

Python for Numerical Integration

Python Codes for Different Methods of Numerical Integration of a Particular Function $f(x) = \frac{1}{1+x^2}$

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Abstract: This study has been undertaken to learn coding of methods of Numerical Integration. Here we study the methods in detail and relate the formulae for each method by coding. The integral of a function is normally described as the "area under the curve." In engineering and science, the integral has many applications for modeling, predicting and understanding physical systems. However in practice, finding an exact solution for the integral of a function is difficult or impossible. This paper describes several methods of numerically integrating functions. By the end of this paper, one should understand these methods and their Python codes.

Keywords: Python, integration points, float, int, curve

1. Introduction

Numerical integration methods can generally be described as combining evaluations of the integrand to get an approximation to the integral. The integrand is evaluated at a finite set of points called **integration points** and a weighted sum of these values is used to approximate the integral. Here we study 5 rules of numerical integration of $f(x) = \frac{1}{1+x^2}$ and hence study simple Python codes that can be easily related to the respective formulae

2. Mid Point Rule

Assume that $f(x)$ is continuous on $[a, b]$. Let n be a positive integer and $h = \frac{b-a}{n}$. If $[a, b]$ is divided into n subintervals, each of length h , and $m_i = \frac{x_i+x_{i-1}}{2}$ is the midpoint of the i th subinterval,

Then

$$\int_a^b f(x) dx = h \sum_{i=1}^n y_i$$

Where $y_i = f(\frac{x_i+x_{i-1}}{2}), i = 1, 2, 3, \dots, n$

Example

Solve

$$\int_0^1 \frac{1}{1+x^2} dx$$

divide $[0, 1]$ into 6 subintervals of length h i.e. $h = \frac{1}{6}$

where $h = \frac{b-a}{6}, a = 0$ and $b = 1$

Solution

| | | | | | | | |
|-----|-----------|---------------------|---------------------|---------------------|---------------------|---------------------|-----------|
| x | $x_0 = 0$ | $x_1 = \frac{1}{6}$ | $x_2 = \frac{2}{6}$ | $x_3 = \frac{3}{6}$ | $x_4 = \frac{4}{6}$ | $x_5 = \frac{5}{6}$ | $x_6 = 1$ |
|-----|-----------|---------------------|---------------------|---------------------|---------------------|---------------------|-----------|

| | | | | | | |
|------------|-------------|---------------|-------------|-------------|------|-------------|
| $y = f(x)$ | 0.993103448 | 0.94117647058 | 0.852071005 | 0.746113989 | 0.64 | 0.543396226 |
|------------|-------------|---------------|-------------|-------------|------|-------------|

$$\int_a^b f(x) dx = h \sum_{i=1}^n y_i = \frac{1}{6} (y_1 + y_2 + y_3 + \dots + y_6) = 0.7859768568056095$$

Python code

```
#mid point rule
a=float(input("Enter a:-"))
b=float(input("Enter b:-"))
n=int(input("Enter n:-"))
def f(x):
    return 1/(1+x*x)
h=(b-a)/n
x=[]
y=[]
for i in range (n+1):
    x.append(a+i*h)
for i in range (1,n+1):
    y.append(f((x[i]+x[i-1])/2))
print(x)
print(y)
sum=0
for i in range (n):
    sum =sum +y[i]
print('the integral is', sum*h)

>>>
Enter a:-0
Enter b:-1
Enter n:-6
[0.0, 0.16666666666666666, 0.3333333333333333, 0.5, 0.6666666666666666, 0.8333333333333333, 1.0]
[0.993103448275862, 0.9411764705882353, 0.8520710059171597, 0.7461139896373058, 0.64, 0.5433962264150943]
the integral is 0.7859768568056095
```

>>>

Trapezoidal rule

In mathematics, the trapezoidal rule, also known as the trapezoid rule or trapezium rule is a technique for approximating the definite integral in numerical analysis. The trapezoidal rule is an integration rule used to calculate the area under a curve by dividing the curve into small trapezoids. The summation of all the areas of the small

trapezoids will give the area under the curve. Let us understand the trapezoidal rule formula and its proof using examples in the upcoming sections.

The **trapezoidal rule** is applied to solve the definite integral of the form $\int_a^b f(x) dx$, by approximating the region under the graph of the function $f(x)$ as a trapezoid and calculating its area. Under the trapezoidal rule, we evaluate the area under a curve is by dividing the total area into little trapezoids rather than rectangles.

Trapezoidal Rule Formula

We apply the trapezoidal rule formula to solve a definite integral by calculating the area under a curve by dividing the total area into little trapezoids rather than rectangles. This rule is used for approximating the definite integrals where it uses the linear approximations of the functions. The trapezoidal rule takes the average of the left and the right sum.

Let $y = f(x)$ be continuous on $[a, b]$. We divide the interval $[a, b]$ into n equal subintervals, each of width, $h = (b - a)/n$, such that $a = x_0 < x_1 < x_2 < \dots < x_n = b$

$$Area = \frac{h}{2} [y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n]$$

where,

y_1, y_2, y_3, \dots are the values of function at $x = 1, 2, 3, \dots$ respectively.

Example

Solve

$$\int_0^1 \frac{1}{1+x^2} dx$$

divide $[0,1]$ into 6 subintervals of length h i.e. $h = \frac{1}{6}$

where $h = \frac{b-a}{6}, a = 0$ and $b = 1$

Solution

| | | | | | | | |
|-----|-----------|---------------------|---------------------|---------------------|---------------------|---------------------|-------------|
| x | $x_0 = 0$ | $x_1 = \frac{1}{6}$ | $x_2 = \frac{1}{3}$ | $x_3 = \frac{1}{2}$ | $x_4 = \frac{2}{3}$ | $x_5 = \frac{5}{6}$ | $x_6 = 1$ |
| y | $y_0 = 1$ | $y_1 = 0.9729729$ | $y_2 = 0.8999999$ | $y_3 = 0.8$ | $y_4 = 0.692307$ | $y_5 = 0.5901639$ | $y_6 = 0.5$ |

$$\therefore \int_0^1 \frac{1}{1+x^2} dx = \frac{h}{2} [y_0 + 2(y_1 + y_2 + y_3 + \dots + y_5) + y_6] = 0.7842407666178159$$

Note that

$$x_i = a + ih \text{ and } y_i = f(a + ih) = \frac{1}{1 + (a + ih)^2}, i = 0,1,2,3,4,5,6$$

Python code

```
#trapezoidal rule
def f(x):
    return 1/(1+x**2)
a=float(input("Enter a:-"))
b=float(input("Enter b:-"))
n=int(input("Enter n:-"))
h=(b-a)/n
x=[]
y=[]
for i in range (n+1):
    x.append(a+i*h)
    y.append(f(a+i*h))
print(x)
print(y)
#trapezoidal rule
sum=0
for i in range (1,n):
    sum=sum+y[i]
```

```
print((y[0]+(2*sum)+y[n])*h/2)
>>>
Enter a:-0
Enter b:-1
Enter n:-6
[0.0, 0.16666666666666666, 0.3333333333333333, 0.5,
0.6666666666666666, 0.8333333333333333, 1.0]
[1.0, 0.972972972972973, 0.8999999999999999, 0.8,
0.6923076923076923, 0.5901639344262296, 0.5]
0.7842407666178159
>>>
```

Solve

$$\int_0^1 \frac{1}{1+x^2} dx$$

divide $[0,1]$ into 6 subintervals of length h i.e. $h = \frac{1}{6}$

where $h = \frac{b-a}{6}, a = 0$ and $b = 1$

Solution

| | | | | | | | |
|-----|-----------|---------------------|---------------------|---------------------|---------------------|---------------------|-------------|
| x | $x_0 = 0$ | $x_1 = \frac{1}{6}$ | $x_2 = \frac{1}{3}$ | $x_3 = \frac{1}{2}$ | $x_4 = \frac{2}{3}$ | $x_5 = \frac{5}{6}$ | $x_6 = 1$ |
| y | $y_0 = 1$ | $y_1 = 0.9729729$ | $y_2 = 0.8999999$ | $y_3 = 0.8$ | $y_4 = 0.692307$ | $y_5 = 0.5901639$ | $y_6 = 0.5$ |

$$\therefore \int_0^1 \frac{1}{1+x^2} dx = \frac{h}{2} [y_0 + 2(y_1 + y_2 + y_3 + \dots + y_5) + y_6] = 0.7842407666178159$$

Note

that

$$x_i = a + ih \text{ and } y_i = f(a + ih) = \frac{1}{1 + (a + ih)^2}, i = 0,1,2,3,4,5,6$$

Simpson’s 1/3 Rule

Simpson’s 1/3rd rule is an extension of the trapezoidal rule in which the integrand is approximated by a second-order polynomial. Simpson rule can be derived from the various way using Newton’s divided difference polynomial, Lagrange polynomial and the method of coefficients. Simpson’s 1/3 rule is defined by:

$$\int_a^b f(x) dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots) + 2(y_2 + y_4 + y_6 + \dots)]$$

This rule is known as Simpson’s **One-third rule**.

Simpson’s $\frac{1}{3}$ Rule formula for n=2

We can get a quick approximation for definite integrals when we divide a small interval [a, b] into two parts. Therefore, after dividing the interval, we get;

$$x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$$

Hence, we can write the approximation as;

$$\int_a^b f(x) dx \approx S_2 = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)]$$

$$S_2 = \frac{h}{3} [f(a) + 4f((a+b)/2) + f(b)]$$

Where $h = (b - a)/2$

This is the Simpson’s $\frac{1}{3}$ rule for integration for $n = 2$.

Example

Solve

$$\int_0^1 \frac{1}{1+x^2} dx$$

divide [0,1] into 6 subintervals of length h i.e. $h = \frac{1}{6}$

where $h = \frac{b-a}{6}, a = 0$ and $b = 1$

Solution

| | | | | | | | |
|---|-----------|---------------------|---------------------|---------------------|---------------------|---------------------|-------------|
| x | $x_0 = 0$ | $x_1 = \frac{1}{6}$ | $x_2 = \frac{1}{3}$ | $x_3 = \frac{1}{2}$ | $x_4 = \frac{2}{3}$ | $x_5 = \frac{5}{6}$ | $x_6 = 1$ |
| y | $y_0 = 1$ | $y_1 = 0.9729729$ | $y_2 = 0.8999999$ | $y_3 = 0.8$ | $y_4 = 0.692307$ | $y_5 = 0.5901639$ | $y_6 = 0.5$ |

$$\therefore \int_0^1 \frac{1}{1+x^2} dx = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4 + y_6)] = 0.7842407666178159$$

Note

that

$$x_i = a + ih \text{ and } y_i = f(a + ih) = \frac{1}{1 + (a + ih)^2}, i = 0,1,2,3,4,5,6$$

Python code

```
#simpson's 1/3 rule
def f(x):
    return 1/(1+x**2)
a=float(input("Enter a:-"))
b=float(input("Enter b:-"))
n=int(input("Enter n:-"))
h=(b-a)/n
x=[]
y=[]
for i in range (n+1):
    x.append(a+i*h)
    y.append(f(a+i*h))
print(x)
print(y)
#trapezoidal rule
sum=0
for i in range (1,n):
    if i%2==0:
        sum=sum+2*y[i]
    else:
```

```
sum=sum+4*y[i]
print((y[0]+sum+y[n])*h/3)

>>>
Enter a:-0
Enter b:-1
Enter n:-6
[0.0, 0.16666666666666666, 0.3333333333333333, 0.5,
0.6666666666666666, 0.8333333333333333, 1.0]
[1.0, 0.972972972972973, 0.8999999999999999, 0.8,
0.6923076923076923, 0.5901639344262296, 0.5]
0.7853979452340107
>>>
```

Simpson’s 3/8 Rule

Another method of numerical integration is called “Simpson’s 3/8 rule”. It is completely based on the cubic interpolation rather than the quadratic interpolation. Simpson’s 3/8 or three-eight rule is given by:

$$\int_a^b f(x) dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5) + \dots + 2(y_3 + y_6 + y_9 + \dots)]$$

This rule is more accurate than the standard method, as it uses one more functional value. For 3/8 rule, the composite Simpson's 3/8 rule also exists which is similar to the generalized form. The 3/8 rule is known as Simpson's second rule of integration.

Example

Solution

| | | | | | | | |
|---|-----------|---------------------|---------------------|---------------------|---------------------|---------------------|-------------|
| x | $x_0 = 0$ | $x_1 = \frac{1}{6}$ | $x_2 = \frac{1}{3}$ | $x_3 = \frac{1}{2}$ | $x_4 = \frac{2}{3}$ | $x_5 = \frac{5}{6}$ | $x_6 = 1$ |
| y | $y_0 = 1$ | $y_1 = 0.9729729$ | $y_2 = 0.8999999$ | $y_3 = 0.8$ | $y_4 = 0.692307$ | $y_5 = 0.5901639$ | $y_6 = 0.5$ |

$$\therefore \int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)] = 0.7853958624450428$$

Note that

$$x_i = a + ih \text{ and } y_i = f(a + ih) = \frac{1}{1 + (a + ih)^2}, i = 0,1,2,3,4,5,6$$

Python code

```
#simpson's 3/8
def f(x):
    return 1/(1+x**2)
a=float(input("Enter a:-"))
b=float(input("Enter b:-"))
n=int(input("Enter n:-"))
h=(b-a)/n
x=[]
y=[]
for i in range (n+1):
    x.append(a+i*h)
    y.append(f(a+i*h))
print(x)
print(y)
sum=0
for i in range (1,n):
    if i%3==0:
        sum=sum+2*y[i]
    else:
        sum=sum+3*y[i]
print((y[0]+sum+y[n])*3*h/8)

>>>
Enter a:-0
Enter b:-1
Enter n:-6
```

Solution

| | | | | | | | |
|---|-----------|---------------------|---------------------|---------------------|---------------------|---------------------|-------------|
| x | $x_0 = 0$ | $x_1 = \frac{1}{6}$ | $x_2 = \frac{1}{3}$ | $x_3 = \frac{1}{2}$ | $x_4 = \frac{2}{3}$ | $x_5 = \frac{5}{6}$ | $x_6 = 1$ |
| y | $y_0 = 1$ | $y_1 = 0.9729729$ | $y_2 = 0.8999999$ | $y_3 = 0.8$ | $y_4 = 0.692307$ | $y_5 = 0.5901639$ | $y_6 = 0.5$ |

$$\therefore \int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{10} [(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)] = 0.7842407666178159$$

Note that

$$x_i = a + ih \text{ and } y_i = f(a + ih) = \frac{1}{1 + (a + ih)^2}, i = 0,1,2,3,4,5,6$$

Solve

$$\int_0^1 \frac{1}{1+x^2} dx$$

divide [0,1] into 6 subintervals of length h i.e. $h = \frac{1}{6}$

where $h = \frac{b-a}{6}, a = 0$ and $b = 1$

```
[0.0, 0.16666666666666666, 0.3333333333333333, 0.5,
0.6666666666666666, 0.8333333333333333, 1.0]
[1.0, 0.972972972972973, 0.8999999999999999, 0.8,
0.6923076923076923, 0.5901639344262296, 0.5]
0.7853958624450428
>>>
```

Weddle's Rule

$$\int_a^b f(x) dx = \frac{3h}{10} [(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6) + (y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) + \dots]$$

Example

Solve

$$\int_0^1 \frac{1}{1+x^2} dx$$

Divide [0,1] into 6 subintervals of length h i.e. $h = \frac{1}{6}$

where $h = \frac{b-a}{6}, a = 0$ and $b = 1$

Python code

```

#weddle's rule
a=float(input("Enter a:-"))
b=float(input("Enter b:-"))
n=int(input("Enter n:-"))
import math
def f(x):
    return 1/(1+x**2)
h=(b-a)/n
x=[]
y=[]
for i in range (n+1):
    x.append(a+i*h)
    y.append(round(f(a+i*h),4))
if n%6==0:
    print(x)
    print(y)
    sum=0
    sum =sum
+((3*h)/10)*(y[0]+5*y[1]+y[2]+6*y[3]+y[4]+5*y[5]+y[n])
    print('the integral is', sum)
else:
    print('error')

>>>
Enter a:-0
Enter b:-1
Enter n:-6
[0.0, 0.16666666666666666, 0.3333333333333333, 0.5,
0.6666666666666666, 0.8333333333333333, 1.0]
[1.0, 0.973, 0.9, 0.8, 0.6923, 0.5902, 0.5]
the integral is 0.7854150000000001

```

Acknowledgment

I thank the references mentioned.

References

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