On Convexity of Fuzzy Sets and Fuzzy Relations

Trupti Mohite

Department of Mathematics, KMC College, Khopoli, Dist.Raigad, Tal. Khalapur, 410203, Maharashtra, India

Abstract: This paper explores the convexity of fuzzy sets and fuzzy relations defined on the set of real numbers and their cartesian product respectively. Using the concept of ß-cut, we establish that a fuzzy set is convex if and only if the ß-cut of the fuzzy set is convex. The paper further extends this concept to fuzzy relations. The findings have significant implications for the application of convexity in various fields.

Keywords: Fuzzy sets, Convex fuzzy sets, Fuzzy relations, Convex fuzzy relations, ß-cuts

1. Introduction

Many researchers have conducted numerous studies in the area of convexity by examining the significance of its numerous challenges in the real world. Prof. Zadeh first time coined the terms fuzzy set, convex fuzzy set, fuzzy relation, ß-cuts [1] in 1965. A fuzzy set is a function \( F: R \rightarrow [0,1] \), called as membership function and \( F(x) \) is called membership grade at \( x \) in \( F \); \( x \in R \); values of membership grades lie in \([0,1]\). Fuzzy relation \( S \) is a fuzzy set defined on the cartesian product of the set of real numbers.

Xinmin Yang [2], and Syau [5] proposed the notion of closed fuzzy sets and convex fuzzy sets and studied the connection between them. Nadaban and Dzitac [4] defined the convex fuzzy relation in their paper and established special types of fuzzy relations. Chen-Wei-Xu [6] introduced new fuzzy relations from old ones and proved results on the convexity of fuzzy relations. In [1], a definition of the convexity of fuzzy sets using the notion of \( ß \)-cut is given. In this paper we will study that fuzzy set \( F \) is convex if and only if \( ß \)-cut of fuzzy set, \( F_ß \) is convex; \( \forall ß \in (0,1) \). We will extend the convexity of a fuzzy Set with respect to \( ß \)-cut to Fuzzy Relation and try to prove that fuzzy relation \( S \) is convex if and only if \( ß \)-cut of the fuzzy relation, \( S_ß \) is convex; \( \forall ß \in (0,1) \). We will study that \( ß \)-cut of a fuzzy set is an interval if \( F \) is a convex fuzzy set and study its connection with a strongly convex fuzzy set and strictly convex fuzzy set.

2. Preliminaries

Throughout this paper, \( F \) denotes fuzzy set defined on \( R \) and \( S \) denotes fuzzy relation defined on \( R^2 \). Here are some definitions that will be useful in this paper.

2.1 Definition [6]:

A fuzzy set \( F \) defined on \( R \) is a function; \( F: R \rightarrow [0,1] \) is called as membership function and \( F(x) \) is called membership grade of \( F \) at \( x \).

2.2 Definition [7]:

A Fuzzy relation \( S \) is a fuzzy set defined on the cartesian product of crisp sets \( X_1 \times X_2 \times X_3 \times \ldots \times X_n \) where tuples \( (x_1,x_2,x_3, \ldots, x_n) \) that may have varying degree of membership value is usually represented by a real number for closed interval \([0,1]\) and indicate the strength of the present relation between elements of the topic. Consider \( S: X \times Y \rightarrow [0,1] \) then the fuzzy relation on \( X \times Y \) denoted by \( S \) or \( (S,y) \) is defined as the set \( S(X, Y) = \{(x,y), S(x,y)) / (x,y) \in X \times Y \} \). Where \( S(x,y) \) is the strength of the relation in two variables called membership function. It gives the degree of membership of the ordered pair \((x,y)\) in \( X \times Y \) a real number in the interval \([0,1]\).

2.3 Definition [3]:

\( S \) be fuzzy relation on \( X \times Y \). Then \( S \) is convex if and only if
\[ S(\mu(x,y_1) + (1 - \mu)(x,y_2)) \geq S(x_1,y_1) \land \land S(x_2,y_2) \forall (x_1,y_1),(x_2,y_2) \in X \times Y \text{ and } \mu \in [0,1]. \]

2.4 Definition [6]:

Let \( S \) be a fuzzy relation defined on \( X \times Y \) and \( ß \) be such that \( 0 < ß \leq 1 \). Then \( ß \)-cut of \( S \) is denoted by \( S_ß \) defined by
\[ S_ß = \{x,y \in X \times Y / \mu(x,y) \geq ß \}. \]

2.5 Definition [1]:

A be a fuzzy set defined on \( R \) and \( ß \) be such that \( 0 < ß \leq 1 \).
Then \( ß \)-cut of \( F \), is denoted by \( F_ß \) defined by \( F_ß = \{x \in R / F(x) \geq ß \} \) is crisp set.

2.6 Definition [1]:

\( F \) be a fuzzy set defined on \( R \). Then \( F \) is convex if and only if
\[ F(\mu x_1 + (1 - \mu)x_2) \geq \min[F(x_1), F(x_2)] \forall x_1,x_2 \in R \text{ and } \mu \in (0,1). \]

2.7 Definition [2]:

A fuzzy set \( F \) on \( R \) is said to be strongly convex fuzzy set if
\[ F(\mu x_1 + (1 - \mu)x_2) > \max[F(x_1), F(x_2)] \forall x_1,x_2 \in R, x_1 \neq x_2 \text{ and } \mu \in (0,1). \]

2.8 Definition [2]:

A fuzzy set \( F \) on \( R \) is said to be strictly convex fuzzy set if
\[ F(\mu x_1 + (1 - \mu)x_2) > \min[F(x_1), F(x_2)] \forall x \in R \text{ and } \mu \in (0,1). \]
3. Main Results

3.1 Theorem

\( F \) be a fuzzy set defined on \( R \) then \( F \) is convex if and only if for every \( \alpha \in (0, 1) \), \( F_\alpha \) is convex.

**Proof:**

Suppose \( F \) is a convex fuzzy set defined on \( R \).

To prove that \( F_\alpha \) is convex, \( \forall \alpha \in (0, 1) \).

Let, if possible, for some \( \alpha \in (0, 1) \), \( F_\alpha \) is not convex.

That is there exist \( x, y \in F_\alpha \) such that, 
\[ \mu x + (1 - \mu) y \notin F_\alpha; \forall \mu \in [0, 1]. \]

Implies that, \( F(\mu x + (1 - \mu)y) < \alpha \)
Since, \( x \in F_\alpha, F(x) \geq \alpha. \)
and \( y \in F_\alpha, F(y) \geq \alpha. \)

Given \( F \) is a convex fuzzy set.

Therefore \( F(\mu x + (1 - \mu)y) \geq \min(F(x), F(y)) \)
\[ = \alpha. \]

By definition of \( \alpha \)-cut of fuzzy set,
\[ \mu x + (1 - \mu)y \notin F_\alpha. \]

A contradiction to our assumption that \( F_\alpha \) is not convex, for some \( \alpha \in (0, 1) \).

Therefore \( F_\alpha \) is convex, \( \forall \alpha \in (0, 1) \).

Conversely, suppose that \( F_\alpha \) is convex; \( \forall \alpha \in (0, 1) \).

To prove that \( F \) is Convex Fuzzy Set on \( R \).

Let if possible, for some \( x, y \in R \).

\[ F(\lambda x + (1 - \lambda)y) < \min(F(x), F(y)) \quad (1) \]

Choose \( \alpha = \min(F(x), F(y)) \).

Then, clearly \( x, y \in F_\alpha \) implies that, 
\[ F(\mu x + (1 - \mu)y) \geq \alpha = \min(F(x), F(y)). \]

A contradiction to (1).

Our assumption is wrong.

Therefore \( F \) is Convex Fuzzy Set on \( R \).

3.2 Corollary:

\( F \) be a strongly (strictly) convex fuzzy set defined on \( F \) if and only if \( F_{\alpha+} \) is convex; for all \( \alpha \in (0, 1] \), where \( F_{\alpha+} \) is strong \( \alpha \)-cut of \( F \).

3.3 Theorem

\( S \) be a fuzzy relation defined on \( R \times R \) then \( S \) is convex if and only if for every \( \alpha \in (0, 1] \), \( S_\alpha \) is convex.

**Proof**

Suppose that \( S \) is convex fuzzy relation defined on \( R \times R \).

To prove that \( S_\alpha \) is convex.

Let if possible, for some \( \alpha \in (0, 1] \), \( S_\alpha \) is not convex.

Then \( \exists (x_1, y_1), (x_2, y_2) \in S_\alpha \) such that \( \mu(x_1, y_1) + (1 - \mu)(x_2, y_2) \notin S_\alpha; \mu \in [0, 1] \).

Implies that \( S(\mu(x_1, y_1) + (1 - \mu)(x_2, y_2)) < \alpha. \)

Since, \( (x_1, y_1) \in S_\alpha; S(x_1, y_1) \geq \alpha. \) \quad (1)
and \( (x_2, y_2) \in S_\alpha; S(x_2, y_2) \geq \alpha. \)

Since \( S \) is convex fuzzy relation.

Consider,
\[ S(\mu(x_1, y_1) + (1 - \mu)(x_2, y_2)) \geq \min(S(x_1, y_1), S(x_2, y_2)). \]
\[ \geq \min(\alpha, \alpha). \quad (by 1) \]
\[ = \alpha. \]

By definition of \( \alpha \)-cut of fuzzy relation,
\[ \mu(x_1, y_1) + (1 - \mu)(x_2, y_2) \in S_\alpha. \]

A contradiction.

\( S_\alpha \) is convex set, for every \( \alpha \in (0, 1] \).

Conversely suppose that \( S_\alpha \) is convex set for every \( \alpha \in (0, 1] \).

We will prove that, \( S \) is convex fuzzy relation on \( R \times R \).

It is enough to prove that \( S \) is convex fuzzy set on \( R \times R \).

Let \( (x_1, y_1), (x_2, y_2) \in S_\alpha. \)

Then,
\[ \mu(x_1, y_1) + (1 - \mu)(x_2, y_2) \in S_\alpha. \]
\[ \text{i.e.} S(\mu(x_1, y_1) + (1 - \mu)(x_2, y_2)) \geq \alpha. \quad (2) \]

Consider left hand side of (2).
\[ S(\mu(x_1, y_1) + (1 - \mu)(x_2, y_2)) = S(\mu x_1 + (1 - \mu)x_2, \mu y_1 + (1 - \mu)y_2). = \min(S(\mu x_1 + (1 - \mu)y_1), S(\mu y_1 + (1 - \mu)y_2)). \]
\[ \geq \alpha. \quad (by 2) \]

Since \( S_\alpha \) is convex \( \alpha \in (0, 1] \).

Choose \( \alpha = \min(S(x_1, y_1), S(x_2, y_2)). \)
Where \( \alpha = S(x_1, y_1) < S(x_2, y_2) \). Then, \( S(\mu(x_1, y_1) + 1 - \mu(x_2, y_2)) \leq \min \{ \alpha x_1 y_1, x_2 y_2 \} \).

Therefore, \( S \) is convex fuzzy set on \( R \times R \), i.e., \( S \) is a convex fuzzy relation on \( R \times R \).

**Theorem 3**

\( S \) be a strongly convex fuzzy relation defined on \( R^2 \) then there exist unique element \( (x_1, x_2) \) such that \( S(x_1, x_2) = \max \{ S(y_1, y_2) / (y_1, y_2) \in R^2 \} \).

**Proof.**

Suppose that there are two elements \( (x_1, x_2) \) and \( (z_1, z_2) \) in \( R^2 \) such that

\[ \alpha = S(x_1, x_2) = S(z_1, z_2) = \max \{ S(y_1, y_2) / (y_1, y_2) \in R^2 \} \]

where, \( (x_1, x_2) \neq (z_1, z_2) \).

\( S_\alpha = \{ (x_1, x_2), (z_1, z_2) \}. \)

Given that \( S \) is strongly convex fuzzy relation on \( R^2 \). therefore \( S_\alpha \) is convex; \( \alpha \in (0, 1). \)

As \( (x_1, x_2) \neq (z_1, z_2). \)

We have, \( \mu(x_1, x_2) + 1 - \mu(z_1, z_2) \in S_\alpha \) as \( S_\alpha \) contains only two elements; for all \( \alpha \in (0, 1) \) and \( \mu \in (0, 1). \)

Therefore \( S_\alpha \) is not convex. a contradiction.

\( \therefore (x_1, x_2) = (z_1, z_2). \)

Therefore, there exist unique element \( (x_1, x_2) \) such that \( S(x_1, x_2) = \max \{ S(y_1, y_2) / (y_1, y_2) \in R^2 \} \)

Converse part of the above theorem is not true in general; we may take non convex fuzzy set and find unique maximum element.

**Theorem 4**

\( S \) is convex fuzzy relation defined on \( R^2 \) then for every \( \alpha \in (0, 1], S_\alpha \) is closed and connected.

**Proof**

First, we prove that \( S_\alpha \) is closed set.

Let \( (x, y) \) be any limit point of \( S_\alpha \) then there exists sequence \( (x_n, y_n) \in S_\alpha \).

\( (x_n, y_n) \rightarrow (x, y) \) \( \forall n \geq N.. \)

As \( (x_n, y_n) \in S_\alpha \) we have \( S(x_n, y_n)) \geq \alpha. \) Therefore \( \lim_{n \to \infty} S(x_n, y_n) = \alpha. \)

\( (x, y) \) is limit point of \( S_\alpha \); \( (x, y) \) is either an interior point or boundary point of \( S_\alpha. \)

Case 1: \( (x, y) \) is an interior point of \( S_\alpha \) then clearly \( S(x, y) \geq \alpha. \)

Case 2: \( (x, y) \) is boundary point of \( S_\alpha \), we can find \( \epsilon > 0 \) such that \( (x-\epsilon, x+\epsilon) \) is an interior point of \( S_\alpha. \)

Given that \( S \) is convex fuzzy relation on \( R^2 \).

Then by definition of convexity of fuzzy relation, \( S_\alpha \) is convex, for all \( \alpha \in (0, 1]. \)

For, \( (x_n, y_n) \), \( (x-\epsilon, x+\epsilon) \) \( \in S_\alpha \): \( \mu(x_n, y_n) + (1 - \mu)(x-\epsilon, x+\epsilon) \in \alpha \).

\[ \lim_{n \to \infty} \mu(x_n, y_n) + (1 - \mu)(x-\epsilon, x+\epsilon) \in S_\alpha. \]

\[ \lim_{n \to \infty} \mu(x_n, y_n) + \lim_{n \to \infty} (1 - \mu)(x-\epsilon, x+\epsilon) \in S_\alpha. \]

\[ \mu(x, y) + (1 - \mu)(x-\epsilon, y-\epsilon) \in S_\alpha. \]

For \( \mu = 1, (x, y) \in S_\alpha. \)

Therefore \( S(x, y) \geq \alpha. \)

Since \( (x, y) \) is an arbitrary limit point of \( S_\alpha. \)

Therefore \( S_\alpha \) is closed.

Now to prove that \( S_\alpha \) is connected.

Let, if possible, it is not connected. Then there exist two non-empty, disjoint, open sets \( A \) and \( B \) such that \( S_\alpha = A \cup B \) and \( A \cap B = \emptyset. \)

We can choose \( (x, y) \in A \) and \( (x', y') \in B. \)

Then \( \mu(x, y) + (1 - \mu)(x', y') \in S_\alpha. \)

\( S_\alpha \) is not convex set. Contradiction. Therefore \( S_\alpha \) is connected.

Converse of the above theorem is not true in general.

For example, \( S(x, y) = 1 \) if \( (x, y) \in \left\{ (x, y) \in R^2 / x^2 + y^2 \geq 1 \text{ and } x^2 + y^2 \leq 4. \right\} \)

\[ = \frac{1}{2} \] if \( (x, y) \in R^2 / x^2 + y^2 < 1 \text{ and } x^2 + y^2 > 4. \)

Then \( S_1 = (x, y) \in (x, y) \in R^2 / x^2 + y^2 \geq 1 \text{ and } x^2 + y^2 \leq 4. \)

which is closed and connected but not convex.

**Theorem 5**

Let \( f : X \rightarrow Y \) is continuous crisp function and \( F \in f(X) \) fuzzy set is connected then \( f(F) \) is connected.

**Proof**

Given that \( F \) is connected then every \( \alpha - \text{cut of } F \) is connected subset of \( X. \)

By extension principle of fuzzy sets and Theorem 2.9 [8].

\[ f(\alpha_F) \subseteq \alpha_{f(F)} \] (1)

Suppose that \( f(F) \) is not connected then \( f(F) = Z \cup T; \) where \( Z \) and \( T \) are two non empty, disjoint, open subsets of \( f(F). \)

Taking strong \( \alpha - \text{cut of both sets.} \)

\[ \alpha \cup f(F) = \alpha + [Z \cup T] \]

\[ f(\alpha + f(F)) = (\alpha + [Z \cup T]) \cup (\alpha + \frac{1}{2}). \] (2)
We know that continuous image of connected set is connected.

Thus, left hand side of equation (2) is connected. Therefore, right hand side must be connected but right-hand side of equation (2) is union of two non-empty disjoint sets hence disconnected.

Thus, is absurd. Contradiction to our assumption. Therefore \( f(F) \) is connected.

Since \( \alpha + F \) is connected \( f(\alpha + F) \) is connected.

4. Conclusion

We demonstrated the relationship between convex fuzzy set (convex fuzzy relation) and \( \alpha \)-cut of fuzzy set (fuzzy relation). \( \alpha \)-cut is work as bridge between crisp sets and fuzzy sets. Because convexity has tremendous applications in various fields, it’s crucial to investigate it using a fuzzy method at several levels. We have used it to look into the convexity of fuzzy sets and fuzzy relations.

5. Future Scope

We can generalize convexity of fuzzy relation for n-dimensional space. Connectivity of fuzzy sets and fuzzy relations using the notion of \( \alpha \)-cut could be studied.

References