The Existence Theorem for Optimal Control with Non-Standard Cost Functional and Analysis

Praveen Kumar

Professor, Department of Mathematics, J V Jain College Saharanpur

Abstract: The existence theorem for optimal control system has been obtained for a general non-standard cost functional of fraction type. As an application of our result we can derive an existence theorem for optimal control given by M.B. Subramanyam for a cost functional, which has been given in the fractional form I.e ratio of two given integral cost functionals.

Keywords: Optimization, Control theory, Regularity, Functionals.

1. Introduction:

Consider the n-dimensional system

$$\bar{x}(t) = A(t).x(t) + B(t).u(t), \quad x(t_0) = 0 \ for\ t\epsilon[t_0, t_1] \qquad t_1 < \infty \qquad (1.1)$$

Where A(t) and B(t) are matrices of nxn and nxr respectively.

There can be further restrictions on the functions x(t) and u(t). It will be discussed later. For the above n-dimensional system we have to optimise(minimise) the following functional-

$$F(x, u) = \frac{F_1(u)}{[F_2(x)]^\alpha} \qquad (1.2)$$

Where $\alpha > 0$ and $u(t)$ is a measurable control. We make the following assumptions-

2. Assumptions:

2.1. $A(t)$ and $B(t)$ are continuous matrix functions.

2.2. $F_1(u)$ is convex function with $|F_1(u)| < \infty$ in the domain.

2.3. $F_1(u)$ and $F_2(x)$ both are continuous in the respective domains.

2.4. $F_1(u) \leq a\ |u|^p$ for $p > 1$, $a > 0$; $F_2(x) \geq 0$ along any $x(t)$ which is responsible for some admissible value $u(t)$.

2.5. For each $k < \infty$, if $||\ u\ ||_p < k$ then $F_2(x) < \infty$ for any admissible value of $u(t)$ in the domain whose trajectories obeys any constraints imposed on this functional.

2.6. There exists a $k > 0$ such that for every $c \geq 0$,

$F_1(cu) = c^k.F_1(u) \quad F_2(cx) = c^{k/\alpha}$. by equation (1.2) we get

F(cx, cu)=F(x, u) for every c > 0.

2.7. There exists an admissible control, the trajectories of which satisfy the imposed constraints and is such that $F_2(x) \geq 0$.

2.8. If $<u^i>$ is a sequence of functions on $L_p[t_0, t_1]$ converging weakly to u^0 in $L_p[t_0, t_1]$ then

$$F_1(u^0) \leq \lim_{i \to \infty} \inf F_1(u^i)] \qquad (2.1)$$

3. **Definition:** We call a constraints regular if the following conditions holds:

3.1. If (x, u) satisfies the constraints then (cu, cx) also satisfies the constraints for every $c > 0$.

3.2. Let $(x^1, u^1), (x^2, u^2), \ldots$ be admissible pair such that $(u^i \to u^0)$ weakly in $L_p[t_0, t_1]$. Suppose that (x^n, u^n) satisfies the constraints for each $n \geq 1$. Then (x^0, u^0) obeys the constraints.

4. **Lemma 1:** Consider all values of x & u in the respected domains that obey equation (1.1) and the constraints. Assuming that all the constraints are regular in nature and let

$$\lambda = \inf F(x, u) = \inf \frac{F_1(u)}{[F_2(x)]^{\alpha}} \qquad (4.1)$$

(λ is well defined by assumption 2.2, 2.3 & 2.7)

Also Let

$$\inf F_1(u\ ) = J \quad \text{subject to the } [F_1(x)]\alpha$$

Then $\lambda = J/M$.

***Remark:-** Observe that in the proof of the theorem-1 it has been shown that u^0 is necessarily admissible. First we prove the lemma-1 which is being used later in the sequel.*

Proof: One can easily see that $\lambda \leqslant J/M$. To prove the reverse inequality, let the variable function u() be such that

$$F(x^*, u^*) \leqslant \lambda + \varepsilon \qquad for\ some\ \varepsilon \geq 0$$

Let $[F_1(x)]\alpha = M < \infty$ (by assumption 2.2, 2.4 & 2.5) and $\mu = (M/M^*)^{1/k}$.

Then $(\mu x^*, \mu u^*)$ obeys all the constraints by regularity of the constraints. Now by the assumption 2.5

$[F(\mu x)]\alpha = M$ and $F(\mu x^*, \mu u^*) \leqslant \lambda + \varepsilon$ $\quad for\ some\ \varepsilon \geq 0$ this implies that $J/M \leqslant \lambda + \varepsilon$ Since ε is arbitrary the conclusion of the lemma-1 follows the following theorem.

Theorem: Consider the control system represented by the equation (1.1) and (1.2) along with all the assumptions discussed above 2.1 to 2.8. Also Assume that the constraints x(), u() are regular. Then there exists a control among all admissible controls that minimises equation (1.2).

Proof: By the previous Lemma-1 it is sufficient to exhibits a minimizing control over all admissible controls for which $[F_2(x)]^\alpha = M > 0$ and trajectories of which satisfy equation (1.1) and all the constraints.

$$Let\ J = inf\, F_1(u)\ \ subject\ to\ \ [F_2(x)]^\alpha = M > 0$$

Choose $\{(x^i, u^i)\}$ such that $\lim_{i \to \infty} F_1(u^i) = J$ with $[F_2(x)]^\alpha = M$ for each i. By assumption 2.3, $\{u^i\}$ from a bounded sequence in $L_p[t_0, t_1]$, and hence the subsequence in , and hence a subsequence, still denoted by $\{u^i\}$, converges weakly to some u^0 in $L_p[t_0, t_1]$. Let x^0 be the response of equation (1.1) to u^0. By assumption 2.1 and by weakly convergence $x^i(t), x^0(t)$ for all $t \in [t_0, t_1]$ {Ref.[1]}. By regularity of constraints $x^0(t)$ obeys all the constraints. Assumption 2.2 implies $F_2(x^i) \to F_2(x^0)$ as i.

Since $\|u\|_p < K$ for some $k < \infty$, hence by assumption 2.4,

$$[F_2(x^0)]^\alpha = \lim_{i \to \infty} inf\, [F_2(x^i)]^\alpha = M$$

Now by assumption 2.6,

$$[F_1(u^0) \leq \lim_{i \to \infty} inf\, F_1(u^i) = J$$

Application: If we specialised the functional

$$F(x, u) = \frac{\int_{t_0}^{t_1} \psi_1(x(t), t)\, dt}{[\int_{t_0}^{t_1} \psi_2(u(t), t)\, dt]^\alpha}$$

Where Ψ_1 and Ψ_2 satisfy the conditions given in {Ref.[4], Th-1.1}, we get the existence of the theorem of [4] as a particular case of the present work.

Discussion: The general method that has been discussed can be extended to cover other variants of the functional given by Subramanayam[4]. For further study we can consider the functional of the form

$$F(x,\, u) = \frac{\int_{t_0}^{t_1} \psi_1(x(t), t)\, dt}{\prod_{i=1}^{i=n} \left(\int_{t_0}^{t_1} \psi_i(u(t), t)\, dt] ^{\alpha_i} \right)}$$

Where all to satisfy certain constraints or restrictions analogous to given as in Ref.[4].

References:

1. Lee, E.B. and Markus, L., Foundations of the Optimal Control Theory, John Wiley, New York, 1967.
2. Subhramanyam, M.B., On Necessary condition for minimum in problems with non-standard cont functions, Journal of Mathematical Analysis and Application, Vol. 60, pp. 601-616, 1977.
3. Subhramanyam, M.B., On application of control theory to integral inequalities, Journal of Mathematical Analysis and Application, Vol. 77, pp. 47-59, 1980.
4. Subhramanaym, M.B., On application of control theory to integral inequalities-II, SIAM Journal of Control and Optimization, vol. 19, pp. 479-489, 1981.
5. Miller B., The generalised solutions of nonlinear optimization problems with impulse control, SIAM Journal on Control and Optimization, Vol. 34 (4), pp. 1420-1440, 1996.
6. Karamzin D., De Oliveira V., Pereira F., Silva G., On some extension of optimal control theory, European Journal of Control, Vol. 20 (6), pp. 284-291, 2014.