Comprehensive Semi-Analytical Approach to Quadratic Optimal Control Problems Governed by Ordinary Differential Equations

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Abstract: ***The primary focus of this research is on a semi-analytical method for solving a generalized quadratic optimal control problem constrained by Ordinary Differential Equation (ODE). The solutions are obtained by applying the first order optimality conditions on the Hamiltonian function, which is a necessary condition for optimality. The Adomian Decomposition Method (ADM), which views the approximation of a non-linear equation as an infinite series that typically converges to the exact solution, is used to solve the associated general Riccati differential equation. This leads to the optimal state, control and adjoint variables which yields the optimal objective functional value. Solutions to two examples of optimal control problems constrained by ODE are given.***

Keywords: Optimal Control, Semi-analytical Solution, Adomian Decomposition Method, Ordinary Differential Equation, Ricatti Differential Equation

1. Introduction

Optimization and optimal control pervade mathematics and science as they are the main tools in decision making [1]. In firms and businesses, people take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is to either minimize the effort required or to maximize the desired benefit. This effort required or benefit desired in any practical situation can be expressed as a function of certain decision variables [2].

Optimal control problems are mathematical programming problems involving two variables; the state and control variables. The state variable defines the system while the control variable describes the behaviour of the system [3]. The general quadratic optimal control problems are a class of optimal control problems whose cost functional is quadratic, and they arise in a wide range of applications [6]. Of a special interest is the Lagrange formulation of the general quadratic optimal control problem formulated as

$$\min I(x,u) = \int_0^T \left(px^2(t) + qu^2(t) \right) dt \quad (1)$$

$$Subject\ to\ \dot{x}(t) = ax(t) + bu(t) \quad (2)$$

$$x(0) = x_0,\ 0 \le t \le T \quad (3)$$

We note that a, b are real constants and $p, q > 0$. In quadratic optimal control problems, we desire to find the control law $u : \left[t_0, t_f \right] \subseteq \Re \rightarrow \Re^m$ that minimizes the given quadratic cost functional in Equation (1) subject to the constraints in Equations (2) and (3) [9].

The advances in many fields of life such as remote sensing, data mining, machine learning, epistemology and disease control has led to formulating arising problems into an optimization formulation [11]. This therefore, requires solving an optimization problem in their core as fast and efficient as possible [4]. The proposed algorithms for handling such optimization problems should remain efficient as the size (dimension) of the problems increases. Unfortunately, many optimization schemes are void of this scalability criterion because of their complicated iterations [7].

Optimal control problems constrained by Ordinary Differential Equations (ODEs) has a lot of applications in engineering, economics, biology and medicine but are often too complex to solve analytically [5]. There is therefore the need to develop semi-analytic approaches to solve problems at a faster rate.

2. Materials and Methods

The general form of a constrained dynamic continuous optimal control problem is defined as:

$$\min J\left(x(t), u(t)\right) = \int_{t_0}^{t_f} f\left(t, x(t), u(t)\right) dt \quad (4)$$

$$Subject\ to\ \dot{x}(t) = h\left(t, x(t), u(t)\right) \quad (5)$$

$$x(t_0) = x_0,\ t_0 \le t \le t_f \quad (6)$$

where t represents the independent time variable, t_0 and t_f are the initial and terminal times respectively, $x(t) \in \Re^n$ is a vector of state variables and $u(t) \in \Re^m$ is a vector of control variables which are going to be optimized,

$f:\Re\times\Re^n\times\Re^m\rightarrow\Re$ is the functional and $h:\Re\times\Re^n\times\Re^m\rightarrow\Re^p$ is a smooth vector field. Both f and h are continuously differentiable functions, that is, $f\in C^2\left[t_0,t_f\right]$ and $h\in C^1\left[t_0,t_f\right]$. x_0 is the known initial state and the final state $x\left(t_f\right)$ could be free (unrestricted) or fixed $\left(x\left(t_f\right)=x_f\right)$.

2.1 Necessary Conditions for a General Optimal Control Problem with ODE Constraint

The general form of an optimal control problem with ODE constraint is given as

$$\min J\left(x(t),u(t)\right)=\int_{t_0}^{t_f} f\left(t,x(t),u(t)\right)dt \qquad (7)$$

$$Subject\ to\ \dot{x}(t)=h\left(t,x(t),u(t)\right) \qquad (8)$$

$$x(t_0)=x_0,\ t_0\le t\le t_f \qquad (9)$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $f:\Re\times\Re^n\times\Re^m\rightarrow\Re$ and $h:\Re\times\Re^n\times\Re^m\rightarrow\Re^p$ are continuously differentiable functions.

The Hamiltonian function [8], from which the necessary conditions are derived is formed by introducing the Lagrange multiplier to change the constrained problem to an unconstrained problem

$$H(t,x,u,\lambda)=f(t,x,u)+\lambda h(t,x,u) \qquad (10)$$

where $H:\Re\times\Re^n\times\Re^m\times\Re^n\rightarrow\Re$ and the Lagrange multiplier $\lambda\in\Re^n$.

The Euler-Lagrange equations for (10) are given as

$$\frac{\partial H}{\partial x}-\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right)=0 \qquad (11)$$

$$\frac{\partial H}{\partial u}-\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{u}}\right)=0 \qquad (12)$$

$$\frac{\partial H}{\partial \lambda}-\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{\lambda}}\right)=0 \qquad (13)$$

Applying Equations (11), (12) and (13) on (10) yields a system of equations

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial h}{\partial x}-\dot{\lambda}=0 \qquad (14)$$

$$\frac{\partial f}{\partial u}+\lambda\frac{\partial h}{\partial u}=0 \qquad (15)$$

$$h(x,u,t)-\dot{x}(t)=0 \qquad (16)$$

Hence the Adjoint or Costate Equation is given as

$$\dot{\lambda}=-\frac{\partial H}{\partial x}\Longrightarrow\dot{\lambda}=-(f_x+\lambda h_x), \qquad (17)$$

the optimality condition as

$$\frac{\partial H}{\partial u}=0\ at\ u^*\Longrightarrow f_u+\lambda h_u=0 \qquad (18)$$

and the dynamics of the state equation as

$$\dot{x}(t)=h(t,x,u)\qquad x(t_0)=x_0 \qquad (19)$$

Equations (17) and (19) are a system of first order ordinary differential equations that can be solved simultaneously in order to get the values of the two constants by applying the boundary conditions. If only one boundary condition is given, then the free-end condition is applied to get the second boundary condition. The free-end condition is given as

$$\lambda(t_f)=0 \qquad (20)$$

For the solution to be optimal, all the conditions given by equations (17), (18), (19) and (20) must be satisfied. If one or more of the conditions are not satisfied, then the solution is not optimal. The state and adjoint (costate) equations are called dynamic equations while the optimal control equation is a stationary (static) equation. The boundary conditions lead to two-point boundary value problem. The state equation develops forward whereas the costate equation develops backward. Traditionally, the two-point boundary value problems demand computationally intensive iterative numerical procedures. These iterative numerical procedures lead to open loop control solutions [10].

2.2 The Semi-Analytical Solution

The Hamiltonian function is given as

$$H(t,x,u,\lambda)=(x^TAx+u^TBu)+\lambda^T(\dot{x}-Cx(\dot{t})-Du(t)) \qquad (21)$$

The Euler-Lagrange (E-L) system of equations yields

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right)=\frac{\partial H}{\partial x}\Leftrightarrow\dot{\lambda}(t)=2Ax(t)-C^T\lambda(t) \qquad (22)$$

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{u}}\right)=\frac{\partial H}{\partial u}\Leftrightarrow 2Bu(t)-D^T\lambda(t)=0 \qquad (23)$$

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{\lambda}}\right)=\frac{\partial H}{\partial \lambda}\Leftrightarrow\dot{x}(t)=Cx(t)-Du(t) \qquad (24)$$

From (23),

$$u(t)=\frac{B^{-1}D^T\lambda(t)}{2} \qquad (25)$$

Substituting (25) into (24) yields

$$\dot{x}(t)=Cx(t)+\frac{DB^{-1}D^T\lambda(t)}{2} \qquad (26)$$

The two-point boundary value problem can be written in matrix form as

$$\dot{N}=\begin{pmatrix}\dot{x}(t)\\ \dot{\lambda}(t)\end{pmatrix}=\begin{pmatrix}C & \frac{DB^{-1}D^T}{2}\\ 2A & -C^T\end{pmatrix}\begin{pmatrix}x(t)\\ \lambda(t)\end{pmatrix} \qquad (27)$$

Let

$$M=\begin{pmatrix}C & \frac{DB^{-1}D^T}{2}\\ 2A & -C^T\end{pmatrix} \qquad (28)$$

This yields

$$\dot{N}=MN\Leftrightarrow\dot{N}-MN=0 \qquad (29)$$

which is a linear Ordinary Differential Equation so we solve Equation (29) by using the Method of Integrating Factors.

The Integrating Factor is given as

$$e^{\int -Mdt}=e^{-Mt} \qquad (30)$$

Multiplying through by the integrating factor yields

$$\frac{d}{dt}\left(Ne^{-Mt}\right)=0 \qquad (31)$$

Integrating both sides gives

$$Ne^{-Mt}=c\Rightarrow N=ce^{Mt} \qquad (32)$$

Therefore

$$N = \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = ce^{Mt} \qquad (33)$$

This yields

$$N = \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \\ \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \\ \vdots \\ \lambda_n(t) \end{pmatrix} = e^{Mt} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ \vdots \\ x_n(0) \\ \lambda_1(0) \\ \lambda_2(0) \\ \lambda_3(0) \\ \vdots \\ \lambda_n(0) \end{pmatrix} \qquad (34)$$

Since the initial condition for x is given as $x(0) = x_0$, the next step is to choose $\lambda(0)$ such that the transversality condition is satisfied. A smart way to do that is to find the optimal control in a linear feedback form by looking for a function

$$K(\cdot):[0,T] \to \Re \qquad (35)$$

where

$$\lambda(t) = K(t)x(t) \qquad (36)$$

Now, assuming a linear function of the form:

$$\lambda(t) = K(t)x(t) \qquad (37)$$

where $K(t)$ is $n \times n$ symmetric, negative semidefinite matrix with element varying over time.

Differentiating (37) gives

$$\dot{\lambda}(t) = K(t)\dot{x}(t) + \dot{K}(t)x(t) \qquad (38)$$

From (22) and (38), we get

$$2Ax(t) - C^T\lambda(t) = K(t)\dot{x}(t) + \dot{K}(t)x(t) \qquad (39)$$

Substituting (26) into (39) yields

$$2Ax(t) - C^T\lambda(t) = K(t)\left[Cx(t) + \frac{DB^{-1}D^T\lambda(t)}{2}\right] + \dot{K}(t)x(t) \qquad (40)$$

$$2Ax(t) - C^T\lambda(t) = K(t)Cx(t) + \frac{K(t)DB^{-1}D^T\lambda(t)}{2} + \dot{K}(t)x(t) \qquad (41)$$

Putting (37) into (41) gives

$$2Ax(t) - C^TK(t)x(t) = K(t)Cx(t) + \frac{K(t)DB^{-1}D^TK(t)x(t)}{2} + \dot{K}(t)x(t) \qquad (42)$$

$$\Rightarrow \dot{K}(t)x(t) = 2Ax(t) - C^TK(t)x(t) - K(t)Cx(t) - \frac{K(t)DB^{-1}D^TK(t)x(t)}{2} \qquad (43)$$

This leads to the matrix Riccati nonlinear differential equation

$$\dot{K}(t) = 2A - C^TK(t) - K(t)C - \frac{K(t)DB^{-1}D^TK(t)}{2} \qquad (44)$$

Since $\lambda(T) = 0$, the terminal condition $K(T) = 0$ since $x(T) \neq 0$.

Equation (44) is solved by means of the Backward RungeKutta method of Order 4 to get $K(0)$. The solution to the matrix Riccati nonlinear differential equation (44) is now obtained by the Adomian Decomposition Method.

3. Results

Example 3.1.
Minimize

$$J = \int_0^1 (2x_1^2(t) + 2x_1(t)x_2(t) + 2x_2^2(t) + 2u_1^2(t) + 2u_1(t)u_2(t) + 2u_2^2(t)dt \qquad (45)$$

$$\dot{x_1} = x_1 - x_2 + 2u_1 + u_2 \qquad (46)$$

$$\dot{x_2} = x_1 + x_2 - u_2, \quad X(0) = (1 \ \ 1) \qquad (47)$$

Solution

$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

$$M = \begin{bmatrix} C & \frac{DB^{-1}D^T}{2} \\ 2A & -C^T \end{bmatrix} \qquad (48)$$

This implies that

$$M = \begin{bmatrix} 1.000 & -1.000 & 1.000 & 0.000 \\ 1.000 & 1.000 & 0.000 & 0.333 \\ 4.000 & 2.000 & -1.000 & -1.000 \\ 2.000 & 4.000 & 1.000 & -1.000 \end{bmatrix} \qquad (49)$$

The matrix riccati nonlinear differential equation in Equation (44) leads to the following system of nonlinear first order differential equations

$$\dot{k}_{11} = 4 - {k_{11}}^2 - 2k_{11} - k_{12} - k_{21} - \frac{k_{12}k_{21}}{3} \qquad (50)$$

$$\dot{k}_{12} = 2 + k_{11} - 2k_{12} - k_{22} - k_{11}k_{12} - \frac{k_{12}k_{22}}{3} \qquad (51)$$

$$\dot{k}_{21} = 2 + k_{11} - 2k_{21} - k_{22} - k_{11}k_{21} - \frac{k_{21}k_{22}}{3} \qquad (52)$$

$$\dot{k}_{22} = 4 - \frac{{k_{11}}^2}{3} - 2k_{22} + k_{12} + k_{21} - k_{12}k_{21} \qquad (53)$$

Solving Equations (50),(51),(52) and (53) by means of Backward Runge-Kutta 4 method gives

$$K(0) = \begin{bmatrix} -\frac{34596}{10000} & -\frac{9015}{10000} \\ -\frac{9015}{10000} & -\frac{43229}{10000} \end{bmatrix} \qquad (54)$$

With our initial condition, $K(0)$, the matrix Riccati differential equation is solved byAdomain Decomposition method.

Applying $L = \frac{d}{dt}$ *and* $L^{-1} = \int_0^t [.]\,dt$ to Equations (50) to (53) gives

$$L^{-1}Lk_{11} = L^{-1}\left[4 - k_{11}^2 - 2k_{11} - k_{12} - k_{21} - \frac{k_{12}k_{21}}{3}\right] \qquad (55)$$

$$L^{-1}Lk_{12} = L^{-1}\left[2 + k_{11} - 2k_{12} - k_{22} - k_{11}k_{12} - \frac{k_{12}k_{22}}{3}\right] \qquad (56)$$

$$L^{-1}Lk_{21} = L^{-1}\left[2 + k_{11} - 2k_{21} - k_{22} - k_{11}k_{21} - \frac{k_{21}k_{22}}{3}\right] \quad (57)$$

$$L^{-1}Lk_{22} = L^{-1}\left[4 - \frac{k_{11}^2}{3} - 2k_{22} + k_{12} + k_{21} - k_{12}k_{21}\right] \quad (58)$$

This yields
$k_{11,0} = k_{11}(0) + 4t$ and

$$k_{11,n+1}(t) = -\int_0^t \left[2k_{11,n} + k_{12,n} + k_{21,n} + \sum_{j=0}^{n} k_{11,j}k_{11,n-j} + \frac{1}{3}\sum_{j=0}^{n} k_{12,j}k_{21,n-j}\right] dt \quad (59)$$

$k_{12,0} = k_{12}(0) + 2t$and

$$k_{12,n+1}(t) = -\int_0^t \left[-k_{11,n} + 2k_{12,n} + k_{22,n} + \sum_{j=0}^{n} k_{11,j}k_{12,n-j} + \frac{1}{3}\sum_{j=0}^{n} k_{12,j}k_{22,n-j}\right] dt \quad (60)$$

$k_{21,0} = k_{21}(0) + 2t$and

$$k_{21,n+1}(t) = -\int_0^t \left[-k_{11,n} + 2k_{21,n} + k_{22,n} + \sum_{j=0}^{n} k_{11,j}k_{21,n-j} + \frac{1}{3}\sum_{j=0}^{n} k_{21,j}k_{22,n-j}\right] dt \quad (61)$$

$k_{22,0} = k_{22}(0) + 4t$and

$$k_{22,n+1}(t) = -\int_0^t \left[-k_{12,n} - k_{21,n} + 2k_{22,n} + \sum_{j=0}^{n} k_{12,j}k_{21,n-j} + \frac{1}{3}\sum_{j=0}^{n} k_{11,j}k_{11,n-j}\right] dt \quad (62)$$

Approximations to the solutions with four terms gives the solution to the Riccati equation as

$$k_{11}(t) = -\frac{155312}{8505}t^7 + \frac{62369939}{1012500}t^6 - \frac{2639851661011}{40500000000}t^5 + \frac{327769341811391}{108000000000000}t^4 + \frac{65390870896557607}{30000000000000000}t^3 + \frac{1012892869139}{1000000000000}t^2 + \frac{48246709}{100000000}t - \frac{8649}{2500} \quad (63)$$

$$k_{12}(t) = -\frac{14464}{1215}t^7 + \frac{51622399}{1215000}t^6 - \frac{346592742089}{6750000000}t^5 + \frac{7561315700463023}{3240000000000000}t^4 + \frac{1412390985523971}{4000000000000000}t^3 - \frac{517459037733}{400000000000}t^2 + \frac{4968783}{20000000}t - \frac{1803}{2000} \quad (64)$$

$$k_{21}(t) = -\frac{14464}{1215}t^7 + \frac{51622399}{1215000}t^6 - \frac{346592742089}{6750000000}t^5 + \frac{327769341811391}{324000000000000}t^4 + \frac{1412390985523971}{40000000000000000}t^3 - \frac{517459037733}{400000000000}t^2 + \frac{4968783}{100000000}t - \frac{8649}{2500} \quad (65)$$

$$k_{22}(t) = -\frac{93584}{8505}t^7 + \frac{4486297}{112500}t^6 - \frac{135941512919}{2700000000}t^5 + \frac{50418912157513}{2160000000000}t^4 + \frac{219014360382378407}{90000000000000000}t^3 + \frac{5011698938087}{1000000000000}t^2 + \frac{604048703}{100000000}t - \frac{43229}{10000} \quad (66)$$

The rest of the component of the iteration formulae can be obtained using MATLAB package. Hence by Equation (37)

$$\lambda(0)^T = \left(\frac{43611}{10000} \; -\frac{52244}{10000}\right) \quad (67)$$

Therefore, the optimal state and adjoint variables are given as

$$x_1(t) = \left(\frac{17042}{1000000}e^{\frac{913}{500}t} + \frac{983}{1000}e^{-\frac{913}{500}t}\right)cos\left(\frac{1633}{2000}\right)t - \left(\frac{1243}{6250}e^{\frac{913}{500}t} + \frac{1491}{500}e^{-\frac{913}{500}t}\right)sin\left(\frac{1633}{2000}\right)t \quad (68)$$

$$x_2(t) = \left(\frac{161}{625}e^{\frac{913}{500}t} + \frac{464}{625}e^{-\frac{913}{500}t}\right)cos\left(\frac{1633}{2000}\right)t - \left(\frac{1277}{25000}e^{\frac{913}{500}t} - \frac{363}{250}e^{-\frac{913}{500}t}\right)sin\left(\frac{1633}{2000}\right)t \quad (69)$$

$$\lambda_1(t) = \left(\frac{2731}{25000}e^{\frac{913}{500}t} - \frac{447}{100}e^{-\frac{913}{500}t}\right)\cos\left(\frac{1633}{2000}\right)t - \left(\frac{573}{2500}e^{\frac{913}{500}t} - \frac{4539}{500}e^{-\frac{913}{500}t}\right)\sin\left(\frac{1633}{2000}\right)t \quad (70)$$

$$\lambda_2(t) = \left(\frac{2309}{5000}e^{\frac{913}{500}t} - \frac{2843}{500}e^{-\frac{913}{500}t}\right)\cos\left(\frac{1633}{2000}\right)t - \left(\frac{4019}{25000}e^{\frac{913}{500}t} + \frac{259}{50}e^{-\frac{913}{500}t}\right)\sin\left(\frac{1633}{2000}\right)t \quad (71)$$

From Equation (25), the optimal control variables are given as

$$u_1(t) = \left(\frac{6579}{50000}e^{\frac{913}{500}t} - \frac{1591}{500}e^{-\frac{913}{500}t}\right)\cos\left(\frac{1633}{2000}\right)t - \left(\frac{707}{5000}e^{\frac{913}{500}t} + \frac{919}{250}e^{-\frac{913}{500}t}\right)\sin\left(\frac{1633}{2000}\right)t \quad (72)$$

$$u_2(t) = \left(-\frac{1539}{10000}e^{\frac{913}{500}t} - \frac{9477}{5000}e^{-\frac{913}{500}t}\right)\cos\left(\frac{1633}{2000}\right)t - \left(\frac{2679}{50000}e^{\frac{913}{500}t} + \frac{4317}{2500}e^{-\frac{913}{500}t}\right)\sin\left(\frac{1633}{2000}\right)t \quad (73)$$

Example 3.2.
Minimize

$$J = \int_0^1 \big(2x_1^2(t) + x_1(t)x_2(t) + x_2^2(t) + 2x_1(t)x_3(t) + x_2(t)x_3(t) + 2x_3^2(t) + 3u_1^2(t) + 4u_1(t)u_2(t) + 2u_2^2(t) + u_1(t)u_3(t) + 2u_1(t)u_3(t) + u_3^2(t)\big)dt \quad (74)$$

$$\dot{x_1} = 2x_1 + x_2 + x_3 + 2u_1 + 2u_2 + u_3, \quad (75)$$

$$\dot{x_2} = -x_1 + x_2 + 2x_3 - u_2 + 2u_3, \quad (76)$$

$$\dot{x_3} = x_1 - x_3 + u_1 + \mu_2 - \mu_3; \quad X(0) = 1_{3\times1}, \quad (77)$$

Solution

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}, A = \begin{bmatrix} 2 & 0.5 & 1 \\ 0.5 & 1 & 0.5 \\ 1 & 0.5 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 3 & 2 & 0.5 \\ 2 & 2 & 1 \\ 0.5 & 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}, D = \begin{bmatrix} 2 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

Now, $M = \begin{bmatrix} C & \frac{DB^{-1}D^T}{2} \\ 2A & -C^T \end{bmatrix}$ (78)

This implies that

$$M = \begin{bmatrix} 2.00 & 1.00 & 1.00 & 1.00 & -0.50 & 0.50 \\ -1.00 & 1.00 & 2.00 & -0.50 & 18.75 & -9.25 \\ 1.00 & 0.00 & -1.00 & 0.50 & -9.25 & 4.75 \\ 4.00 & 1.00 & 2.00 & -2.00 & 1.00 & -1.00 \\ 1.00 & 2.00 & 1.00 & -1.00 & -1.00 & 0.00 \\ 2.00 & 1.00 & 4.00 & -1.00 & -2.00 & 1.00 \end{bmatrix} \quad (79)$$

The matrix riccati nonlinear differential equation in (44) leads to the following system of nonlinear first order differential equations

$\dot{k}_{11} = 4 - 4k_{11} + k_{12} - k_{13} + k_{21} - k_{31} - \frac{k_{11}}{2}(2k_{11} -$
$k12+k13-k312k11-372k12+192k13+k212k11-752k12+372k13$ (80)

$\dot{k}_{12} = 1 - 3k_{12} - k_{11} + k_{22} - k_{32} - \frac{k_{12}}{2}(2k_{11} - k_{12} +$
$k13-k322k11-372k12+192k13+k222k11-752k12+372k13$ (81)

$\dot{k}_{13} = 2 - k_{13} - k_{11} - 2k_{12} + k_{23} - k_{33} - \frac{k_{13}}{2}(2k_{11} -$
$k12+k13-k332k11-372k12+192k13+k232k11-752k12+372k13$ (82)

$\dot{k}_{21} = 1 - 3k_{21} - k_{11} - k_{23} + k_{22} - \frac{k_{11}}{2}(2k_{21} - k_{22} +$
$k23-k312k21-372k22+192k23+k212k21-752k22+372k23$ (83)

$$\dot{k}_{22} = 2 - 2k_{22} - k_{12} - k_{21} - \frac{k_{12}}{2}(2k_{21} - k_{22} + k_{23}) - \frac{k_{32}}{2}\left(k_{21} - \frac{37}{2}k_{22} + \frac{19}{2}k_{23}\right) - \frac{k_{22}}{2}\left(k_{21} - \frac{75}{2}k_{22} + \frac{37}{2}k_{23}\right) \quad (84)$$

$$\dot{k}_{23} = 1 - k_{13} - k_{21} - 2k_{22} - \frac{k_{13}}{2}(2k_{21} - k_{22} + k_{23}) - \frac{k_{33}}{2}\left(k_{21} - \frac{37}{2}k_{22} + \frac{19}{2}k_{23}\right) + \frac{k_{23}}{2}\left(k_{21} - \frac{75}{2}k_{22} + \frac{37}{2}k_{23}\right) \quad (85)$$

$\dot{k}_{31} = 2 - k_{31} - k_{11} - 2k_{21} + k_{32} - k_{33} - \frac{k_{11}}{2}(2k_{31} -$
$k32+k33-k312k31-372k32+192k33+k212k31-752k32+372k33$ (86)

$\dot{k}_{32} = 1 - k_{12} - 2k_{22} - k_{31} - \frac{k_{12}}{2}(2k_{31} - k_{32} + k_{33}) - \frac{k_{32}}{2}\left(k_{31} - \frac{37}{2}k_{32} + \frac{19}{2}k_{33}\right) + \frac{k_{22}}{2}\left(k_{31} - \frac{75}{2}k_{32} + \frac{37}{2}k_{33}\right)$
(87)

$\dot{k}_{33} = 4 + 2k_{33} - k_{13} - 2k_{23} - k_{31} - 2k_{32} - \frac{k_{13}}{2}(2k_{31} - k_{32} + k_{33}) - \frac{k_{33}}{2}\left(k_{31} - \frac{37}{2}k_{32} + \frac{19}{2}k_{33}\right) + \frac{k_{23}}{2}\left(k_{31} - \frac{75}{2}k_{32} + \frac{37}{2}k_{33}\right)$(88)

Solving the system of equations from (80)-(88) by means of Backward Runge-Kutta 4 method gives

$$K(0)=\begin{bmatrix} -\frac{40037}{10000} & -\frac{11269}{10000} & -\frac{16124}{10000} \\ -\frac{11269}{10000} & -\frac{12778}{10000} & -\frac{16135}{10000} \\ -\frac{16124}{10000} & -\frac{16135}{10000} & -\frac{27979}{10000} \end{bmatrix} \quad (89)$$

With our initial condition,K(0), the matrix Riccatidifferential equation is solved by Adomian Decomposition method. Applying L=$\frac{d}{dt}$, and L^{-1}= $\int_0^t [.]\, dt$ *to Equations* (80) to (88) gives

$L^{-1}Lk_{11} = L^{-1}\Big[4 - 4k_{11} + k_{12} - k_{13} + k_{21} - k_{31} -$
k1122k11−k12+k13
−k312k11−372k12+192k13+k212k11−752k12+372k13 (90)

$L^{-1}Lk_{12} = L^{-1}\Big[1 - 3k_{12} - k_{11} + k_{22} - k_{32} -$
k1222k11−k12+k13−k322k11−372k12+192k13+k222k11−752k12+372k13
(91)

$L^{-1}Lk_{13} = L^{-1}\Big[2 - k_{13} - k_{11} - 2k_{12} + k_{23} - k_{23} -$
k1322k11−k12+k13−k332k11−372k12+192k13+k232k11−752k12+372k13 (92)

$L^{-1}Lk_{21} = L^{-1}\Big[1 - 3k_{21} - k_{11} - k_{23} + k_{22} -$
k1122k21−k22+k23−k312k21−372k22+192k23+k212k21−752k22+372k23
(93)

$L^{-1}Lk_{22} = L^{-1}[2 - 2k_{22} - k_{12} - k_{21} - \frac{k_{12}}{2}(2k_{21} - k_{22} +$
k23−k322k21−372k22+192k23−k222k21−752k22+372k23] (94)

$$L^{-1}Lk_{23} = L^{-1}\left[1 - k_{13} - k_{21} - 2k_{22} - \frac{k_{13}}{2}(2k_{21} - k_{22} + k_{23}) - \frac{k_{33}}{2}\left(k_{21} - \frac{37}{2}k_{22} + \frac{19}{2}k_{23}\right) + \frac{k_{23}}{2}\left(k_{21} - \frac{75}{2}k_{22} + \frac{37}{2}k_{23}\right)\right] \quad (95)$$

$$L^{-1}Lk_{31} = L^{-1}\left[2 - k_{31} - k_{11} - 2k_{21} + k_{32} - k_{33} - \frac{k_{11}}{2}(2k_{31} - k_{32} + k_{33}) - \frac{k_{31}}{2}\left(k_{31} - \frac{37}{2}k_{32} + \frac{19}{2}k_{33}\right) + \frac{k_{21}}{2}\left(k_{31} - \frac{75}{2}k_{32} + \frac{37}{2}k_{33}\right)\right] \quad (96)$$

$$L^{-1}Lk_{32} = L^{-1}\left[1 - k_{12} - 2k_{22} - k_{31} - \frac{k_{12}}{2}(2k_{31} - k_{32} + k_{33}) - \frac{k_{32}}{2}\left(k_{31} - \frac{37}{2}k_{32} + \frac{19}{2}k_{33}\right) + \frac{k_{22}}{2}\left(k_{31} - \frac{75}{2}k_{32} + \frac{37}{2}k_{33}\right)\right]$$
(97)

$$L^{-1}Lk_{33} = L^{-1}\left[4 + 2k_{33} - k_{13} - 2k_{23} - k_{31} - 2k_{32} - \frac{k_{13}}{2}(2k_{31} - k_{32} + k_{33}) - \frac{k_{33}}{2}\left(k_{31} - \frac{37}{2}k_{32} + \frac{19}{2}k_{33}\right) + \frac{k_{23}}{2}\left(k_{31} - \frac{75}{2}k_{32} + \frac{37}{2}k_{33}\right)\right] \quad (98)$$

This yields

$k_{11,0} = k_{11}(0) + 4t$ and

$$k_{11,\,n+1}(t) = -\int_0^t \Bigg[4k_{11,n} - k_{12,n} + k_{13,n} - k_{21,n} + k_{31,n} + \sum_{j=0}^{n} k_{11,j}k_{11,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{11,j}k_{12,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{11,j}k_{13,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{11,j}k_{31,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{12,j}k_{31,n-j} + \frac{19}{4}\sum_{j=0}^{n} k_{13,j}k_{31,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{11,j}k_{21,n-j} + \frac{75}{4}\sum_{j=0}^{n} k_{12,j}k_{21,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{13,j}k_{21,n-j}\Bigg]dt \quad (99)$$

$k_{12,0} = k_{12}(0) + t$ and

$$k_{12,n+1}(t) = -\int_0^t \Bigg[k_{11,n} + 3k_{12,n} - k_{22,n} + k_{32,n} + \sum_{j=0}^{n} k_{11,j}k_{12,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{12,j}k_{12,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{12,j}k_{13,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{11,j}k_{32,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{12,j}k_{32,n-j} + \frac{19}{4}\sum_{j=0}^{n} k_{13,j}k_{32,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{11,j}k_{22,n-j} + \frac{75}{4}\sum_{j=0}^{n} k_{12,j}k_{22,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{13,j}k_{22,n-j}\Bigg]dt \quad (100)$$

$k_{13,0} = k_{13}(0) + 2t$ and

$$k_{13,\,n+1}(t) = -\int_0^t \Bigg[k_{11,n} + 2k_{12,n} + k_{13,n} - k_{23,n} + k_{33,n} + \sum_{j=0}^{n} k_{11,j}k_{13,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{12,j}k_{13,\,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{13,j}k_{13,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{11,j}k_{33,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{12,j}k_{33,n-j} + \frac{19}{4}\sum_{j=0}^{n} k_{13,j}k_{33,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{11,j}k_{23,n-j} + \frac{75}{4}\sum_{j=0}^{n} k_{12,j}k_{23,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{13,j}k_{23,n-j}\Bigg]dt \quad (101)$$

$k_{21,0} = k_{21}(0) + t$ and

$$k_{21,n+1}(t) = -\int_0^t \Bigg[k_{11,n} + 3k_{21,n} - k_{22,n} + k_{23,n} + \sum_{j=0}^{n} k_{11,j} k_{21,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{11,j} k_{22,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{11,j} k_{23,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{21,j} k_{31,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{22,j} k_{31,n-j} + \frac{19}{4}\sum_{j=0}^{n} k_{23,j} k_{31,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{21,j} k_{21,n-j} + \frac{75}{4}\sum_{j=0}^{n} k_{21,j} k_{22,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{21,j} k_{23,n-j} \Bigg] dt \qquad (102)$$

$k_{22,0} = k_{22}(0) + 2t$ and

$$k_{22,n+1}(t) = -\int_0^t \Bigg[k_{12,n} + k_{21,n} + 2k_{22,n} + \sum_{j=0}^{n} k_{12,j} k_{21,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{12,j} k_{22,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{12,j} k_{23,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{21,j} k_{32,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{22,j} k_{32,n-j} + \frac{19}{4}\sum_{j=0}^{n} k_{23,j} k_{32,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{21,j} k_{22,n-j} - \frac{75}{4}\sum_{j=0}^{n} k_{22,j} k_{22,n-j} + \frac{37}{4}\sum_{j=0}^{n} k_{22,j} k_{23,n-j} \Bigg] dt \qquad (103)$$

$k_{23,0}=k_{23}(0)+t$ and

$$k_{23,n+1}(t) = -\int_0^t \Bigg[k_{13,n} + k_{21,n} + 2k_{22,n} + \sum_{j=0}^{n} k_{13,j}\, k_{21,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{13,j}\, k_{22,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{13,j}\, k_{23,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{21,j}\, k_{33,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{22,j}\, k_{33,n-j} + \frac{19}{4}\sum_{j=0}^{n} k_{23,j}\, k_{33,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{21,j}\, k_{23,n-j} + \frac{75}{4}\sum_{j=0}^{n} k_{22,j}\, k_{23,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{23,j}\, k_{23,n-j} \Bigg] dt \qquad (104)$$

$k_{31,0}=k_{31}(0)+2t$ and

$$k_{31,n+1}(t) = -\int_0^t \Bigg[k_{11,n} + 2k_{21,n} + k_{31,n} - k_{32,n} + k_{33,n} + \sum_{j=0}^{n} k_{11,j}\, k_{31,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{11,j} k_{32,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{11,j}\, k_{33,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{31,j}\, k_{31,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{31,j}\, k_{32,n-j} + \frac{19}{4}\sum_{jj=0}^{n} k_{31,j}\, k_{33,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{21,j}\, k_{31,n-j} + \frac{75}{4}\sum_{j=0}^{n} k_{21,j}\, k_{32,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{21,j}\, k_{33,n-j} \Bigg] dt \qquad (105)$$

$k_{32,0}=k_{32}(0)+t$ and

$$k_{32,n+1}(t) = -\int_0^t \Bigg[k_{12,n} + 2k_{22,n} + k_{31,n} + \sum_{j=0}^{n} k_{12,j}\, k_{31,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{12,j} k_{32,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{12,j}\, k_{33,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{31,j}\, k_{32,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{32,j}\, k_{32,n-j} + \frac{19}{4}\sum_{jj=0}^{n} k_{32,j}\, k_{33,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{22,j}\, k_{31,n-j} + \frac{75}{4}\sum_{j=0}^{n} k_{22,j}\, k_{32,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{22,j}\, k_{33,n-j} \Bigg] dt \qquad (106)$$

$k_{33,0}=k_{33}(0)+4t$ and

$$k_{33,n+1}(t) = -\int_0^t \Bigg[k_{13,n} + 2k_{23,n} + k_{31,n} + 2k_{32,n} - 2k_{33,n} + \sum_{j=0}^{n} k_{13,j}\, k_{31,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{13,j} k_{32,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{13,j}\, k_{33,n-j} + \frac{1}{2}\sum_{j=0}^{n} k_{31,j}\, k_{33,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{32,j}\, k_{33,n-j} + \frac{19}{4}\sum_{j=0}^{n} k_{33,j}\, k_{33,n-j} - \frac{1}{2}\sum_{j=0}^{n} k_{23,j}\, k_{31,n-j} + \frac{75}{4}\sum_{j=0}^{n} k_{23,j}\, k_{32,n-j} - \frac{37}{4}\sum_{j=0}^{n} k_{23,j}\, k_{33,n-j}\Bigg] dt \quad (107)$$

Approximations to the solutions with four terms gives the solution to the Riccati equation as

$$k_{11}(t) = -\frac{1994}{35}t^7 - \frac{2477394787}{38400000}t^6 + \frac{1422195273119}{2400000000}t^5 + \frac{547729768011977777}{160000000000000}t^4 - \frac{440772182243114906 81}{2000000000000000 0}t^3 - \frac{421547629690369}{2000000000000}t^2 + \frac{180605659}{100000000}t - \frac{8649}{2500} \quad (108)$$

$$k_{12}(t) = \frac{101471}{2016}t^7 - \frac{4939603777}{57600000}t^6 - \frac{46683532121379}{32000000000}t^5 - \frac{24493151134534607}{9600000000000}t^4 - \frac{35112195491924803}{25600000000000}t^3 + \frac{1763173131492233}{4000000000000}t^2 - \frac{670575697}{25000000}t - \frac{1803}{2000} \quad (109)$$

$$k_{13}(t) = -\frac{56272}{315}t^7 - \frac{1979001511}{12800000}t^6 + \frac{267760640524857}{64000000000}t^5 + \frac{706991682884294273}{192000000000000}t^4 - \frac{7631270778421188359}{4000000000000000}t^3 - \frac{34035587051919}{200000000000}t^2 + \frac{19635187}{20000000}t - \frac{1803}{2000} \quad (110)$$

$$k_{21}(t) = \frac{101471}{2016}t^7 - \frac{4939603777}{57600000}t^6 - \frac{46683532121379}{32000000000}t^5 - \frac{24493151134534607}{9600000000000}t^4 - \frac{35112195491924803}{25600000000000}t^3 + \frac{1763173131492233}{4000000000000}t^2 - \frac{670575697}{25000000}t - \frac{1803}{2000} \quad (111)$$

$$k_{22}(t) = \frac{12162011}{2520}t^7 - \frac{120443012197}{1536000}t^6 + \frac{345839261379744 57}{64000000000}t^5 - \frac{3462413202763632 99253}{192000000000000}t^4 + \frac{963247574629126394 05743}{400000000000000000}t^3 - \frac{1177284082676347 7}{400000000000}t^2 + \frac{142941207}{400000}t - \frac{43229}{10000} \quad (112)$$

$$k_{23}(t) = \frac{897749}{2016}t^7 - \frac{235713057349}{57600000}t^6 + \frac{1001798147035957}{64000000000}t^5 - \frac{325827279864876 3611}{640000000000000}t^4 + \frac{1921398394918606369}{2000000000000000}t^3 + \frac{126927760685747}{400000000000}t^2 - \frac{13241839}{625000}t - \frac{1803}{2000} \quad (113)$$

$$k_{31}(t) = -\frac{56272}{315}t^7 - \frac{1979001511}{12800000}t^6 + \frac{267760640524857}{64000000000}t^5 + \frac{706991682884294273}{192000000000000}t^4 - \frac{7631270778421188359}{4000000000000000}t^3 - \frac{34035587051919}{200000000000}t^2 + \frac{19635187}{20000000}t - \frac{1803}{2000}$$

(114)

$$k_{32}(t) = \frac{897749}{2016}t^7 - \frac{235713057349}{57600000}t^6 + \frac{1001798147035957}{64000000000}t^5 - \frac{3258272798648763611}{64000000000000}t^4 + \frac{1921398394918606369}{2000000000000000}t^3 + \frac{126927760685747}{400000000000}t^2 - \frac{13241839}{625000}t - \frac{1803}{2000}$$

(115)

$$k_{33}(t) = -\frac{30134}{35}t^7 + \frac{42552322763}{23040000}t^6 + \frac{249496097907749}{19200000000}t^5 + \frac{33827135708450851}{9600000000000}t^4 - \frac{393629056989711343}{240000000000000}t^3 - \frac{1093895194219}{8000000000}t^2 + \frac{5459573}{200000}t - \frac{1803}{2000}$$

(116)

The rest of the components of the iteration formulae can be obtained using MATLAB package. Hence by the equation (37),

$$\lambda(0)^T = \left(-\frac{67430}{10000} \quad -\frac{40182}{10000} \quad -\frac{60238}{1000}\right)$$

(117)

Therefore, the optional state and the adjoint variables are given as

$$x_1(t) = \frac{237}{10000000}e^{\frac{5983}{1000}t} - \frac{5191}{10000}e^{-\frac{5983}{1000}t} - \frac{247}{2000}e^{\frac{3083}{5000}t} - \frac{803}{2500}e^{-\frac{3083}{5000}t} + \frac{8197}{1000000}e^{\frac{3581}{1000}t} + \frac{489}{250}e^{-\frac{3581}{1000}t}$$

(118)

$$x_2(t) = \frac{1746}{10000000}e^{\frac{5983}{1000}t} + \frac{4731}{1000}e^{-\frac{5983}{1000}t} + \frac{823}{5000}e^{\frac{3083}{5000}t} - \frac{77}{5000}e^{-\frac{3083}{5000}t} + \frac{669}{100000}e^{\frac{3581}{1000}t} - \frac{3887}{1000}e^{-\frac{3581}{1000}t}$$

(119)

$$x_3(t) = -\frac{6952}{100000000}e^{\frac{5983}{1000}t} - \frac{2767}{1000}e^{-\frac{5983}{1000}t} + \frac{5189}{100000}e^{\frac{3083}{5000}t} + \frac{1907}{2500}e^{-\frac{3083}{5000}t} - \frac{5736}{10000000}e^{\frac{3581}{1000}t} - \frac{3073}{1000}e^{-\frac{3581}{1000}t}$$

(120)

$$\lambda_1(t) = -\frac{2537}{100000000}e^{\frac{5983}{1000}t} + \frac{121}{100}e^{-\frac{5983}{1000}t} - \frac{5273}{1250}e^{\frac{3083}{5000}t} + \frac{143}{200}e^{-\frac{3083}{5000}t} + \frac{6883}{1000000}e^{\frac{3581}{1000}t} - \frac{8457}{1000}e^{-\frac{3581}{1000}t}$$

(121)

$$\lambda_2(t) = \frac{3982}{100000000}e^{\frac{5983}{1000}t} - \frac{2491}{2500}e^{-\frac{5983}{1000}t} + \frac{184}{625}e^{\frac{3083}{5000}t} - \frac{553}{500}e^{-\frac{3083}{5000}t} + \frac{3083}{1000000}e^{\frac{3581}{1000}t} - \frac{2213}{1000}e^{-\frac{3581}{1000}t}$$

(122)

$$\lambda_3(t) = -\frac{3232}{100000000}e^{\frac{5983}{1000}t} + \frac{9439}{10000}e^{-\frac{5983}{1000}t} + \frac{16}{25}e^{\frac{3083}{5000}t} - \frac{211}{100}e^{-\frac{3083}{5000}t} + \frac{2999}{1000000}e^{\frac{3581}{1000}t} - \frac{5501}{1000}e^{-\frac{3581}{1000}t}$$

(123)

From Equation (25), the optimal control variables are given as

$$u_1(t) = \frac{1878}{10000000}e^{\frac{5983}{1000}t} - \frac{4903}{1000}e^{-\frac{5983}{1000}t} + \frac{7047}{100000}e^{\frac{3083}{5000}t} - \frac{7069}{10000}e^{-\frac{3083}{5000}t} + \frac{629}{100000}e^{\frac{3581}{1000}t} + \frac{5053}{10000}e^{-\frac{3581}{1000}t}$$

(124)

$$u_2(t) = -\frac{3611}{10000000}e^{\frac{5983}{1000}t} + \frac{52}{5}e^{-\frac{5983}{1000}t} - \frac{1652}{100000}e^{\frac{3083}{5000}t} + \frac{9671}{10000}e^{-\frac{3083}{5000}t} - \frac{7617}{1000000}e^{\frac{3581}{1000}t} - \frac{7167}{1000}e^{-\frac{3581}{1000}t}$$

(125)

$$u_3(t) = \frac{3359}{10000000}e^{\frac{5983}{1000}t} - \frac{881}{100}e^{-\frac{5983}{1000}t} - \frac{307}{2000}e^{\frac{3083}{5000}t} - \frac{123}{400}e^{-\frac{3083}{5000}t} + \frac{9497}{1000000}e^{\frac{3581}{1000}t} + \frac{403}{125}e^{-\frac{3581}{1000}t}$$

(126)

4. Conclusion

This paper presents a semi-analytical solution of generalized continuous quadratic optimal control problems constrained by ordinary differential equations. The proposed method gives the optimal solution of the state, control and adjoint variables which gives rise to the optimal objective functional value, hence it is recommended to be used to solve real life problems.

Conflict of Interest

The authors declare that there are no conflicts of interest.

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