International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2022): 7.942

Dynamical Behavior of an Eco-Epidemiological Model with Disease in Predator

Palash Mandal

Department of Mathematics, Hooghly Mohsin College, Chinsurah-712101, Hooghly, West Bengal, India

Abstract: ***A predator-prey model with parasitic infection spread in predator population is considered. The predator population is divided into two groups, namely susceptible predator, infected predator whereas the prey population remains free from infection. The existence of various equilibrium points and local stability analysis at those equilibrium points has been discussed. It has been observed that a Hopf-bifurcation may occur about the interior equilibrium point taking rate of parasitic infection parameter is bifurcation parameter. All the important analytical findings are numerically verified.***

Keywords: Eco-epidemiological model, Infected Predator, Stability, Oscillation, Hopf- bifurcation

1. Introduction

The effect of disease in ecological system is an important issue from mathematical as well as ecological point of view. So, in recent time ecologists and researchers are paying more and more attention to the development of important tool along with experimental ecology and describe how ecological species are infected. However, the first breakthrough in modern mathematical ecology was done by Lotka and Volterra for a predator- prey competing species. On the other hand, most models for the transmission of infectious diseases originated from the classic work of Kermack and Mc Kendrick [1].After these pioneering works in two different fields, lots of research works have been done both in theoretical ecology and epidemiology. Anderson and May [2] were the first who merged the above two fields and formulated a predator-prey model where prey species were infected by some disease. In the subsequent time many authors [3, 4, 5, 6] proposed and studied different predator- prey models in presence of disease. Venturino [7], Haque and Venturino [8], Haque et al. [9, 10, 11], Xiao and Chen [12, 13], Tewa [14], Shaikh [15], Pal [16], Chattopadhyay et. al [17, 18] discussed the dynamics of prey-predator system with disease in prey population. But some researchers [19, 20, 21] discussed the dynamics of prey-predator system with disease in predator population.

In this paper, predator-prey model with disease in predator is considered. Some similar kinds of models have appeared in the recent literature, but the main new distinctive feature is the inclusion of an infectious disease in the predator population and also the inclusion of susceptible predator and infected predator consuming prey by Holling type-III functional response. Under this additional effect the model becomes more realistic than the existing models in ecological as well as epidemiological point of view.

2. Mathematical Model

To construct the mathematical model, we make the following assumptions:

1) Let x denote the population density of the prey, y the population density of the susceptible predator and z the density of the infected predator, respectively, in time t.
2) We assume that the prey is capable of reproducing in logistic law with carrying capacity k and intrinsic birth rate r :
$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right).$$
3) The parasite is assumed to be horizontally transmitted. We further assume that the parasite attacks the predator population only.
4) The susceptible predator (x) becomes infected following the mass action law at constant rate of infection α.

Considering the above basic assumptions we can now write down the following dynamical system:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - c\frac{x^2}{a+x^2}(y+bz),\\ \frac{dy}{dt} &= c\frac{x^2}{a+x^2}(y+bz) - \alpha yz - dy,\\ \frac{dz}{dt} &= \alpha yz - (d+d_1)z, \end{aligned} \tag{1}$$

where ‘c’ is the predation rate of susceptible predator, ‘bc’ is the predation rate of infected predator (0<b<1) and ‘a’ is the half saturation constant. The constant ‘d’ is the parasite independent mortality rate of predator and d_1 denotes additional mortality rate of infected predator due to infection.

The system (1) has to be analyzed with the following initial conditions,

$$x(0) \geq 0, y(0) \geq 0, z(0) \geq 0 \tag{2}$$

3. Qualitative analysis

3.1 Boundedness of the system

Theorem-1: All the solutions of the system (1) are bounded.

Proof: Consider the function
$$u(t) = x(t) + y(t) + z(t)$$
Now using the system (1) we have
$$\frac{du}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}$$

$$= rx\left(1-\frac{x}{k}\right) - dy - (d+d_1)z$$

Therefore,

$$\frac{du}{dt} + \mu u = x\left(r + \mu - \frac{r}{k}x\right) - (d-\mu)y - (d+d_1-\mu)z$$

$\leq k\frac{(r+\mu)^2}{4r}$, if d> μ.

Now we can choose in such a way that $d > \mu$, then the right hand side of the above inequality is bounded. Then we can find a constant $P > 0$, such that

$\frac{du}{dt} + \mu u < P$,

Now by the theory of differential inequality [22] we have

$$0 \leq u(t) \leq \frac{P}{\mu}(1-e^{-\mu t}) + u(0)e^{-\mu t}.$$

As $t \to \infty$, then $0 \leq u(t) \leq \frac{P}{\mu}$. Hence u(t) is bounded quantity.

3.2 Equilibrium Points

- The equilibria $E_0(0,0,0)$ and $E_1(k,0,0)$ exists for all parametric values.
- The infection free equilibrium point $E_2(x_2, y_2, 0)$, where $x_2 = \sqrt{\frac{ad}{c-d}}$, $y_2 = \frac{rx_2}{d}\left(1-\frac{x_2}{k}\right)$ exist if $c - d > 0$ and $k > x_2$.
- The positive interior equilibrium point $E^*(x^*, y^*, z^*)$, where $y^* = \frac{d+d_1}{\alpha}$,

$z^* = \frac{(c-d){x^*}^2 - ad}{(\alpha y^* - bc){x^*}^2 + a\alpha y^*} y^*$ and x^* is the positive root of the equation

$$Ax^3 + Bx^2 + Cx + D = 0,$$

where $A = r(bc - \alpha L), B = -kr(bc - \alpha L), C = dkL(bc - \alpha L) - \alpha kL^2(c-d) - raL\alpha,$

$D = rk\alpha aL,\ L = \frac{d+d_1}{\alpha}$.

4. Stability Analysis

4.1 E_0

The jacobian matrix of the system (1) at $E_0(0,0,0)$ is given by

$$J(E_0) = \begin{bmatrix} r & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & -(d+d_1) \end{bmatrix}$$

So, eigenvalues of $J(E_0)$ are $r, -d, \ -(d+d_1)$. So the equilibrium point E_0 is unstable because one eigenvalue is positive.

4.2 E_1

The jacobian matrix of the system (1) at $E_1(k,0,0)$ is given by

$$J(E_1) = \begin{bmatrix} -r & -\frac{ck^2}{a+k^2} & -\frac{bck^2}{a+k^2} \\ 0 & \frac{ck^2}{a+k^2} - d & 0 \\ 0 & 0 & -(d+d_1) \end{bmatrix}$$

So, eigen values of $J(E_1)$ are $-r, -(d+d_1)$ and $\frac{ck^2}{a+k^2} - d$.

Thus, the equilibrium point E_1 is stable if $\frac{d(a+k^2)}{ck^2} > 1$.

4.3 E_2

The jacobian matrix of the system (1) at $E_2(x_2, y_2, 0)$is given by

$$J(E_2) = \begin{bmatrix} r - \frac{2r}{k}x_2 - \frac{2acx_2y_2}{(a+{x_2}^2)^2} & -\frac{c{x_2}^2}{a+{x_2}^2} & -\frac{bc{x_2}^2}{a+{x_2}^2} \\ \frac{2acx_2y_2}{(a+{x_2}^2)^2} & 0 & -\alpha y_2 \\ 0 & 0 & \alpha y_2 - (d+d_1) \end{bmatrix}$$

So, the eigenvalues of $J(E_2)$ are $\alpha y_2 - (d+d_1)$and the roots of the equation

$$\lambda^2 + \left\{\frac{2r}{k}x_2 + \frac{2acx_2y_2}{(a+{x_2}^2)^2} - r\right\}\lambda + \frac{2acx_2^3y_2}{(a+{x_2}^2)^3} = 0,$$

Thus, the equilibrium point E_2 is stable if $\frac{\alpha y_2}{d+d_1} < 1$ and $\frac{2r}{k}x_2 + \frac{2acx_2y_2}{(a+{x_2}^2)^2} - r > 0$.

4.4 Stability of positive interior equilibrium point $E^*(x^*, y^*, z^*)$,

Theorem-2: The positive interior equilibrium is locally asymptotically stable if and only if $p_1 > 0, p_2 > 0,\ p_1p_2 - p_3 > 0$, where $p_i's$ are given in the proof of the theorem.

Proof: The jacobian matrix of the system (1) around $E^*(x^*, y^*, z^*)$is

$$J(E^*) = \begin{bmatrix} -m_{11} & -m_{12} & -m_{13} \\ m_{21} & m_{22} & -m_{23} \\ 0 & m_{32} & 0 \end{bmatrix},$$

Where $m_{11} = \left\{\frac{2r}{k}x^* + 2ac\frac{x^*(y^*+bz^*)}{(a+{x^*}^2)^2} - r\right\} > 0, m_{12} = c\frac{{x^*}^2}{a+{x^*}^2},\ m_{13} = bc\frac{{x^*}^2}{a+{x^*}^2};$

$$m_{21} = 2ac\frac{x^*(y^*+bz^*)}{(a+{x^*}^2)^2}, m_{22} = c\frac{{x^*}^2}{a+{x^*}^2} - \alpha z^* - d > 0, m_{23} = \alpha y^*, m_{32} = \alpha z^*.$$

The characteristic equation of $J(E^*)$ at E^*

$$\rho^3 + p_1\rho^2 + p_2\rho + p_3 = 0,$$

where $p_1 = m_{11} - m_{22}, p_2 = m_{12}m_{21} + m_{23}m_{32} - m_{11}m_{22}$,

$p_3 = m_{13}m_{21}m_{32} + m_{11}m_{23}m_{32}$,

$p_1p_2 - p_3 = (m_{11} - m_{22})(m_{12}m_{21} + m_{23}m_{32} - m_{11}m_{22}) - (m_{13}m_{21}m_{32} + m_{11}m_{23}m_{32})$.

By the Routh-Hurwitz criteria, all roots of the above equation have negative real parts if and only if $p_1 > 0, p_2 > 0, p_1p_2 - p_3 > 0$. Thus, the positive interior equilibrium point $E^*(x^*, y^*, z^*)$ is asymptotically stable if and only if $p_1 > 0, p_2 > 0,\ p_1p_2 - p_3 > 0$.

5. Numerical Simulations

Table 1: A set of parameter values

Parameter	Definition	Value	Dimension
r	Growth rate of prey	1	1/time
k	Carrying capacity	6	mass/volume
α	The infectious rates in predator populations	0.086	1/time
a	Half saturation constants in prey population	1	-
c	Predation rate of susceptible predator	0.4	1/time
b	Constant, 0<b<1	0.8	-
d	Mortality rate of predators	0.24	1/time
d_1	Additional mortality rate of infected predator	0.1	1/time

In this portion, concentrated on the occurrence and termination of the disease is studied. For the set of parametric values in Table 1, the existence conditions of the coexistence equilibrium point E^* is satisfied and the coexistence equilibrium point E^*= (1.2553,3.9535,0.1290) is locally asymptotically stable with eigen values$-0.01437 \pm 0.389i$, -0.005045(see figure 1(a)).Next to observe the effects of some parameters on system (1), firstly, consider c= 0.23 and other set of parametric values in Table 1, observe that the infection free equilibrium point E_2 is stable (Figure 1(b)). Again, if d = 0.42 and other set of parametric values in Table 1, the predator free equilibrium pointE_1 is stable (Figure 1 (c)). Next, if α is increased from 0.086 to 0.1 then it is observed that the solution of (1) changes from stable behavior to oscillatory behavior (see figure 2(a)) Finally, for a clear understanding of the dynamical changes of system (1) due to change the value of the parameter α from 0.085 to 0.095, a bifurcation diagram is plotted as shown in the bifurcation diagram (see figure 2(b)).

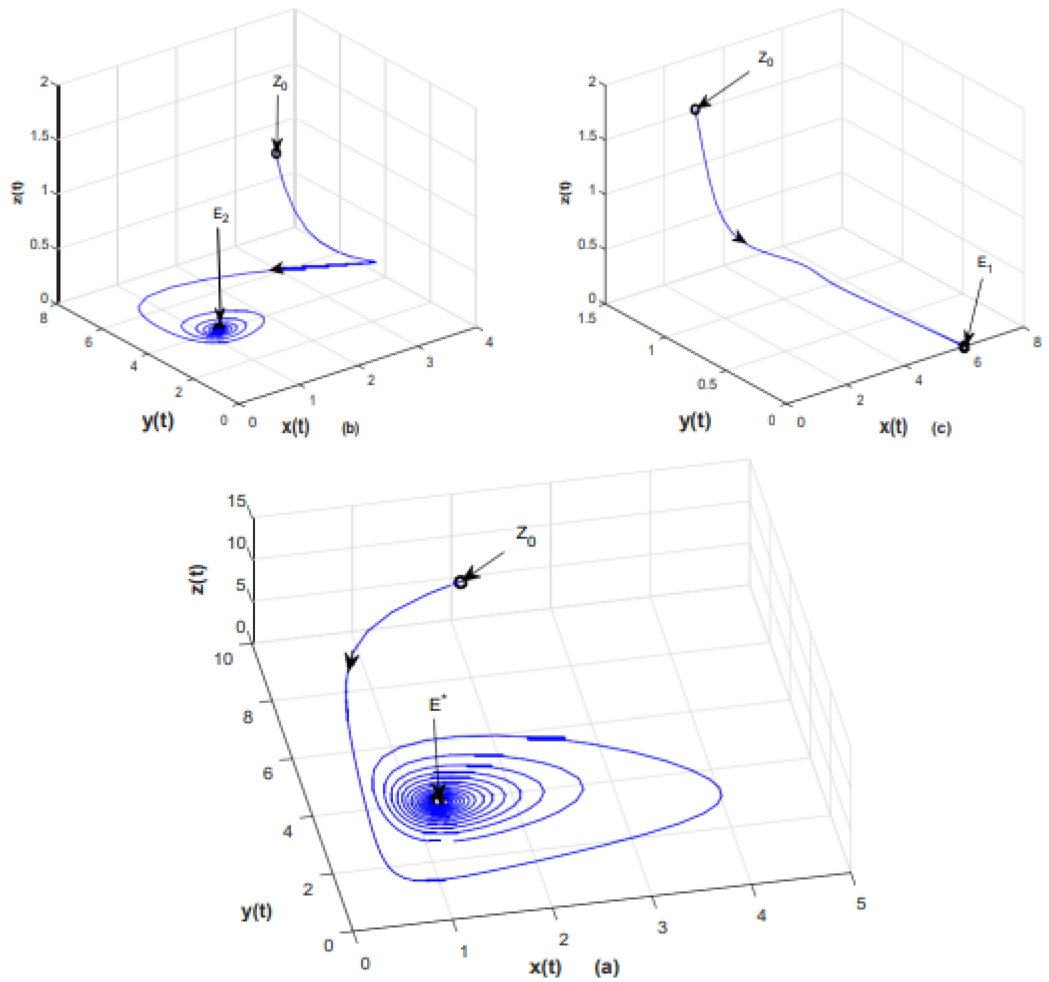

Figure 1: (a) Phase diagram denotes the equilibrium point E^* is locally asymptotically stable for the set of parameter in the Table 1. (b) Phase diagram denotes the equilibrium point E_2 is locally asymptotically stable for c=0.23,withother set of parameter fixed as given in Table1.(c) Phase diagram denotes the equilibrium point E_1 is locally asymptotically stable for d=0.42, with other set of parameter fixed as given in Table1.

Paper ID: SR23625161749 DOI: 10.21275/SR23625161749

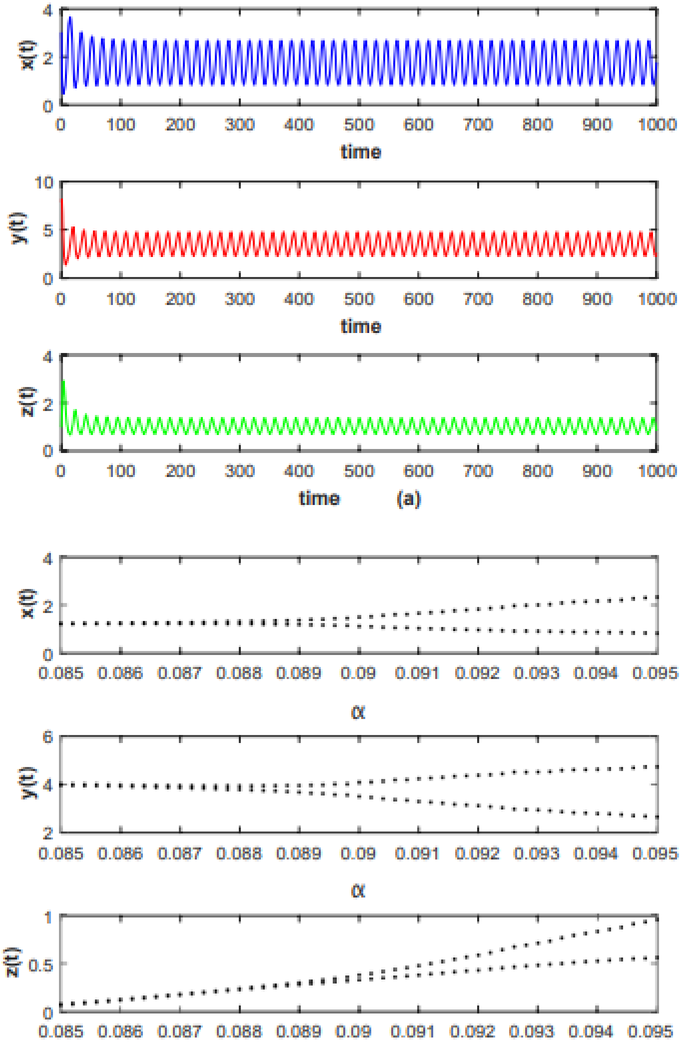

Figure 2: (a) The figure depicts oscillatory behaviour of three species for $\alpha = 0.15$, with other set of parameter fixed as given in Table1.(b) The bifurcation diagram of three species for α.

6. Conclusions

An ecological predator-prey model and an epidemiological model have been partially applied in this work. Through the analysis an eco-epidemiological predator- prey mathematical model has been established in which only predator population is affected by an infectious disease. Predator population is divided into two categories, namely susceptible predator and infected predator. In this paper, there are four equilibrium points, namely, one trivial equilibrium E_0, predator free equilibrium point E_1, infection free equilibrium point E_2 and an interior equilibrium E^*. Here E_0 always exists but unstable. Next, E_1exists and it is stable under some conditions. Also, E_2 exists under some conditions and it is stable under some conditions. The interior equilibrium E^* exists and it is asymptotically stable under some conditions. The stability switching and a Hopf-bifurcation may occur at the interior equilibrium taking rate of infection parameter (α) is bifurcation parameters. Without numerical verification the analytical results cannot be completed. So all important analytical findings are numerically verified using Matlab here.

References

[1] W. O. Kermack and A. G. Mc Kendrick, "Contributions to the mathematical theory of epidemics, part 1," *Proceedings of the Royal Society Series A*, vol.115, pp. 700-721, 1927.

[2] R. M. Anderson and R. M. May, "Regulation and stability of host-parasite population interactions," *Journal of Animal Ecology*, vol. 47, pp. 219- 249, 1978.

[3] J. Chattopadhyay, O. Arino," A predator- prey model with disease in the prey," *Non linearAnal.*, vol. 36, pp.747-766,1999.

[4] H. I. Freedman, "A model of predator-prey dynamics as modified by the action of a parasite," *Mathematical Biosciences*, vol.99, no.2, pp.143-155, 1990.

[5] H. Hethcote, W. Wang, L. Han L and Z. Ma, A predator-prey model with infected prey, *Theor. Popul Biol*, vol. 66, pp: 259-268, 2004.

[6] Y. Xiao and L. Chen, "Modeling and analysis of a predator-prey model with disease in theprey," It Mathematical Biosciences, vol.171, no.1, pp.59-82, 2001.

[7] E. Venturino, The influence of diseases on Lotka-Volterra systems, *Rocky MtJMath*, vol.24, pp: 381-

402,1994.

[8] M. Haque, S. Sarwardi, S. Preston and E.Venturino, Effect of delay in a Lotka- Volterra type predator- prey model with a transmission disease in the predator species. *Math Biosci*, vol. 234(1), pp: 47-57, 2011.

[9] M. Haque, A predator-prey model with disease in the predator species only, *Nonlinear Anal Real World Appl*, vol.11, pp: 2224-2236, 2010.

[10] M. Haque and D. Greenhalgh, when Predator avoids infected prey: a model based theoretical studies, *IMAJ Math MedBiol*, vol.27, pp: 75-94, 2009.

[11] M. Haque, J. Zhen and E. Venturino, Rich dynamics of Lotka- Volterra type predator-prey model system with viral disease in Prey species, *Math Methods Appl Sci*,vol.32, pp: 875-898,2009.

[12] Y. N. Xiao and L.S. Chen, Analysis of a three species eco-epidemiological model, *JMathAnal Appl*, vol.258(2), pp: 733-754,2001.

[13] Y. N. Xiao and L.S. Chen, Modelling and analysis of a predator-prey model with disease in the prey, *MathBiosci*,vol.171, pp: 59-82,2001.

[14] J. J. Tewa,V. Y. Djeumen and S. Bowong, Predator-prey model with Holling response function of type II and SIS infectious disease, *Appl Math Model*, vol.37,pp: 4825-4841,2013.

[15] Absos Ali. Shaikh, H. Das and S. Sarwardi, Dynamics of eco-epidemiological systemwith desease in competitive prey species, *Journal of Applied Mathematics and Computing*, https://doi.org/10.1007/s12190-019-01295-6, 2019.

[16] Nazmul Sk, S.Pal, Dynamics of an infected prey-generalised predator system with the effect of fear, refuge and harvesting: deterministic and stochastic approach, *The European Physical Joural Plus*. 2022.

[17] K. Das, K. Kundu, J. Chattopadhyay,"A predator-prey mathematical model with both the populations affected by diseases," *Ecol.Complex.*vol.8, pp.68-80, 2011.

[18] J. Chattopadhyay, S. Pal, A.E. Abdllaoui,"Classical predator-prey system with in-fection of prey population-a mathematical model,"*Math. Methods Appl.Sci.*, vol. 26,no.14,pp.1211-1222,2003.

[19] E. Venturino, Epidemics in predator- prey models: disease in the predators, *IMAJMathApplMedBiol*,vol.19,pp: 285-305, 2002.

[20] M. Haque and E. Venturino, "Increase of the prey may decrease the healthy predator population in presence of a disease in the predator," Hermis,vol.7,pp.38-59,2006.

[21] M. Haque and E. Venturino, "An eco-epidemiological model with disease in predator:the ratio- dependent case," Mathematical Methods in the Applied Sciences, vol. 30, no.14,pp.1791-1809,2007.

[22] Birkhoff, G., Rota, G. C." Ordinary Differential equations."itGinnBoston,1982.