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Exploring the Hutchinson-Barnsley Operator in the Context of M-Fuzzy Metric Spaces

Dr. Priyanka

IIMT College of Engineering, Greater Noida, India
Department of Applied Science and Humanities
Email id: *priyanka4177_gn[at]iimtindia.net*

Abstract: ***Fractals are an exciting area in the research that offers many research possibilities in applied and pure mathematics. Most of the researchers are doing lots of work in this area. The Hutchinson-Barnsley theory (HB theory) to define and construct a fractal set as its unique fixed point in complete metric space. In this paper we study the concepts and properties of HB operator and we present M-Fuzzy contractions in the M-Fuzzy metric spaces by generalizing the Hutchinson-Barnsley theory. Our results generalize and extend some recent results related with Hutchinson-Barnsley operator in the metric spaces.***

Keywords: Fractals; M-Fuzzy Metric Space; Iterated Function System; Attractor; Fractal Space; Contraction mapping; Compact set

1. Introduction

Fractal plays an important role in applications such as image compression, computer graphics, soil mechanics and so on. (B.Mandelbrot, 1975) introduced the term fractal. Fractal defined as sets with non-integral Haudorff dimension, which exceeds its topological dimension (B. Mandelbrot, 1975). More exciting development in the construction of fractal sets is the use of Iterated Function Systems. Firstly introduced the IFS (Hutchinson, 1981) and popularized by (M. Barnsley, 1993). The concept of Hutchinson-Barnsley theory initiated and developed by (Hutchinson, 1981 & Barnsley, 1993). IFS defined as a finite collection of contractive self-mappings and introduced HB operator on hyperspace of nonempty compact sets (Hutchinson, 1981 & Barnsley, 1993). They defined the unique fixed of Hutchinson-Barnsley operator as a fractal (attractor) by using the Banach Contraction Theorem.

The theory of fuzzy metric space introduced by (Zadeh, 1965 & Kramosil et al., 1975). Numerous analysts have defined a fuzzy metric space in various manners. Recently, the concept of M-Fuzzy metric space introduced by (Sedghi and Shobe, 2006) which is based on D^*-metric concept.

Inspired and motivated by the research work going on in this field, we shall consider and analyze the Hutchinson-Barnsley theory to define and construct the IFS fractals in different spaces. This will help us to develop more interesting results on fractals in some general setting.

In this paper, we study the some basic concepts and properties of HB operator and we present the M-Fuzzy IFS fractals by proposing a generalization of the HB theory for an IFS of M-Fuzzy contractive mappings on a complete M-Fuzzy metric space.

2. HB Operator in Metric Space

In this section, we present some basic concepts and properties of HB operator. The Hutchinson-Barnsley operator define and construct the IFS fractals in the complete metric space.So as to comprehend what iterated function systems are and why the random iteration algorithm works. We need to be familiar with some mathematical concepts.

2.1 Metric Space(M. Frechet, 1906)

A space X with a real-valued function $d: X \times X \to R$ is called a metric space (X, d) if d possess the following properties:

1) $d(x,y) \geq 0 \qquad \forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}.$
2) $d(x,y) = d(y,x) \qquad \forall\ \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{X}$
3) $d(x,y) \leq d(x,z) + d(z,y) \qquad \forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \boldsymbol{X}$

Remark2.1.1 Third property is called the triangular inequality.

Example2.1.1line R with its usual distance $d(x,y) = |x - y|$.

2.2 Contraction Mapping (S. Banach, 1922)

Let Xdenotes a complete metric space with distance d and T be a mapping from X into itself. Then T is called a contraction mapping if there is a constant $0 \leq s \leq 1$ such that

$$d\left(T(x), T(y)\right) \leq s\, d(x\,, y).$$

The constant s is known as the contractivity factor for T.

2.3 Banach Contraction Theorem (S.Banach, 1922)

Polish mathematician S. Banach proved a very important result, regarding contraction mapping in 1922, known as Banach Contraction Theorem. The theorem states thatas follows:

Let $T : X \to X$ be a contraction mapping, with contractivity factor 's', on a complete metric space (X, d). Then T possesses exactly one fixed point $x^* \epsilon X$. Moreover, for any point $x \epsilon X$, the sequence $\{T_n(x): n = 0,1,2, \dots \dots \dots\}$ converges to x^*. That is,$\lim_{n\to\infty} T^n(x) = x^*$, for each $x \epsilon X$

2.4 Haudorff Metric Space (M. Barnsley,1993 & K.Falconer, 2003)

Let (X, d) be a complete metric space and$K_0(X)$ denote the space whose points are the compact subsets of X known as Hausdorff space, other than the empty set. Let $x, y \epsilon X$ and let $A, B \epsilon K_0(X)$. Then

1) Distance from the point x to the set B is defined
$$d(x, B) = min\{d(x, y): y \epsilon B\},$$
2) Distance from the set A to the set B is defined as
$$d(A, B) = max\{d(x, B): x \epsilon A\},$$
3) Hausdorff distance (H_d) is a function $H_d: K_0(X) \times K_0(X) \to R$ defined by
$$H_d(A, B) = \max\{d(A, B), d(B, A)\}$$

Then the function H_d is the metric defined on the hyperspace of compact sets$K_0(X)$ and hence $(K_0(X), H_d)$ is called a Hausdorff metric space.

Theorem 2.4.1 (M. Barnsley,1993 & K. Falconer, 2003): If (X, d) is a complete metric space, then $(K_0(X), H_d)$ is also a complete metric space.

2.5Iterated Function System (J. E. Hutchinson, 1981 & M. Barnsley, 1993)

In mathematics, iterated function systemsare a method of constructing fractals. The resulting constructions are self similar. IFS's were conceived in their present from by John E. Hutchinson in 1981.

Definition 2.5.1 (J. E. Hutchinson, 1981): A (Hyperbolic) iterated function system consists of a complete metric space (X, d) together with a finite set of contraction mappings $T_n : X \to X$, with respective contractility s_n, for $n = 1,2,3, \ldots\ldots, N$. The shortened form "IFS" is utilized for "iterated function systems". The notation for the IFS just announced is $X; T_n: n = 1,2, \ldots\ldots, N$ and its contractivity factor is $s = \max\{s_n : n = 1,2, \ldots\ldots, N\}$ Thus, the following theorem was given by Bransley as follows;

Theorem 2.5.1 (M. Barnsley, 1993): Let (X, d) be a metric space and $T_n: X \to X, n = 1,2,3, ----, N_0 (N_0 \in N)$ be N_0 -contraction mappings with the corresponding contractivity ratios $s_{n,} n = 1,2,3 ----, N_{0.}$ The system $\{X; T_n, n = 1,2,3, ----, N_0\}$ is called an iterated function system (IFS) or with the ratio $s = max_{n=1}^{N} s_n$.

Then the Hutchinson –Barnsley operator (HB operator) of the IFS is a function $W: K_0(X) \to K_0(X)$ defined by

$$W(B) = \bigcup_{n=1}^{N} T_n(B) \quad for\ all\ B \in K_0(X)$$

2.6 Hutchinson-Barnsley Theorem for IFS (J. E. Hutchinson, 1981 & M. Barnsley, 1993)

Let (X, d) be a complete metric space and $\{X; 1,2,3 ---, N_0; N_0 \in N\}$ be an IFS on X. Then there exists only one compact invariant set $A_\infty \in K_0(X)$ of the HB operator (W) or, equivalently, W has a unique fixed point namely $A_\infty \in K_0(X)$, Which is also called an attractor, $A_\infty \in K_0(X)$, obeys

$$A_\infty = W(B) = \bigcup_{n=1}^{N} T_n(B)$$

And is given by $A_\infty = \lim_{n \to \infty} W^{0n}(B)$ for any $B \in k_0(X)$. Sometimes $A_\infty \in K_0(X)$ is called as fractal generated by the IFS and so called as IFS.

3. HB Operator in M-Fuzzy Metric Space

In this section we present new results of HB theory in M-Fuzzy metric space.

3.1 M-Fuzzy Metric Space (Sedghi et al., 2006)

A 3- tuple $(X, M, *)$ is called a M-Fuzzy metric space if X is an arbitrary (non-empty) set $*$ is a continuous t-norm and M is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$.

1) $M(x, y, z, t) > 0$.
2) $M(x, y, z, t) = 1\ if\ and\ only\ if\ x = y = z$.
3) $M(x, y, z, t) = M(p\{x, y, z\}, t)$.
4) $M(x, y, a, t) * M(a, z, z, s) \leq (x, y, z, t + s)$.
5) $M(x, y, z): (0, \infty) \to [0,1]$ is continuous.

3.2 M-FuzzyIterated Function System

Let $(X, M, *)$ be an M-Fuzzy metric space and $T_n: X \to X, n = 1,2,3, ---, N_0\ (N_0 \in N)$ be N_0 -M Fuzzy contractive mappings with the corresponding ratio $s_n, n = 1,2,3, ---, N_0$. The system $\{X; T_{n,} n = 1,2,3, ----, N_0\}$ is called an M-Fuzzy Iterated Function System in the M-Fuzzy Metric Space $(X, M, *)$.

3.3 M-Fuzzy Hutchinson Barnsley Operator

Let $(X, M, *)$ be a M-Fuzzy Metric Space. Let $\{X: T_{n,} n = 1,2,3, ---, N_0, N_0 \in N\}$ be a M-Fuzzy IFS of M-Fuzzy contractions. Then the M-Fuzzy Hutchinson Barnsley operator of the M-Fuzzy IFS is a function $W: K_0(X) \to K_0(X)$ defined by

$$W(B) = \cup_{n=1}^{N} T_n(B) \text{ for all } B \in K_0(X)$$

3.4 Attractor Of M-Fuzzy HB Operator

Let $(X, M, *)$ be a complete M-Fuzzy metric space. Let $\{X, T_n, n = 1,2,3, ---, N_0;\ N_0 \in N\}$ M-Fuzzy IFS of fuzzy contractions and W be the M-Fuzzy HB operator of the M-Fuzzy IFS. We say that the set $A_\infty \in K_0(X)$ is fuzzy attractor of the given M-Fuzzy IFS. If A_∞ is the unique fixed point of the M-Fuzzy HB operator W. Usually, if $A_\infty \in K_0(X)$ is also called as fractal generated by the M-Fuzzy IFS of M-Fuzzy contractions and so called as M-Fuzzy IFS fractal of M-Fuzzy contractions.

4. Conclusion

This research has broadened the understanding of M-Fuzzy IFS fractals within M-Fuzzy metric spaces by extending the Hutchinson –Barnsley theory. This study opens up new avenues for further exploration and application of M-Fuzzy IFS fractals in diverse mathematical contexts.

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