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An Elementary Proof of the Poincaré

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Abstract: ***We provide a comprehensive analysis and (simple) proof of Poincare´'s conjecture, a fundamental problem in topology. The conjecture asserts that any compact, smooth 3-manifold with the same homotopy type as the 3-sphere (S^3) is necessarily diffeomorphic to S^3 itself. Index Terms-Poincare´'s conjecture, compact manifolds, smooth manifolds, homotopy equivalence, diffeomorphism, 3-manifold, 3-sphere, topology, differential topology, algebraic topology.***

Keywords: Poincare's Conjecture, topology, homotopy, sphere

1. Introduction

In its original formulation, the Poincare´ hypothesis postulates that any closed three-dimensional manifold that is simply connected is topologically equivalent to the three-sphere, represented as S^3. The three-sphere can be envisioned as an extension of the familiar sphere to a higher dimension. Put simply, the hypothesis asserts that the only bounded three-dimensional space without any cavities is the three-sphere. This conjecture was initially proposed by H. Poincare´ in1904 and later extended to include the generalized statement that any compact n-manifold is homotopy-equivalent to then-sphere if and only if it is homeomorphic to then-sphere and the statement reduces to the primary form when n=3.

The research by Perelman (2002, 2003; Robinson 2003) proved a broader theorem known as Thurston's geometrization conjecture, which implies the Poincare´ conjecture as a direct consequence. Perelman's findings have since been corrobo-rated, solidifying the conjecture's validity.

But the proof we've ruled down in this paper follows a different approach from Poincare´ by implementing Lickorich's proof on Alexander branched cover theorem on three dimensional manifolds.

2. General Statement of the Conjecture

The conjecture posits that if every closed curve within the manifold, such as a loop, can be continuously deformed to a single point, then the manifold must be a 3-sphere. Visualizing 3-dimensional manifolds is challenging, so we consider analogous 2-manifolds in the provided figure, with loops represented by blue and green lines. As observed, any loop on the sphere can be tightened and smoothly moved off the surface by sliding it upwards or sideways. However, on the torus, although the blue loop can be tightened and removed, the green loop cannot be without cutting the torus. Therefore, we conclude that the torus is not homeomorphic to the sphere.

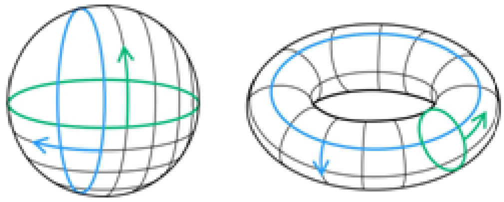

Figure 1: The blue and green loops present on the sphere be continuously contracted / tightened until they become single points. In contrast, the green loop on the torus can't undergo such contraction. This distinction implies that the torus and The spheres are not homeomorphic to each other.

Theorem 2.1 *[1] If V is a closed, simply connected three-dimensional smooth manifold, then V is diffeomorphic to the three-sphere S^3, as Poincare´ Conjecture holds.*

3. Elementary Proof

Theorem 3.1*[5] If M is a closed, orientable 3-manifold, then it can be represented as a branched cover over the three-sphere (S^3) with branching index at most two. This means there exists a smooth map f: $M \to S^3$ and a linkLinS^3suchthat the map f: $M - f^{-1}(L) \to S^3 - L$ is a covering, and when restricted to a neighborhood of any component of $f^{-1}(L)$, f is either an embedding or a standard branched double covering map of $D^2 \times S^1$by$D^2 \times S^1$.Additionally,the link $L = \{L_1,\ldots,L_n\}$ is an unknot and unlink, meaning that each L_i bounds an embedded disk and these disks do not intersect each other.*

In layman terms, the above states that any closed, 3-dimensional shape can be transformed into the shape of a sphere with some special conditions. It's like stretching and bending the shape without tearing it. The shape's points might overlap or branch out, but it remains connected.

Proof of Theorem 2.1: *The theorem states that if we have a closed, simply connected 3-dimensional space, denoted as M, and we have a map f: $M \to S^3$, then M can be smoothly stretched and folded over S^3 in a way that every point in M is covered by exactly one point in S^3, except for a special set of points on S^3. These special points are called the branching points. The map f is called a branched cover map because it covers most of S^3smoothly, but at the branching* *points, it behaves differently.*

To prove this theorem, we start with a map g_1: $M \rightarrow S^3$ *that is a "homotopy equivalence". This means that it preserves the basic shape of* M *while mapping it to* S^3. *Then, we find a map* g: $S^3 \rightarrow M$ *that is the "inverse" of* g_1. *Next, we combine* f *and* g *to get a new map* h: $S^3 \rightarrow S^3$. *We show that* h *can be homotoped (smoothly deformed) into a branched cover map* h_0: $S^3 \rightarrow S^3$, *such that* h_0 *behaves like* f *on a neighborhood of the branching points. Finally, we use* h_0 *to construct a diffeomorphism* $\bar{h}_0$: $S^3 \rightarrow M$, *which means that we can smoothly stretch and fold* S^3 *onto* M *in a way that preserves all the shapes and structures. This proves the* ***Theorem 2.1***.

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