quad + s\left[\frac{\mathcal{S}(w,w,w^*)\mathcal{S}(w,w,\mathcal{T}w)}{2\mathcal{S}(w,w,w^*)+\mathcal{S}(w^*,w^*,\mathcal{T}w^*)}\right] + t\left[\frac{\mathcal{S}(w,w,w^*)\mathcal{S}(w^*,w^*,\mathcal{T}w^*)}{2\mathcal{S}(w,w,w^*)+\mathcal{S}(w^*,w^*,\mathcal{T}w^*)}\right].
\end{aligned}$$

As w and w^* are fixed points, so we have

$$\begin{aligned}
\mathcal{S}(w,w,w^*) &\leq p\,\mathcal{S}(w,w,w^*) + q\,[\mathcal{S}(w,w,w)+\mathcal{S}(w^*,w^*,w^*)]\\
&\quad + r\,[\mathcal{S}(w,w,w^*)+\mathcal{S}(w^*,w^*,w)]\\
&\quad + s\left[\frac{\mathcal{S}(w,w,w^*)\mathcal{S}(w,w,w)}{2\mathcal{S}(w,w,w^*)+\mathcal{S}(w^*,w^*,\mathcal{T}w^*)}\right] + t\left[\frac{\mathcal{S}(w,w,w^*)\mathcal{S}(w^*,w^*,w^*)}{2\mathcal{S}(w,w,w^*)+\mathcal{S}(w^*,w^*,\mathcal{T}w^*)}\right]\\
&\leq p\,\mathcal{S}(w,w,w^*) + 2r\,\mathcal{S}(w,w,w^*)
\end{aligned}$$

$$\mathcal{S}(w,w,w^*) \leq (p+2r)\mathcal{S}(w,w,w^*)$$

$$(1-p-2r)\mathcal{S}(w,w,w^*) \leq 0,$$

a contradiction as $p+2r<1$.

Hence, $\mathcal{S}(w,w,w^*) = 0$ which implies that $w = w^*$.
Thus, the fixed point is unique.
This completes the proof.

Theorem 2.2: Let $(\mathcal{X},\mathcal{S})$ be a complete S-metric space and $\mathcal{T}$ be a self mapping on $\mathcal{X}$ into itself satisfying the following condition

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) \le p\,\mathcal{S}(u,u,v) + q\,[\mathcal{S}(u,u,\mathcal{T}u) + \mathcal{S}(v,v,\mathcal{T}v)]\left[\frac{2\mathcal{S}(u,u,v)+\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,\mathcal{T}v)}\right] + r\,[\mathcal{S}(u,u,\mathcal{T}v)+\mathcal{S}(v,v,\mathcal{T}u)]\left[\frac{\mathcal{S}(u,u,v)+\mathcal{S}(v,v,\mathcal{T}v)+\mathcal{S}(u,u,\mathcal{T}v)}{\mathcal{S}(u,u,\mathcal{T}v)}\right], \quad (2.3)$$

for all $u,v \in \mathcal{X}$ with $u \ne v$ and $p,q,r \in [0,1)$ and such that

$$p+2q+5r<1 \text{ and } p+4r<1$$

then, $\mathcal{T}$ contains a fixed point which is also unique in $\mathcal{X}$.

Proof: Let a sequence be $\{u_n\}$ in $\mathcal{X}$ defined for $u_0 \in \mathcal{X}$ such that $\mathcal{T}u_n = u_{n+1}$ for all $n = 0,1,2,\ldots$. From (2.3) we have

$$\begin{aligned}\mathcal{S}(u_n,u_n,u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n)\\ &\le p\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + q\,[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1}) + \mathcal{S}(u_n,u_n,\mathcal{T}u_n)]\left[\frac{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)}\right]\\ &\quad + r\,[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)+\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})]\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,\mathcal{T}u_n)+\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)}\right]\\ &\le p\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + q\,[\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_n,u_n,u_{n+1})]\left[\frac{2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})}\right]\\ &\quad + r\,[\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})+\mathcal{S}(u_n,u_n,u_n)]\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})+\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})}\right]\\ &\le p\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + q\,[\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_n,u_n,u_{n+1})]\\ &\quad + r\,[\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_n,u_n,u_{n+1}) + \mathcal{S}un{-}1,un{-}1,un{+}1.\end{aligned}$$

$$\mathcal{S}(u_n,u_n,u_{n+1}) \le (p+q)\mathcal{S}(u_{n-1},u_{n-1},u_n) + q\mathcal{S}(u_n,u_n,u_{n+1}) + r\,[\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1}) + 2\mathcal{S}(u_{n-1},u_{n-1},u_n)+\mathcal{S}(u_n,u_n,u_{n+1})].$$

$$(1-q-2r)\mathcal{S}(u_n,u_n,u_{n+1}) \le (p+q+3r)\mathcal{S}(u_{n-1},u_{n-1},u_n).$$

$$\mathcal{S}(u_n,u_n,u_{n+1}) \le \left(\frac{p+q+3r}{1-q-2r}\right)\mathcal{S}(u_{n-1},u_{n-1},u_n),$$

which implies that

$$\mathcal{S}(u_n,u_n,u_{n+1}) \le k\,\mathcal{S}(u_{n-1},u_{n-1},u_n),$$

where $k = \left(\frac{p+q+3r}{1-q-2r}\right) < 1$, as $p+2q+5r<1$.

Repeating the iteration, we have

$$\mathcal{S}(u_n,u_n,u_{n+1}) \le k\,\mathcal{S}(u_{n-1},u_{n-1},u_n) \le \cdots \le k^n\mathcal{S}(u_0,u_0,u_1)$$

As $0 \le k \le 1$, so for $n \to \infty, k^n \to 0$ and hence $\mathcal{S}(u_n,u_n,u_{n+1}) \to 0$.

Thus, $\{u_n\}$ is a Cauchy sequence in a complete S-metric space. Therefore, there exists a point $w \in \mathcal{X}$ such that $u_n \to w$.

Next, we show that $\mathcal{T}$ has a fixed point. From (2.3) we have

$$\begin{aligned}\mathcal{S}(u_n,u_n,\mathcal{T}w) &= \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}w)\\ &\le p\,\mathcal{S}(u_{n-1},u_{n-1},w) + q\,[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1}) + \mathcal{S}(w,w,\mathcal{T}w)]\left[\frac{2\mathcal{S}(u_{n-1},u_{n-1},w)+\mathcal{S}(w,w,\mathcal{T}w)}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w)}\right]\\ &\quad + r\,[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w)+\mathcal{S}(w,w,\mathcal{T}u_{n-1})]\left[\frac{\mathcal{S}(u_{n-1},u_{n-1},w)+\mathcal{S}(w,w,\mathcal{T}w)+\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w)}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w)}\right].\end{aligned}$$

(2.4)

Taking lim as $n \to \infty$ in (2.4), we have

$$\begin{aligned}\mathcal{S}(w,w,\mathcal{T}w) &\le p\,\mathcal{S}(w,w,w) + q\,[\mathcal{S}(w,w,\mathcal{T}w) + \mathcal{S}(w,w,\mathcal{T}w)]\left[\frac{2\mathcal{S}(w,w,w)+\mathcal{S}(w,w,\mathcal{T}w)}{\mathcal{S}(w,w,\mathcal{T}w)}\right]\\ &\quad + r\,[\mathcal{S}(w,w,\mathcal{T}w) + \mathcal{S}(w,w,\mathcal{T}w)]\left[\frac{\mathcal{S}(w,w,w)+\mathcal{S}(w,w,\mathcal{T}w)+\mathcal{S}(w,w,\mathcal{T}w)}{\mathcal{S}(w,w,\mathcal{T}w)}\right]\end{aligned}$$

$$\mathcal{S}(w,w,\mathcal{T}w) \le (2q+4r)\mathcal{S}(w,w,\mathcal{T}w),$$

which implies that $\mathcal{S}(w,w,\mathcal{T}w) = 0$ and hence $\mathcal{T}(w) = w$.
Therefore, $\mathcal{T}$ has a fixed point.
Now we claim that the fixed point is unique.
If possible, let us suppose w^* be another fixed point of $\mathcal{T}$, then we have $\mathcal{T}(w^*) = w^*$.
Using (2.3), we have

$$\begin{aligned}\mathcal{S}(w,w,w^*) &= \mathcal{S}(\mathcal{T}w,\mathcal{T}w,\mathcal{T}w^*)\\ &\le p\,\mathcal{S}(w,w,w^*) + q\,[\mathcal{S}(w,w,\mathcal{T}w) + \mathcal{S}(w^*,w^*,\mathcal{T}w^*)]\left[\frac{2\mathcal{S}(w,w,w^*)+\mathcal{S}(w^*,w^*,\mathcal{T}w^*)}{\mathcal{S}(w,w,\mathcal{T}w^*)}\right]\\ &\quad + r\,[\mathcal{S}(w,w,\mathcal{T}w^*) + \mathcal{S}(w^*,w^*,\mathcal{T}w)]\left[\frac{\mathcal{S}(w,w,w^*)+\mathcal{S}(w^*,w^*,\mathcal{T}w^*)+\mathcal{S}(w,w,\mathcal{T}w^*)}{\mathcal{S}(w,w,\mathcal{T}w^*)}\right]\end{aligned}$$

As w and w^* are fixed points, so we have

$$\begin{aligned}\mathcal{S}(w,w,w^*) &\le p\,\mathcal{S}(w,w,w^*) + q\,[\mathcal{S}(w,w,w) + \mathcal{S}(w^*,w^*,w^*)]\left[\frac{2\mathcal{S}(w,w,w^*)+\mathcal{S}(w^*,w^*,w^*)}{\mathcal{S}(w,w,w^*)}\right]\\ &\quad + r\,[\mathcal{S}(w,w,w^*) + \mathcal{S}(w^*,w^*,w)]\left[\frac{\mathcal{S}(w,w,w^*)+\mathcal{S}(w^*,w^*,w^*)+\mathcal{S}(w,w,w^*)}{\mathcal{S}(w,w,w^*)}\right]\\ &\le p\,\mathcal{S}(w,w,w^*) + 4r\,\mathcal{S}(w,w,w^*)\end{aligned}$$

$$\mathcal{S}(w,w,w^*) \le (p+4r)\mathcal{S}(w,w,w^*),$$

a contradiction.
Hence, $\mathcal{S}(w,w,w^*) = 0$ which implies that $w = w^*$.
Thus, the fixed point is unique.
This completes the proof.

Theorem 2.3: Let us take a complete S-metric space $(\mathcal{X},\mathcal{S})$. Let $\mathcal{P},\mathcal{T}: \mathcal{X} \to \mathcal{X}$ then, it satisfies the below condition:

1) $\mathcal{T}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$.
2) $\mathcal{T}$ and $\mathcal{P}$ are continuous.
3) $\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{T}v) \le \alpha\,\mathcal{S}(u,u,v) + \beta\,[\mathcal{S}(u,u,\mathcal{P}u) + \mathcal{S}(v,v,\mathcal{T}v)]\left[\frac{2\mathcal{S}(u,u,v)+\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,\mathcal{T}v)}\right] + \gamma\,[\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{P}u)]\left[\frac{\{2\mathcal{S}(u,u,v)+\mathcal{S}(v,v,\mathcal{T}v)+\mathcal{S}(u,u,\mathcal{T}v)\}^2}{\{\mathcal{S}(u,u,\mathcal{T}v)\}^2}\right],$

for all $u,v \in \mathcal{X}$ and $\alpha,\beta,\gamma > 0$ with $\alpha+2\beta+8\gamma<1$.

Then show that $\mathcal{P}$ and $\mathcal{T}$ contains a common fixed point which is unique.

Proof: Let there be an arbitrary point $u_0 \in \mathcal{X}$, defining the sequence $\{u_j\}_{j\in N}$

$$u_1 = \mathcal{P}(u_0)$$

$$u_2 = \mathcal{T}(u_1)$$

$$\cdots\cdots\cdots$$

$$u_{2j+1} = \mathcal{P}(u_{2j})$$

$$u_{2j} = \mathcal{T}(u_{2j-1})$$

We have

$$\begin{aligned}\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2}) &= \mathcal{S}(\mathcal{P}u_{2j+1},\mathcal{P}u_{2j+1},\mathcal{T}u_{2j+1})\\ &\le \alpha\,\mathcal{S}(u_{2j},u_{2j},u_{2j+1})\\ &+\beta\left[\mathcal{S}(u_{2j},u_{2j},\mathcal{P}u_{2j})+\mathcal{S}(u_{2j+1},u_{2j+1},\mathcal{T}u_{2j+1})\right]\left[\frac{2\mathcal{S}(u_{2j},u_{2j},u_{2j+1})+\mathcal{S}(u_{2j+1},u_{2j+1},\mathcal{T}u_{2j+1})}{\mathcal{S}(u_{2j},u_{2j},\mathcal{T}u_{2j+1})}\right]\\ &+\gamma\left[\mathcal{S}(u_{2j},u_{2j},\mathcal{T}u_{2j+1})+\mathcal{S}(u_{2j+1},u_{2j+1},\mathcal{P}u_{2j})\right]\left[\frac{\{2\mathcal{S}(u_{2j},u_{2j},u_{2j+1})+\mathcal{S}(u_{2j+1},u_{2j+1},\mathcal{T}u_{2j+1})+\mathcal{S}(u_{2j},u_{2j},\mathcal{T}u_{2j+1})\}^2}{\{\mathcal{S}(u_{2j},u_{2j},\mathcal{T}u_{2j+1})\}^2}\right]\\ &\le \alpha\,\mathcal{S}(u_{2j},u_{2j},u_{2j+1})\\ &+\beta\left[\mathcal{S}(u_{2j},u_{2j},u_{2j+1})+\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2})\right]\left[\frac{2\mathcal{S}(u_{2j},u_{2j},u_{2j+1})+\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2})}{\mathcal{S}(u_{2j},u_{2j},u_{2j+2})}\right]\\ &+\gamma\left[\mathcal{S}(u_{2j},u_{2j},u_{2j+2})+\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+1})\right]\left[\frac{\{2\mathcal{S}(u_{2j},u_{2j},u_{2j+1})+\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2})+\mathcal{S}(u_{2j},u_{2j},u_{2j+2})\}^2}{\{\mathcal{S}(u_{2j},u_{2j},u_{2j+2})\}^2}\right]\\ &\le \alpha\,\mathcal{S}(u_{2j},u_{2j},u_{2j+1})\\ &+\beta\left[\mathcal{S}(u_{2j},u_{2j},u_{2j+1})+\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2})\right]\\ &+4\gamma\ \mathcal{S}(u_{2j},u_{2j},u_{2j+2}).\end{aligned}$$

$$\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2}) \le \alpha\,\mathcal{S}(u_{2j},u_{2j},u_{2j+1}) +\beta\left[\mathcal{S}(u_{2j},u_{2j},u_{2j+1})+\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2})\right]$$

$+4\gamma\,[\,\mathcal{S}(u_{2j},u_{2j},u_{2j+1})+\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2})]$,
which implies that

$$(1-\beta-4\gamma)\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2}) \le (\alpha+\beta+4\gamma)\mathcal{S}(u_{2j},u_{2j},u_{2j+1}).$$

$$\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2}) \le \left(\frac{\alpha+\beta+4\gamma}{1-\beta-4\gamma}\right)\mathcal{S}(u_{2j},u_{2j},u_{2j+1})$$

$\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2}) \le k\mathcal{S}(u_{2j},u_{2j},u_{2j+1})$, where $k = \left(\frac{\alpha+\beta+4\gamma}{1-\beta-4\gamma}\right) < 1$.

Continuing this way, we have
$\mathcal{S}(u_{2j+1},u_{2j+1},u_{2j+2}) \le k^{2j}\mathcal{S}(u_{2j},u_{2j},u_{2j+1}), 0 < k < 1$, and $k^{2j} \to 0$ as $n \to \infty$.
So, by Lemma 1.10, sequence $\{u_j\}_{j\in N}$ is a Cauchy sequence. Thus, there exists $w_1^* \in \mathcal{X}$ such that $\{u_j\}$ converges to w_1^*.
Further, the subsequence $\{\mathcal{P}u_{2j}\} \to w_1^*$ and $\{\mathcal{T}u_{2j}\} \to w_1^*$.
Since $\mathcal{P}$ and $\mathcal{T}$ are continuous, we have
$\mathcal{P}w_1^* = w_1^*$ and $\mathcal{T}w_1^* = w_1^*$. Thus, $\mathcal{P}$ and $\mathcal{T}$ has a fixed point w_1^*.
Now we claim that the fixed point for $\mathcal{P}$ and $\mathcal{T}$ are unique.
If possible, suppose w_1^* and w_2^* be another two fixed point of $\mathcal{P}$ and $\mathcal{T}$, then we have

$$\begin{aligned}\mathcal{S}(w_1^*,w_1^*,w_2^*) &= \mathcal{S}(\mathcal{P}w_1^*,\mathcal{P}w_1^*,\mathcal{T}w_2^*)\\ &\le \alpha\,\mathcal{S}(w_1^*,w_1^*,w_2^*)\\ &+\beta\left[\mathcal{S}(w_1^*,w_1^*,\mathcal{P}w_1^*)+\mathcal{S}(w_2^*,w_2^*,\mathcal{T}w_2^*)\right]\left[\frac{2\mathcal{S}(w_1^*,w_1^*,w_2^*)+\mathcal{S}(w_2^*,w_2^*,\mathcal{T}w_2^*)}{\mathcal{S}(u,u,\mathcal{T}v)}\right]\\ &+\gamma\left[\mathcal{S}(w_1^*,w_1^*,\mathcal{T}w_2^*)+\mathcal{S}(w_2^*,w_2^*,\mathcal{P}w_1^*)\right]\left[\frac{\{2\mathcal{S}(w_1^*,w_1^*,w_2^*)+\mathcal{S}(w_2^*,w_2^*,\mathcal{T}w_2^*)+\mathcal{S}(w_1^*,w_1^*,\mathcal{T}w_2^*)\}^2}{\{\mathcal{S}(w_1^*,w_1^*,\mathcal{T}w_2^*)\}^2}\right]\\ &\le \alpha\,\mathcal{S}(w_1^*,w_1^*,w_2^*)\\ &+\gamma\left[\mathcal{S}(w_1^*,w_1^*,w_2^*)+\mathcal{S}(w_2^*,w_2^*,w_1^*)\right]\left[\frac{\{2\mathcal{S}(w_1^*,w_1^*,w_2^*)+\mathcal{S}(w_2^*,w_2^*,w_2^*)+\mathcal{S}(w_1^*,w_1^*,w_2^*)\}^2}{\{\mathcal{S}(w_1^*,w_1^*,w_2^*)\}^2}\right].\end{aligned}$$

$$\mathcal{S}(w_1^*,w_1^*,w_2^*) \le (\alpha+9\gamma)\mathcal{S}(w_1^*,w_1^*,w_2^*) + 9\gamma\mathcal{S}(w_2^*,w_2^*,w_1^*) \quad . \qquad (2.5)$$

Similarly,

$$\mathcal{S}(w_2^*,w_2^*,w_1^*) \le (\alpha+9\gamma)\mathcal{S}(w_2^*,w_2^*,w_1^*) + 9\gamma\mathcal{S}(w_1^*,w_1^*,w_2^*) \qquad (2.6)$$

Subtracting equation (2.5) and (2.6), we obtain

$|\mathcal{S}(w_1^*, w_1^*, w_2^*) - \mathcal{S}(w_2^*, w_2^*, w_1^*)| \leq |\alpha||\mathcal{S}(w_1^*, w_1^*, w_2^*) - \mathcal{S}(w_2^*, w_2^*, w_1^*)|$ (2.7)

Clearly, $|\alpha| < 1$.

So, above inequality holds.

Hence, $\mathcal{S}(w_1^*, w_1^*, w_2^*) - \mathcal{S}(w_2^*, w_2^*, w_1^*) = 0$ (2.8)

From (2.5), (2.7) and (2.8), we have

$\mathcal{S}(w_1^*, w_1^*, w_2^*) = 0$ and $\mathcal{S}(w_2^*, w_2^*, w_1^*) = 0$,

which implies that $w_1^* = w_2^*$.

This completes the proof.

Theorem 2.4: Let $(\mathcal{X}, \mathcal{S})$ be a complete S-metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a self mapping satisfying the following condition:

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq b_1\mathcal{S}(u,u,v) + b_2\mathcal{S}(u,u,\mathcal{T}u) + b_3\mathcal{S}(v,v,\mathcal{T}v) + b_4\mathcal{S}(u,u,\mathcal{T}v) + b_5\mathcal{S}(v,v,\mathcal{T}u), \quad (2.9)$$

for all $u, v \in \mathcal{X}$ with $u \neq v$ and $b_1, b_2, b_3, b_4, b_5 \in [0,1]$ such that $1 - b_1 - b_2 - b_3 - 3b_4 > 0$ and $b_1 + b_3 + b_4 + b_5 < 1$. Then, $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$.

Proof: Let $u_0 \in \mathcal{X}$, choose $u_1 \in \mathcal{X}$ such that $\mathcal{T}u_0 = u_1$. Continuing this process, we can define a sequence $\mathcal{T}u_n = u_{n+1}$, $n \geq 1, n \in \mathbb{N}$. Without loss of generality, we suppose that $u_{n+1} \neq u_n$, for all $n \geq 1, n \in \mathbb{N}$. By condition (2.9) we have

$$\begin{aligned}
\mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\
&\leq b_1\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + b_2\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1}) + b_3\mathcal{S}(u_n, u_n, \mathcal{T}u_n) \\
&\quad + b_4\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + b_5\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1}) \\
&\leq b_1\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + b_2\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + b_3\mathcal{S}(u_n, u_n, u_{n+1}) \\
&\quad + b_4\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + b_5\mathcal{S}(u_n, u_n, u_n) \\
&\leq b_1\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + b_2\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + b_3\mathcal{S}(u_n, u_n, u_{n+1}) \\
&\quad + b_4[2\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})] + b_5\mathcal{S}(u_n, u_n, u_n).
\end{aligned}$$

Thus we have,

$$(1 - b_3 - b_4)\mathcal{S}(u_n, u_n, u_{n+1}) \leq (b_1 + b_2 + 2b_4)\mathcal{S}(u_{n-1}, u_{n-1}, u_n)$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{b_1 + b_2 + 2b_4}{1 - b_3 - b_4}\right)\mathcal{S}(u_{n-1}, u_{n-1}, u_n)$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq t\,\mathcal{S}(u_{n-1}, u_{n-1}, u_n),$$

where

$t = \left(\frac{b_1+b_2+2b_4}{1-b_3-b_4}\right) < 1$.

Hence, by Lemma 1.10, we conclude that sequence $\{u_n\}$ is a Cauchy sequence. Also, as $(\mathcal{X}, \mathcal{S})$ is complete so there exists a point $w \in \mathcal{X}$ such that $u_n \to w$.

Next, we claim that $\mathcal{T}$ has a fixed point. From (2.9) we have

$$\begin{aligned}
\mathcal{S}(u_n, u_n, \mathcal{T}w) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}w) \\
&\leq b_1\mathcal{S}(u_{n-1}, u_{n-1}, w) + b_2\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1}) + b_3\mathcal{S}(w, w, \mathcal{T}w) \\
&\quad + b_4\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}w) + b_5\mathcal{S}(w, w, \mathcal{T}u_{n-1}) \\
&\leq b_1\mathcal{S}(u_{n-1}, u_{n-1}, w) + b_2\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + b_3\mathcal{S}(w, w, \mathcal{T}w) \\
&\quad + b_4\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}w) + b_5\mathcal{S}(w, w, \mathcal{T}w) \qquad (2.10)
\end{aligned}$$

Taking lim as $n \to \infty$ in (2.10), we obtain,

$$\mathcal{S}(w, w, \mathcal{T}w) \leq b_1\mathcal{S}(w, w, w) + b_2\mathcal{S}(w, w, w) + b_3\mathcal{S}(w, w, \mathcal{T}w) + b_4\mathcal{S}(w, w, \mathcal{T}w) + b_5\mathcal{S}(w, w, w)$$

$$\mathcal{S}(w, w, \mathcal{T}w) \leq (b_3 + b_4 + b_5)\mathcal{S}(w, w, \mathcal{T}w)$$

$$\mathcal{S}(w, w, \mathcal{T}w) \leq (b_1 + b_3 + b_4 + b_5)\mathcal{S}(w, w, \mathcal{T}w).$$

which implies that

$\mathcal{S}(w, w, \mathcal{T}w) < \mathcal{S}(w, w, \mathcal{T}w)$, a contradiction.

Hence, we get

$$\mathcal{S}(w, w, \mathcal{T}w) = 0 \Rightarrow \mathcal{T}w = w.$$

Thus, $\mathcal{T}$ has a fixed point.

Next, we claim that the fixed point is unique.

If possible, let us consider w^* to be other fixed point of $\mathcal{T}$ with $w \neq w^*$.

Therefore, from (2.9), we have

$$\begin{aligned}
\mathcal{S}(w, w, w^*) &= \mathcal{S}(\mathcal{T}w, \mathcal{T}w, \mathcal{T}w^*) \\
&\leq b_1\mathcal{S}(w, w, w^*) + b_2\mathcal{S}(w, w, \mathcal{T}w) + b_3\mathcal{S}(w^*, w^*, \mathcal{T}w^*) \\
&\quad + b_4\mathcal{S}(w, w, \mathcal{T}w^*) + b_5\mathcal{S}(w^*, w^*, \mathcal{T}w) \\
&\leq b_1\mathcal{S}(w, w, w^*) + b_2\mathcal{S}(w, w, w) + b_3\mathcal{S}(w^*, w^*, w^*) \\
&\quad + b_4\mathcal{S}(w, w, w^*) + b_5\mathcal{S}(w^*, w^*, w)
\end{aligned}$$

$$\mathcal{S}(w, w, w^*) \leq (b_1 + b_4 + b_5)\mathcal{S}(w, w, w^*).$$

a contradiction as $b_1 + b_3 + b_4 + b_5 < 1$, which further implies that $\mathcal{S}(w, w, w^*) = 0$, and hence $w^* = w$.

Therefore, $\mathcal{T}$ has a unique fixed point.

This completes the proof.

Theorem 2.5: Let $(\mathcal{X}, \mathcal{S})$ be a complete S-metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a self mapping satisfying the following condition:

$$\begin{aligned}
\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) &\leq b_1\mathcal{S}(u, u, v) + b_2[\mathcal{S}(u, u, \mathcal{T}u) + \mathcal{S}(v, v, \mathcal{T}v)] \\
&\quad + b_3[\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}u)] + b_4[\mathcal{S}(u, u, \mathcal{T}u) + \mathcal{S}(u, u, v)] \\
&\quad + b_5[\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(u, u, v)], \qquad (2.11)
\end{aligned}$$

for all $u, v \in \mathcal{X}$ with $u \neq v$ and $b_1, b_2, b_3, b_4, b_5 \in [0,1]$ such that $b_1 + 2b_2 + 3b_3 + 2b_4 + 3b_5 < 1$ and $1 - (b_2 + b_3 + b_5) > 0$. Then, $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$.

Proof: Let $u_0 \in \mathcal{X}$, choose $u_1 \in \mathcal{X}$ such that $\mathcal{T}u_0 = u_1$. Continuing this process, we can define a sequence $\mathcal{T}u_n = u_{n+1}$ for all $n \geq 1, n \in \mathbb{N}$.

By using condition (2.11), we have

$$\begin{aligned}
\mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\
&\leq b_1\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + b_2[\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)] \\
&\quad + b_3[\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})] \\
&\quad + b_4[\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1}) + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)] \\
&\quad + b_5[\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n) + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)] \\
&\leq b_1\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + b_2[\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})] \\
&\quad + b_3[\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_n, u_n, u_n)] \\
&\quad + b_4[\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)] \\
&\quad + b_5[\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1}) + \mathcal{S}(u_{n-1}, u_{n-1}, u_n)]
\end{aligned}$$

$$\leq b_1\mathcal{S}(u_{n-1},u_{n-1},u_n) + b_2[\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_n,u_n,u_{n+1})] + b_3[2\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_n,u_n,u_{n+1})] + 2b_4\mathcal{S}(u_{n-1},u_{n-1},u_n) + b_5[2\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_n,u_n,u_{n+1}) + \mathcal{S}(u_{n-1},u_{n-1},u_n)].$$

Thus, we have

$$(1-b_2-b_3-b_5)\mathcal{S}(u_n,u_n,u_{n+1}) \leq (b_1+b_2+2b_3+2b_4+3b_5)\mathcal{S}(u_{n-1},u_{n-1},u_n)$$

$$\mathcal{S}(u_n,u_n,u_{n+1}) \leq \left(\frac{b_1+b_2+2b_3+2b_4+3b_5}{1-b_2-b_3-b_5}\right)\mathcal{S}(u_{n-1},u_{n-1},u_n)$$

$$\mathcal{S}(u_n,u_n,u_{n+1}) \leq t\,\mathcal{S}(u_{n-1},u_{n-1},u_n),$$

where

$$t = \left(\frac{b_1+b_2+2b_3+2b_4+3b_5}{1-b_2-b_3-b_5}\right) < 1.$$

Hence, by Lemma 1.10, we conclude that sequence $\{u_n\}$ is a Cauchy sequence. Also, as $(\mathcal{X},\mathcal{S})$ is complete so there exists a point $w \in \mathcal{X}$ such that $u_n \to w$.

Next, we show that $\mathcal{T}$ has a fixed point. From (2.11) we have

$$\mathcal{S}(u_n,u_n,\mathcal{T}w) = \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}w) \leq b_1\mathcal{S}(u_{n-1},u_{n-1},w) + b_2[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1}) + \mathcal{S}(w,w,\mathcal{T}w)] + b_3[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w) + \mathcal{S}(w,w,\mathcal{T}u_{n-1})] + b_4[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1}) + \mathcal{S}(u_{n-1},u_{n-1},w)] + b_5[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w) + \mathcal{S}(u_{n-1},u_{n-1},w)]$$

$$\leq b_1\mathcal{S}(u_{n-1},u_{n-1},w) + b_2[\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(w,w,\mathcal{T}w)] + b_3[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w) + \mathcal{S}(w,w,u_n)] + b_4[\mathcal{S}(u_{n-1},u_{n-1},u_n) + \mathcal{S}(u_{n-1},u_{n-1},w)] + b_5[\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}w) + \mathcal{S}(u_{n-1},u_{n-1},w)]. \quad (2.12)$$

Taking lim as $n \to \infty$ in (2.12), we get

$$\mathcal{S}(w,w,\mathcal{T}w) \leq b_1\mathcal{S}(w,w,w) + b_2[\mathcal{S}(w,w,w) + \mathcal{S}(w,w,\mathcal{T}w)] + b_3[\mathcal{S}(w,w,\mathcal{T}w) + \mathcal{S}(w,w,w)] + b_4[\mathcal{S}(w,w,w) + \mathcal{S}(w,w,w)] + b_5[\mathcal{S}(w,w,\mathcal{T}w) + \mathcal{S}(w,w,w)]$$

$$\mathcal{S}(w,w,\mathcal{T}w) \leq (b_2+b_3+b_5)\mathcal{S}(w,w,\mathcal{T}w),$$

a contradiction. Thus, we have $\mathcal{S}(w,w,\mathcal{T}w) = 0$, which implies that $\mathcal{T}w = w$.

Hence $\mathcal{T}$ has a fixed point.

Next, we claim that the fixed point is unique.

If possible, let us consider $w^*$to be other fixed point of $\mathcal{T}$.

Using (2.11), we have

$$\mathcal{S}(w,w,w^*) = \mathcal{S}(\mathcal{T}w,\mathcal{T}w,\mathcal{T}w^*) \leq b_1\mathcal{S}(w,w,w^*) + b_2[\mathcal{S}(w,w,\mathcal{T}w) + \mathcal{S}(w^*,w^*,\mathcal{T}w^*)] + b_3[\mathcal{S}(w,w,\mathcal{T}w^*) + \mathcal{S}(w^*,w^*,\mathcal{T}w)] + b_4[\mathcal{S}(w,w,\mathcal{T}w) + \mathcal{S}(w,w,w^*)] + b_5[\mathcal{S}(w,w,\mathcal{T}w^*) + \mathcal{S}(w,w,w^*)]$$

$$\leq b_1\mathcal{S}(w,w,w^*) + b_2[\mathcal{S}(w,w,w) + \mathcal{S}(w^*,w^*,w^*)] + b_3[\mathcal{S}(w,w,w^*) + \mathcal{S}(w^*,w^*,w)] + b_4[\mathcal{S}(w,w,w) + \mathcal{S}(w,w,w^*)] + b_5[\mathcal{S}(w,w,w^*) + \mathcal{S}(w,w,w^*)].$$

$$\mathcal{S}(w,w,w^*) \leq (b_1+2b_3+b_4+2b_5)\mathcal{S}(w,w,w^*)$$

$$\mathcal{S}(w,w,w^*) < (b_1+2b_2+3b_3+2b_4+3b_5)\mathcal{S}(w,w,w^*),$$

a contradiction as $b_1+2b_2+3b_3+2b_4+3b_5 < 1$, which further implies that $\mathcal{S}(w,w,w^*) = 0$, and hence $w^* = w$.

Therefore, $\mathcal{T}$ has a unique fixed point.

This completes the proof.

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