Hyers-Ulam-Rassias Stability of Third Order Partial Differential Equation

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Abstract: ***In this paper, we prove the Hyers- Ulam- Rassias (HUR) stability of third order partial differential equation:*** $p(x,t)u_{xxx}(x,t)+p_x(x,t)\,u_{xx}(x,t)+p(x,t)u_x(x,t)=g(x,t,u(x,t))$.

Keywords: Hyers- Ulam- Rassias Stability, Banach's contraction principle, partial differential equation, Functional equations.

1. Introduction

S. M. Ulam [17] gave a well-known talk on stability for several functional equations in 1940. Ulam spoke about a problem concerning the stability of group homomorphism. In 1941, D. H. Hyers [5] provided a partial answer to this Ulam's problem. In 1998, Alsina and Ger [3] investigated the HU stability of the differential equation y' = y. In 2002, Takahasi et al. [16] generalized the result for y' = λy. There have been number of publications on stability of solutions to differential equations [6, 7] and partial differential equations [8, 9]. This stability is now known as the Hyers Ulam (HU) stability and its various extensions has been named with additional word. Hyers Ulam Rassias (HUR) stability is one such extension. In [10] and [11], HUR stability of n^{th} order linear differential operators with non-constant coefficients is invested. HUR stability for special types of non-linear equations have been studied in [1, 2, 12, 13]. In 2011, Gordji et al. [4] established the HUR stability of non-linear partial differential equations by applying Banach's Contraction Principle.

In 2019, Sonalkar et. al. [14], using the Laplace transform method, proved the HUR Stability of linear partial differential equations. The result in [14], is extended to n^{th} order linear partial differential equation by Sonalkar et. al.[15]. In this paper, by using the result of [4], we prove the HUR stability of third order partial differential equation:

$p(x,t)u_{xxx}(x,t)+p_x(x,t)\,u_{xx}(x,t)+p(x,t)u_x(x,t)=g(x,t,u(x,t))$. (1.1)

Here $p: J\times J\to R^+$ is a differentiable function atleast once w. r. t. both the arguments and $p(x,t)\neq 0, \forall x,t\in J$, $J=[a,b]$ be a closed interval and $g: J\times J\times R\to R$ be a continuous function.

Definition 1.1: A function $u: J\times J\to R$ is called a solution of equation (1.1) if $u\in C^3(J\times J)$ and satisfies the equation (1.1).

2. Preliminaries

Definition 2.1: The equation (1.1) is said to be HUR stable if the following holds:

Let $\varphi: J\times J\to (0,\infty)$ be a continuous function. Then ∃ a continuous function $\Psi: J\times J\to(0,\infty)$, which depends on φ such that whenever $u: J\times J\to R$ is a continuous function with
$|p(x,t)u_{xxx}(x,t)+p_x(x,t)u_{xx}(x,t)+p(x,t)u_x(x,t)-g(x,t,u(x,t))|\leq\varphi(x,t)$, (2.1)

There exists a solution $u_0: J\times J\to R$ of (1.1) such that
$|u(x,t)-u_0(x,t)|\leq\Psi(x,t), \quad \forall (x,t)\in J\times J$.

We need the following result.

Banach Contraction Principle:

Let (Y, d) be a complete metric space, then each contraction map $T: Y\to Y$ has a unique fixed point, that is, there exists $b\in Y$ such that $Tb=b$. Moreover,

$$d\,(b,w\leq\frac{1}{(1-a)}d(w,Tw), \forall\, w\in Y \text{ and } 0\leq a<1$$

Using the results from Gordji et al. [4], we establish the following result.

3. Main Result

In this section we prove HUR stability of third or derpartial differential equation (1.1).

Theorem 3.1: Let $c\in J$. Let p and g be as in (1.1) with additional conditions:

1) $P\,(x,t)\geq 1, \forall\; x,t\in J$.
2) $\varphi: J\times J\to(0,\infty)$ be a continuous function and $M: J\times J\to[1,\infty)$ be an integrable function.
3) Assume that there exists γ, $0<\gamma<1$suchthat

$$\int_c^x M(\tau,t)\varphi(\tau,t)d\tau\;\leq\gamma\varphi(x,t), \qquad (3.1)$$

$$\int_c^x\int_c^\tau M(\alpha,t)\varphi(\alpha,t)d\alpha d\tau\;\leq\gamma\varphi(x,t) \qquad (3.2)$$

and

$K\big(x,t,u(x,t)\big)=p(x,t)^{-1}\big[p(c,t)u_{xx}(c,t)-$
$cxp(\alpha,t)u\alpha\alpha,td\alpha+cxg\tau,t,u\tau,td\tau.$ (3.3)

Suppose that the following holds:

C1: $|K(\tau,t,l(\tau,t))-K(\tau,t,m(\tau,t))|\leq M(\tau,t)|l(\tau,t)-m(\tau,t)|, \forall\tau, t\in J$ and $l, m\in C(J\times J)$.

C2: $u: J\times J\to R$ be a function satisfying the inequality (2.1).

Then there exists a unique solution $u_0:J\times J\to R$ of the equation (1.1) of the form

$u_0(x,t) = u(c,t) + \int_c^x \int_c^\tau K(\alpha,t,u_0(\alpha,t))d\alpha d\tau$

such that

$$|u(x,t) - u_0(x,t)| \le \frac{\gamma}{(1-\gamma)}\varphi(x,t), \qquad \forall\, x,t \in J.$$

Proof: Consider

$|p(x,t)u_{xxx}(x,t)+p_x(x,t)u_{xx}(x,t)+ p(x,t)u_x(x,t) - g(x,t,u(x,t))|$
$=|\{p(x,t)u_{xx}(x,t)\}_x + p(x,t)\, u_x\,(x,t) - g\,(x,t,u\,(x,t))|$.

From the inequality (2.1), we get

$|\{p(x,t)u_{xx}(x,t)\}_x + p(x,t)u_x(x,t) - g(x,t,u(x,t))| \le \varphi(x,t).$

$\Rightarrow -\varphi(x,t) \le \{p(x,t)u_{xx}(x,t)\}_x + p(x,t)u_x(x,t) - g(x,t,u(x,t)) \le \varphi(x,t).$

$\Rightarrow \{p(x,t)u_{xx}(x,t)\}_x + p(x,t)u_x(x,t) - g(x,t,u(x,t)) \le \varphi(x,t).$

Integrating from c to x we get,

$$p(x,t)u_{xx}(x,t) - p(c,t)u_{xx}(c,t) + \int_c^x p(\alpha,t)u_\alpha(\alpha,t)\,d\alpha - \int_c^x g(\tau,t,u(\tau,t))\,d\tau \le \int_c^x \varphi(\tau,t)d\tau.$$

$$\Rightarrow p(x,t)\Big\{u_{xx}(x,t) - p(x,t)^{-1}\Big[p(c,t)u_{xx}(c,t) - \int_c^x p(\alpha,t)u_\alpha(\alpha,t)\,d\alpha + \int_c^x g(\tau,t,u(\tau,t))\,d\tau\Big]\Big\} \le \int_c^x \varphi(\tau,t)d\tau.$$

$$\Rightarrow \Big\{u_{xx}(x,t) - p(x,t)^{-1}\Big[p(c,t)u_{xx}(c,t) - \int_c^x p(\alpha,t)u_\alpha(\alpha,t)\,d\alpha + \int_c^x g(\tau,t,u(\tau,t))\,d\tau\Big]\Big\} \le p(x,t)^{-1}\int_c^x \varphi(\tau,t)d\tau.$$

$$\Rightarrow \Big\{u_{xx}(x,t) - p(x,t)^{-1}\Big[p(c,t)u_{xx}(c,t) - \int_c^x p(\alpha,t)u_\alpha(\alpha,t)\,d\alpha + \int_c^x g(\tau,t,u(\tau,t))\,d\tau\Big]\Big\} \le \int_c^x \varphi(\tau,t)d\tau, \quad (\because p(x,t) \ge 1).$$

$$\Rightarrow \{u_{xx}(x,t) - K(x,t,u(x,t))\} \le \int_c^x \varphi(\tau,t)d\tau.$$

Where $K(x,t,u(x,t))$isgivenbyequation(3.3).

Since $M{:}J\times J\to[1,\infty)$ be an integrable function, we have

$$\{u_{xx}(x,t) - K(x,t,u(x,t))\} \le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$\{u_{xx}(x,t) - K(x,t,u(x,t))\} \le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \;\le \gamma\varphi(x,t).$$

$\Rightarrow \{u_{xx}(x,t) - K(x,t,u(x,t))\} \le \varphi(x,t), (\because 0 < \gamma < 1).$ (3.5)

Again, integrating from c to x we get,

$$u_x(x,t) - u_x(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \le \int_c^x \varphi(\tau,t)d\tau.$$

Since $M{:}\ J\times J\to[1,\infty)$ be an integrable function, we have

$$u_x(x,t) - u_x(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$u_x(x,t) - u_x(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \quad (3.4)$$
$$\le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \le \gamma\varphi(x,t).$$

$$\Rightarrow\; u_x(x,t) - u_x(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \le \gamma\varphi(x,t).$$

$\Rightarrow\; u_x(x,t) - u_x(c,t) - \int_c^x K(\tau,t,u(\tau,t))d\tau \le \varphi(x,t), \quad (\because 0 < \gamma < 1).$

Again, integrating from c to x we get,

$$u(x,t) - u(c,t) - \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau \le \int_c^x \varphi(\tau,t)d\tau.$$

$$\Rightarrow u(x,t) - \Big[u(c,t) + \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau\Big] \le \int_c^x \varphi(\tau,t)d\tau.$$

Since $M{:}\ J\times J\to[1,\infty)$ be an integrable function, we have

$$u(x,t) - \Big[u(c,t) + \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau\Big] \le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau.$$

Using inequality (3.1) we have,

$$u(x,t) - \Big[u(c,t) + \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau\Big] \le \int_c^x M(\tau,t)\varphi(\tau,t)d\tau \le \gamma\varphi(x,t).$$

$\Rightarrow u(x,t) - \left[u(c,t) + \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau\right] \le \gamma\varphi(x,t).$ (3.6)

In a similar way, from the left inequality of (3.4), we obtain

$-\left\{u(x,t) - \left[u(c,t) + \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau\right]\right\} \le \gamma\varphi(x,t).$ (3.7)

From the inequalities (3.6) and (3.7) we get,

$\left|u(x,t) - \left[u(c,t) + \int_c^x \int_c^\tau K(\alpha,t,u(\alpha,t))d\alpha d\tau\right]\right| \le \gamma\varphi(x,t).$ (3.8)

Let Y be the set of all continuously differentiable functions l: $J\times J\rightarrow \mathbb{R}$. We define a metric d and an operator T on Y as follow: For $l, m \in Y$

$$d(l,m) = sup_{x,t\ \in J}\left|\frac{l(x,t) - m(x,t)}{\varphi(x,t)}\right|$$

And the operator

$(T_m)(x,t) = u(c,t) + \int_c^x \int_c^\tau K\big(\alpha, t, m(\alpha,t)\big) d\alpha d\tau$.(3.9)

Consider,

$$d(Tl,Tm) = sup_{x,t\ \in J}\left|\frac{Tl(x,t) - Tm(x,t)}{\varphi(x,t)}\right|.$$

$$= sup_{x,t\ \in J}\left\{\frac{\int_c^x\int_c^\tau K(\alpha,t,l(\alpha,t))d\alpha d\tau\ -\ \int_c^x\int_c^\tau K(\alpha,t,m(\alpha,t))d\alpha d\tau}{\varphi(x,t)}\right\}.$$

$$\leq sup_{x,t\ \in J}\left\{\frac{\int_c^x\int_c^\tau |K(\alpha,t,l(\alpha,t)) - K(\alpha,t,m(\alpha,t))| d\alpha d\tau}{\varphi(x,t)}\right\}.$$

By using condition C1 we get,

$$d(Tl,\ Tm) \leq sup_{x,t\ \in J}\left\{\frac{\int_c^x\int_c^\tau M(\alpha,t)|\ l(\alpha,t) - m(\alpha,t)| d\alpha d\tau}{\varphi(x,t)}\right\}.$$

$$= sup_{x,t\ \in J}\left\{\frac{\int_c^x\int_c^\tau M(\alpha,t)\varphi(\alpha,t)\frac{|\ l(\alpha,t) - m(\alpha,t)|}{\varphi(\alpha,t)}\ d\alpha d\tau}{\varphi(x,t)}\right\}.$$

$$\leq sup_{x,t\ \in J}\left\{\frac{\int_c^x\int_c^\tau M(\alpha,t)\varphi(\alpha,t)\times sup_{\ \alpha,t\ \in J}\frac{|\ l(\alpha,t) - m(\alpha,t)|}{\varphi(\alpha,t)}\ d\alpha d\tau}{\varphi(x,t)}\right\}.$$

$$= d(l,m)\times sup_{x,t\ \in J}\left\{\frac{\int_c^x\int_c^\tau M(\alpha,t)\varphi(\alpha,t)\ d\alpha d\tau}{\varphi(x,t)}\right\}.$$

By using inequality (3.2) we get,

$d(Tl,Tm) \leq \gamma\ \times d(l,m)$.

By using Banach contraction principle, there exists a unique $u_0 \in Y$ such that $Tu_0 = u_0$, that is

$u(c,t) + \int_c^x\int_c^\tau K\big(\alpha,t,u_0(\alpha,t)\big)d\alpha d\tau = u_0(x,t)$, (by using equation(3.9))

and

$d(u_0,u) \leq \frac{1}{(1-\gamma)} d(u,Tu)$.(3.10)

Now by using inequality (3.8) we get,

$$|u(x,t) - (Tu)(x,t)| \leq \gamma\varphi(x,t).$$

$$\Rightarrow \frac{|u(x,t) - (Tu)(x,t)|}{\varphi(x,t)} \leq \gamma\ .$$

$$\Rightarrow sup_{x,t\ \in J}\left\{\frac{|u(x,t) - (Tu)(x,t)|}{\varphi(x,t)}\right\} \leq \gamma.$$

Thus $d(u,Tu) \leq \gamma$.(3.11)

Again,

$$d(u_0,u) = sup_{x,t\ \in J}\left|\frac{u_0(x,t)\ -\ u(x,t)}{\varphi(x,t)}\right|.$$

From equation (3.10) we get,

$d(u_0,u) \leq \frac{1}{(1-\gamma)} d(u,Tu)$.

$$sup_{x,t\ \in J}\left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \leq \frac{1}{(1-\gamma)} d(u,Tu).$$

$$\left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \leq sup_{x,t\ \in J}\left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \leq \frac{1}{(1-\gamma)} d(u,Tu).$$

$$\Rightarrow \left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \leq \frac{1}{(1-\gamma)} d(u,Tu).$$

From equation (3.11) we get,

$$\left|\frac{u_0(x,t) - u(x,t)}{\varphi(x,t)}\right| \leq \frac{\gamma}{(1-\gamma)}.$$

$$\left|\frac{u(x,t) - u_0(x,t)}{\varphi(x,t)}\right| \leq \frac{\gamma}{(1-\gamma)}.$$

$$|u(x,t) - u_0(x,t)| \leq \frac{\gamma}{(1-\gamma)}\varphi(x,t), \forall\ x,t\ \in J.$$

Hence the result.

4. Conclusion

In this paper we have proved the HUR stability of the third order partial differential equation (1.1) by employing Banach's contraction principle.

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