A Literature Survey on Fixed-Point Results in Metric and Partial Order Metric Spaces

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Abstract: ***The aim of this paper is to investigate some fixed- point results in metric space and partial order metric spaces. Fixed-point results for different types of contractive conditions are established, which generalize some existing fixed-point theorems.Partial b-metric spaces are characterized by a modified triangular inequality and that the self-distance of any point of space may notbe zero and the symmetry is preserved. The spaces with a symmetric property have interesting topological properties. Our results some fixed-point results in the context of metric spaces and partial order metric spaces can be deduced.***

Keywords: Fixed-point theorem; Metric Space; Partial order metric space; Contraction mapping; Banach Space

1. Introduction

In Present time, Fixed-point results are one of the most familiar research domains for its various applications in several fields in Mathematics. In mathematics, this technique plays a prominentrole in the study of statistical models, dynamical systems, game-theoretic models, differential equations, and many others. More clearly, for example, this method is mainly applied in finding the analytical solution to some differential and integral equations, fractional equations, integro-differential equations (IDEs), and functional analysis which facilitates the way to find numerical solutions to such problems.An interesting generalization of Banach's Contraction principal was introduced by Ćirć [1] to establish the existence and uniqueness of fixed-point with multivalued contraction. However, the potentially of fixed-point approaches attracted many scientists and hence there is a wide literature available for interested readers. We give some details on the notions and idea used in this study.

First, the notion of partial metric space was introduced in 1994 by Matthews [2], an attempt to generalize the metric space by replacing the condition $d(X,X)=0$ with the condition $d(x,x) \leq d(x,y)\, \forall x,y$, in the definition of metric, as a part of the study of denotational semantics of data for networks. Clearly, this setting is a generalization of the classical concept of metric space.

In this article, we have extended the concept of metric spaces and partial order metric space. Finally, investigated the conditions of the existence of fixed-point results. We also have given some suitable examples to understand the value of this results.

The b-metric space was introduced by Czerwik [3]. It isobtained by modifying the triangle property of the metricspace. Every metric is a b-metric, but the converse is nottrue. Almost all the fixed-point theorems in the metricspaces have been proved true in the b-metric spaces; forexample, see [4–12] and references therein.Matthews [13] introduced the notion of the partial metricspace as a part of the study of denotational semantics ofthe dataflow network. In this space, the usual metric isreplaced by a partial metric having a property that the self distance of any point of the space may not be zero. Everymetric is a partial metric, but the converse is not true. Matthews[13] also initiated the fixed-point theory in the partialmetric space. He proved the Banach contraction principle inthis space to be applied in program verification. We can findso many fixed-point theorems in the metric spaces whichhave been proved in the partial metric spaces by manyfixed-point theorists ([14,17] and references therein).Shukla [15] introduced the concept of partial b-metric bymodifying the triangle property of the partial metric andinvestigated fixed points of Banach contraction and Kannancontraction in the partial b-metric spaces. Mustafa et al. [16] modified the triangle property of partial b-metric and establisheda convergence criterion and some working rules inpartial b-metric spaces. Moreover, Mustafa et al.

In 2000 Branciari [21] introduced a class of generalized metric spaces by replacing triangular inequality by similar ones which involve four or more points instead of three and improved Banach contraction mapping principle. Recently Azam and Arshad [20] in 2008 extended the Kannan's theorem for this kind of generalized metric spaces. In 2009 [19] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh introduced new classes of contractive functions and established the Banach contractive principle. In the present paper at first, we extend the Kannan's theorem [22] and then extend the theorem due to Azam and Arshad [19] for these new classes of functions.

2. Some Notations and definitions

Let R^+ denote the set of all non-negative real numbers and N denote the set of all positive integers.

Definition 1. ([18]) Let X be a set and $d : X^2 \to R^+$be a mapping such thatfor all $x, y \in X$ and for all distinct points $z, w \in X$ each of them different from $x$and y, one has
(i) $d(x,y) = 0$ if and only if $x = y$,
(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,y) = d(x,z) + d(z,w) + d(w,y)$

then we will say that (X,d) is a generalized metric space (or shortly g.m.s).Any metric space is a g.m.s but the converse is not true([18], see also Example1).As in a metric space, a topology can be defined in a g.m.sX with the helpof the neighbourhood basis given by $B = \{B(x,r);\ x \in X, r \in R \setminus \{0\}\}$ where$B(x,r) = \{y \in X;\ d(x,y) < r\}$ is the open ball with centre x and radius r.

Example 1: Let $X = C$ the set of complex numbers and $x, y \in X$ define d by $d(x,y) = |X - Y|$. Then (X,d) is a metric space.

Example 2: let any non-empty set X with metric d define as $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. Here d is called the discrete metric on X and (X,d) called discreate space.

Definition 2: Let (X,d) be a metric space. Then the map $f: X \to X$ is a Contraction mapping of (X,d) if for some real number $0 \le c < 1$ called constant of contraction
$d\big(f(x), f(y)\big) \le cd(x,y)\ \forall x, y \in X.$

Definition 3: Let (X,d) be a complete metric space, and let $T: X \to X$ satisfy
$d(Tx,Ty) \le kd(x,y)$ where $0 \le k \le 1$ and $x, y \in X$.
Then by Banach's fixed-point theorem T has a unique fixed-point.

In the year 1968, Kannan introduced a contraction mapping which is non-continuous and gave a fixed-point. Such as if X is a complete metric-space and $T: X \to X$ is a mapping satisfying,$d(Tx,Ty) \le \alpha[d(x,Tx) + d(y,Ty)], \forall\ x, y \in X$ and $\alpha \in [0,1)$. Then T has a unique fixed-point.

Also, a mapping T on a metric space (X,d) is called Kannan if there exists $\alpha \in [0,\frac{1}{2}]$, such that, $d(Tx,Ty) \le \alpha d(x,Tx) + \alpha d(y,Ty)\ \forall\ x, y\ \in X$. Kannan proved that if X is complete, then every Kannan mapping has a fixed-point. Kannan's theorem is not an extension of definition (1). The metric space X is complete if and only if every Kannan mapping on X has a fixed-point. Kirk proved that Caristi's fixed-point theorem characterizes the metric space.

Example 3: Take]0,1/2] equipped with the metric of absolute value. This is clearly an incomplete metric space. The mapping $T: X \to X\ \ given\ by\ Tx = x^2$ is a contraction but T has no fixed-point.

Remark: The condition of T being a contraction cannot be replaced by the weaker one: namely, Contractive.

Example 4: Let complete metric space $X = [0,\infty[$, consider the mapping
$T: X \to X$ givenby $Tx = \frac{1}{1+x^2}$ then
(a) The mapping T satisfies $d(Tx,Ty) < d(x,y)$ and hence T is a contractive map, while T is not a contraction.
(b) T has no fixed-point.

Definition 4: The triple $(X, d \preccurlyeq)$ is called a partial ordered metric space, if $(X, \preccurlyeq)$ is a partially ordered set together with (X,d) is a metric space.

Definition 5: If (X,d) is a complete metric space then the triple $(X, d \preccurlyeq)$ is called a partially ordered complete metric space.
Geraghty introduced certain type of contraction maps. Namely Geraghty Contraction maps through which the author developed a technique, where the class of Geraghty contraction maps in one among the generalization of contraction maps.

In 2012, Samet, Vetro and Vetra [9] introduced the concept of $\alpha - \varphi$ Contractive maps where α is an $\alpha -$ admissible map. In 2013, Karapinar, Kumar and Salimi [7] introduced $\alpha - \varphi$ Meir-Keeler Contractive maps in complete metric space via triangular $\alpha -$ admissible mapping.

In 2013, Cho, Bae and Karapinar [4] introduced the notion of α −Geraghty contraction type map.

Theorem 1: Let (X,d) be a complete metric space and let $T: X \to X$ be an F- contraction, Then T has a unique fixed point $x^* \in X$ and for every$x_0 \in X$ a sequence $\{T^n x_0\}_{n \to \mathbb{N}}$ is convergent to x^*, gives the family of contraction which is general are not equivalent.

Now, we also proved Theorem1 are equivalent.

Lemma 1: (see [23]). Let Xbe a nonempty set and themappings T, $f: X \to X$have a unique point of coincidence$v$in X.If $T$and $f$are weakly compatible, then $T$and $f$have a unique common fixed point.

Theorem 2: Let $(X,\ q)$ be a left complete Hausdorffquasicone metric space and let$f\ :X \to X$be a continuous function. Suppose that there exist functionsη, λ, ζ, μ, ξ :$X \to [0, 1)$ which satisfy the following for$x$, $y \in X$:
(1) $\eta(f(x)) \le \eta(x)$, $\lambda(f(x)) \le \lambda(x)$, $\zeta(f(x)) \le \zeta(x)$, $\mu(f(x)) \le \mu(x)$ *and* $\xi(f(x)) \le \xi(x)$;
(2) $\eta(x) + \lambda(x) + \zeta(x) + \mu(x) + 2\xi(x) < 1$;
(3) $q(f(x), f(y)) \le \eta(x)q(x, y) + \lambda(x)q(x, f(x)) + \zeta(x)q(y, f(y)) + \mu(x)q(f(x), y) + \xi(x)q(x, f(y))$.

Then, $f$has a unique fixed point.
Lemma 2:(see [24]). Let$(X,\ d)$ be a complex valued b-metric space and let$\{x_n\}$ be a sequence in$X$. Then$\{x_n\}$ converges to$x$if and only if$|d(x_n, x)| \to 0\ as\ n \to \infty$.

Lemma 3:(see [24]). Let$(X,\ d)$ be a complex valued b-metric space and let$\{x_n\}$ be a sequence in$X$. Then$\{x_n\}$ is a Cauchy sequence if and only if$|d(x_n, x_n{+}m)| \to 0$ as$n \to \infty$, where$m \in$N.

These are some literature surveys on fixed-point theorem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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